Limit Cycles Bifurcations for a Class of Kolmogorov Model in Symmetrical Vector Field

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The problem of limit cycles for Kolmogorov model is interesting and significant both in theory and applications. Our work is concerned with limit cycles bifurcations problem for a class of quartic Kolmogorov model with two positive singular points (i.e. (1, 2) and (2, 1)). The investigated model is symmetrical with regard to \( y = x \). We show that each one of points (1, 2) and (2, 1) can bifurcate five small limit cycles at the same step under a certain condition. Hence, the two positive singular points can bifurcate ten limit cycles in sum, in which six cycles can be stable. In terms of symmetrical Kolmogorov model, published references are less. In terms of the Hilbert Number of Kolmogorov model, our results are new.

Keywords: Kolmogorov model; positive equilibrium points; limit cycles; stable cycles.

1. Introduction

Let \( H(m) \) denote the maximal number of limit cycles of polynomial systems of degree \( m \). It is called the Hilbert Number. The main part of Hilbert’s 16th problem posed in 1902 is to find its value. Study on Hilbert Number continues to attract more and more interest and many good results on limit cycles bifurcations have been published (see [Li & Liu, 2010; Du et al., 2010; Zhang & Zheng, 2012; Han & Li, 2012]). Recently, the research on limit cycles bifurcation from several diverse symmetrical singular points has attracted more attention. Li and Liu [2010] introduced some developments in such work. But almost all published references are studies on equivariant systems with regard to the origin. In this paper, our study is concerned with the limit cycle bifurcations for a class of symmetrical systems with regard to line \( y = x \). Our work focuses on

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C. Du et al.

studying the following system

\[
\begin{align*}
\frac{dx}{dt} &= \frac{1}{270} \left[ 270a + 270(A_{10}x + A_{01}y) + a_{30}x^2 + 270A_{11}x^2y + a_{30}x^3 + a_{21}x^2y + a_{03}y^3 + a_{23}y^3 + a_{12}x^2y^2 + a_{03}y^3 \right] \\
&= xP(x, y), \\
\frac{dy}{dt} &= \frac{1}{270} \left[ 270a + 270(A_{10}x + A_{01}y) + a_{20}y^2 + 270A_{11}x^2y + a_{30}y^3 + a_{21}y^2x + a_{03}y^3 + a_{23}y^3 + a_{12}x^2y^2 + a_{03}y^3 \right] \\
&= yQ(x, y),
\end{align*}
\]  

(1)

in which 

\[
a_{30} = -18(4 + 9a + 4A_{10} + 14A_{01} + 6A_{11}),
\]

\[
a_{01} = 2(-1 + 4a - A_{10} + 4A_{01} - 9A_{11}),
\]

\[
a_{21} = -9(-17 + 18a + 28A_{10} + 8A_{01} + 12A_{11}),
\]

\[
a_{12} = 6(4 + 4a + 4A_{10} - A_{10} - 9A_{11}),
\]

\[
a_{00} = -81 + 44a + 54A_{10} + 24A_{11} + 36A_{11},
\]

where \( a, A_{i0}, A_{10}, A_{11} \in \mathbb{R} \). Here, \( x \) and \( y \) of system (1) denote prey and predator densities, and \( P(x, y), Q(x, y) \) are the intrinsic growth rates or biotic potential of the prey and predators, respectively. Obviously, the above system belongs to a class of famous ecologic models namely the Kolmogorov model, realistic meanings of which can be found in [Wang & Jiang, 2012; Saez & Szanto, 1996].

In mathematical ecology, more and more mathematicians have paid attention to the three most fundamental systems namely the predator-prey, the competition and the cooperation systems (for example [Han & Romanowski, 2013; Wang & Jiang, 2012; Saez & Szanto, 1996; Du et al., 2007; Du & Huang, 2013; Lloyd et al., 2002; Huang & Zhu, 2005; Yuan et al., 2012]). A lot of natural predator-prey systems can be discussed and investigated. Theoretically, these systems can be reduced to some kinds of ecological models. From published references, it can be seen that Kolmogorov model is a class of investigated thermal wall. Kolmogorov system’s equation is described as

\[
\begin{align*}
\frac{dx}{dt} &= xP(x, y),
\frac{dy}{dt} &= yQ(x, y),
\end{align*}
\]  

(2)
in which \( f(x, y) \) and \( g(y, x) \) are polynomials. Such Kolmogorov models are widely used in ecology to describe the interaction between two populations. In that case, attention is restricted to the behavior of orbits in the “realistic quadrant” \( \{(x, y): x > 0, y > 0\} \). Of particular significance in applications is the existence of limit cycles and the number of limit cycles that can arise from positive equilibrium points. Because a limit cycle corresponds to an equilibrium state of the system, while the existence and stability of limit cycles is related to the positive equilibrium points. At the same time, the problem on the number of limit cycles gets closely related to the famous Hilbert 16th problem. Hence, many articles studying Kolmogorov models focus more on the limit cycles bifurcation problem. For example, Han and Romanowski [2013] studied the number of limit cycles of polynomial Liénard systems; Saez and Szanto [1996] studied a class of cubic Kolmogorov system with three limit cycles; Du and Huang [2013] showed this class of Kolmogorov systems could bifurcate five limit cycles including three stable cycles; Du and Huang [2013] showed this class of Kolmogorov systems could bifurcate five limit cycles including four stable cycles; Lloyd et al. [2002] showed a class of cubic Kolmogorov systems could bifurcate six limit cycles; Huang and Zhu [2005], Yuan et al. [2012] studied a general Kolmogorov model and obtained the conditions for the existence and uniqueness of limit cycles, at the same time they classified a series of models. As far as limit cycles of Kolmogorov models are concerned, many good results have been obtained by analyzing the sole positive equilibrium point state. But this kind of result is hardly seen on limit cycles bifurcation from several different equilibrium points which are symmetrical with regard to a line or an axis, perhaps it is difficult to investigate this kind of problem.

Clearly, the studied model (1) has two positive equilibrium points namely (1.2) and (2.1) which are symmetrical with regard to line \( y = x \). In fact, system (1) is symmetrical with regard to line \( y = x \). We will focus on the limit cycles bifurcations of the two positive equilibrium points. With the help of a computer and by carefully calculating, we can
obtain the expression of the first five focal values for each positive equilibrium point. We will show that each one of the positive equilibrium points (1, 2) and (2, 1) can bifurcate five limit cycles at the same step under a certain condition. Hence, the two positive singular points can bifurcate ten limit cycles in sum, in which six cycles can be stable. It is worth pointing out that our work presents an interesting bifurcation behavior, namely the limit cycle bifurcations of the Kolmogorov model with two positive equilibrium points with regard to line \( y = x. \)

This paper concludes three sections. In Sec. 2, a method to study limit cycle bifurcation is given by making use of the relation between focal values and singular point values at the origin which is necessary for investigating bifurcations of limit cycles. At the same time, we give the singular point value recursive formulas. In Sec. 3, we compute the focal values of the positive equilibrium points (1, 2) of model (1) and obtain that the positive equilibrium points (1, 2) of model (1) has fifth focal values. Moreover, we discuss the bifurcations of limit cycles of model (1) and obtain that each one of the two positive equilibrium points (1, 2) and (2, 1) of model (1) can arise five small limit cycles under a certain condition. In sum, model (1) can bifurcate ten small limit cycles from two positive equilibrium points of which six limit cycles can be stable. In terms of the number of stable limit cycles and the results on limit cycles bifurcations for Kolmogorov model, our results are new.

2. Our Method to Study Limit Cycles Bifurcations

In order to study limit cycles bifurcations, computing the focal values or Lyapunov constants is often significant. Next, we use the algorithm of singular point values to compute the focal values (or so called Lyapunov constants) and construct the Poincaré succession function. Liu [2001] and Du et al. [2016] gave the relation between focal values and singular point values. Considering the following real system

\[
\begin{align*}
\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x, y), \\
\frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x, y),
\end{align*}
\]  

in which

\[
\begin{align*}
X_k(x, y) &= \sum_{a+b=k} A_{ab} x^a y^b, \\
Y_k(x, y) &= \sum_{a+b=k} B_{ab} x^a y^b,
\end{align*}
\]

under the polar coordinates \( x = r \cos \theta, y = r \sin \theta, \) system (3) takes the following form:

\[
\frac{dr}{d\theta} = r + \sum_{k=2}^{\infty} r^{k-1} \psi_{k+1}(\theta),
\]

in which

\[
\begin{align*}
\psi_{k+1}(\theta) &= \cos \theta X_k(\cos \theta, \sin \theta) \\
&\quad + \sin \theta Y_k(\cos \theta, \sin \theta),
\end{align*}
\]

For sufficiently small \( h, \) let

\[
d(h) = r(2\pi, h) - h,
\]

\[
r(\theta, h) = \sum_{m=1}^{\infty} v_m(\theta) h^m
\]

be the Poincaré succession function and solution of Eq. (4) which satisfy the initial-value condition \( r|_{\theta=0} = h. \) It is evident that

\[
v_1(\theta) = e^{i\theta} > 0, \quad v_m(0) = 0, \quad m = 2, 3, \ldots
\]

Definition 2.1. If \( v_1(2\pi) \neq 1 \) of (5), then the origin of system (3) is called the rough focus (strong focus); if \( v_1(2\pi) = 1 \) of (3), and \( v_2(2\pi) = v_3(2\pi) = \cdots = v_{2k}(2\pi) = 0, v_{2k+1}(2\pi) \neq 0, \) then the origin of system (3) is called fine focus (weak focus) of order \( k, \) and the quantity of \( v_{2k+1}(2\pi), k = 1, 2, \ldots \) is called the \( k \)th focal value (or Lyapunov constants) at the origin of system (3); if \( v_1(2\pi) = 1 \) of (5) and for any positive integer \( k, v_{2k+1}(2\pi) = 0, \) then the origin of system (3) is called a center.

Using the transformation

\[
z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1},
\]
system \((3)|_{k=0}\) can be transformed into the following complex system:

\[
\begin{aligned}
\frac{dz}{dt} &= z + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w), \\
\frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),
\end{aligned}
\]

in which \(z, w, T\) are complex variables and

\[
\begin{aligned}
Z_k(z,w) &= \sum_{\alpha + \beta = k} a_{\alpha \beta} z^\alpha w^\beta, \\
W_k(z,w) &= \sum_{\alpha + \beta = k} b_{\alpha \beta} w^\alpha z^\beta.
\end{aligned}
\]

Obviously, the coefficients of \((6)\) satisfy the conjugate condition, i.e.,

\[
\overline{a_{\alpha \beta}} = b_{\alpha \beta}, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta \geq 2.
\]

System \((3)|_{k=0}\) and system \((6)\) are called concomitant systems.

**Lemma 2.1** (see [Liu, 2001; Du et al., 2010]). For system \((6)\), we can find a unique formal series as follows:

\[
M = 1 + \sum_{\alpha + \beta = 1}^{\infty} c_{\alpha \beta} z^\alpha w^\beta,
\]

such that

\[
\frac{\partial M}{\partial z} \frac{\partial M}{\partial w} W + \left( \frac{\partial Z}{\partial z} - \frac{\partial W}{\partial w} \right) M = \sum_{m=1}^{\infty} (m + 1) \mu_m (zw)^m.
\]

Let \(c_{11} = 1, c_{20} = c_{02} = 0\), by making use of undetermined coefficients and comparing the coefficients of both sides of the above equation, we obtain \(c_{\alpha \beta} \in R, \alpha = 2, 3, \ldots\) and to any integer \(\alpha, \beta, m, c_{\alpha \beta}\) and \(\mu_m\) are determined by the following formulas:

\[
c_{11} = 1, \quad c_{20} = c_{02} = 0, \\
\text{if } (\alpha = \beta = 0 \text{ and } \beta \neq 1) \text{ or } (\alpha < 0, \text{ or } \beta < 0), \quad \text{then } c_{\alpha \beta} = 0,
\]

else

\[
c_{\alpha \beta} = \frac{1}{\beta - \alpha} \sum_{k+j=\beta}^{\alpha + \beta + 2} \left[ (\alpha - k + 1) c_{\alpha, k-1} \right]
\]

\[
- (\beta - j + 1) c_{\beta, j-1} c_{\alpha, \beta-j+1},
\]

\[
\mu_m = \sum_{k+j=\beta}^{\beta + 2} (m - k - 2) c_{\alpha, \beta-j+1}
\]

\[
- (m - j + 2) c_{\beta, j-1} c_{\alpha, \beta-j+1},
\]

\[
\mu_k \text{ in Lemma 2.1 is called } k\text{-th order singular point value at the origin of system (6).}
\]

**Lemma 2.2** (see [Liu, 2001]). For system \((3)\) and any positive integer \(m\), among \(v_{2m}(2\pi), v_2(2\pi)\) and \(v_1(\pi)\), there exists the following relation

\[
v_{2m}(2\pi) = \frac{1}{1 + v_1(\pi)} \left[ \xi_m^{(2)}(v_1(2\pi) - 1) + \sum_{k=1}^{m-1} \xi_k^{(2)} v_{2k+1}(2\pi) \right],
\]

where \(\xi_m^{(k)}\) are all polynomials of \(v_1(\pi), v_2(\pi), \ldots, v_m(\pi)\) and \(v_1(2\pi), v_2(2\pi), \ldots, v_{m}(2\pi)\) with rational coefficients.

Obviously, we imply that \(v_{2m}(2\pi) = 0\) when \(v_1(2\pi) = 1, v_{2k+1}(2\pi) = 0, k = 1, 2, \ldots, m - 1\).

**Lemma 2.3** (see [Liu, 2001]). For system \((3)|_{k=0}\), \((6)\) and any positive integer \(m\), the following assertion holds:

\[
v_{2k+1}(2\pi) = i\pi \mu_m + \sum_{k=1}^{m-1} \xi_k^{(2)} \mu_k,
\]

where \(\xi_m^{(k)}\) \((k = 1, 2, \ldots, m - 1)\) are polynomial functions of coefficients of system \((6)\).

According to Lemmas 2.2 and 2.3, we have

**Lemma 2.4.** For system \((3)|_{k=0}\) and \((6)\), the following relation holds:

\[
v_{2m+1}(2\pi) = i\pi \mu_m
\]

when \(\mu_k = 0, k = 1, 2, \ldots, m - 1\).

**Lemma 2.5** (see [Liu, 2001]). For system \((3)\), we have the following conclusions:
(a) System (3) can bifurcate in limit cycles at most in a small enough neighborhood at the origin of (3), if the following conditions hold:

$$v_1(2\pi, \epsilon) = 1 = \lambda_{0} e^{\delta + N} + o(e^{\delta + N + 1}), \quad v_{2k+1}(2\pi, \epsilon) = \lambda_{k} e^{\delta + N} + o(e^{\delta + N + 1}), \quad k = 1, 2, \ldots, 0 < |\epsilon| \ll 1, \quad \lambda_{0} \neq 0.$$  

where \(l_0, l_1, \ldots, l_m, N\) are positive integers and \(l_m = 0, \lambda_{0} \neq 0\).

(b) If conditions in (a) hold, and \(\lambda_{k}\lambda_{k-1} < 0\), then \(\lambda_{k}\lambda_{k-1} > l_k - l_{k+1}, \quad (k = 1, 2, \ldots, m - 1)\), then \(\sum_{k=m}^{\infty} l_k \lambda_{k} e^{\delta/k} = 0\) has \(m\) positive solutions, i.e.

$$h_k(\epsilon) = \sqrt{-\frac{\lambda_{k-1}}{\lambda_{k}}} e^{\delta/k-1} + o(e^{\delta/k-1}).$$

Accordingly, system (3) can bifurcate in limit cycles which are near circles \(x^2 + y^2 = \left(\frac{\lambda_{k-1}}{\lambda_{k}}\right)^{l_k-1} - l_k\).

From the discussions, we have the following theorem.

**Theorem 2.1.** If the origin of unperturbed system (3)|\(\epsilon = 0\) is a fine focus of \(n\) order, then the origin of the disturbed system (3) can bifurcate \(n\) limit cycles under a suitable perturbation.

Proof. Let undisturbed system (3) have \(n\) real parameters, they are \(a_1, a_2, \ldots, a_{n-1}, \delta\). Because the origin of the undisturbed system (3)|\(\epsilon = 0\) is a fine focus of \(n\) order, then there exist a group of values \(a_1 = \pi_1, a_2 = \pi_2, \ldots, a_{n-1} = \pi_{n-1}\) such that \(v_3 = v_5 = \cdots = v_{2n-1} = 0, v_{2n+1} = 0 \neq 0\).

Given a suitable perturbation about these parameters, we may as well let

$$v_1(2\pi, \epsilon) = \epsilon_1,$$
$$v_3(2\pi, \epsilon) = \epsilon_2, \ldots, v_{2n-1}(2\pi, \epsilon) = \epsilon_{n-1},$$

in which \(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}\) are a group of arbitrary real numbers. Because the origin of the undisturbed system (3)|\(\epsilon = 0\) is a fine focus of \(n\) order, then the Jacobian of the functions group \((v_3, v_5, \ldots, v_{2n+1})\) with respect to the variables group \((a_1, a_2, \ldots, a_{n-1})\)

\[
J = \begin{vmatrix}
\frac{\partial v_3}{\partial a_1} & \frac{\partial v_3}{\partial a_2} & \cdots & \frac{\partial v_3}{\partial a_{n-1}} \\
\frac{\partial v_5}{\partial a_1} & \frac{\partial v_5}{\partial a_2} & \cdots & \frac{\partial v_5}{\partial a_{n-1}} \\
& \vdots & \ddots & \vdots \\
\frac{\partial v_{2n-1}}{\partial a_1} & \frac{\partial v_{2n-1}}{\partial a_2} & \cdots & \frac{\partial v_{2n-1}}{\partial a_{n-1}}
\end{vmatrix} \neq 0.
\]

Hence, according to the existence theorem of implicit function, Eq. (12) has a group of solutions as follows:

$$a_1 = \pi_1 + a_1(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}),$$
$$a_2 = \pi_2 + a_2(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}),$$

$$\vdots$$

$$a_{n-1} = \pi_{n-1} + a_{n-1}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}).$$

Obviously, given perturbations by (13), we will let (12) hold.

We may as well let \(\epsilon_k, k = 1, 2, \ldots, n - 1\) satisfy the conditions of Lemma 2.5, and let \(\epsilon = \lambda_{0} e^{\delta + N}\), then from Lemma 2.5, the result of Theorem 2.1 holds. This completes the proof.

Of course, the conclusion of Theorem 2.1 has also been proved in [Andronov et al., 1973] by using other methods.

3. Limit Cycles Bifurcations of Model (1)

It is easy to see that model (1) is symmetrical with regard to line \(y = x\), and model (1) has two positive singular points \((1, 2)\) and \((2, 1)\). Of course, the origin model (1) is an equilibrium point, but our work should focus on the positive equilibrium points because of the real significance from the mathematical models. From the symmetrical system quality, positive singular points \((1, 2)\) and \((2, 1)\) have the same topological structure, therefore they have the same bifurcation behavior. Hence, we only need to investigate the limit cycle bifurcations from positive singular points \((1, 2)\), which we next study.

3.1. Focal values of positive singular point \((1, 2)\)

Obviously, the positive singular point \((1, 2)\) is a focus of model (1). In order to compute the focal
values of point \((1, 2)\), we may as well make the following transformation
\[
\begin{align*}
    u &= x - 1, \\
    v &= y - 2,
\end{align*}
\]
model \((1)\) is changed into the following form
\[
\begin{align*}
    \frac{du}{dt} &= (u + 1)P(u + 1, v + 2) \\
    &= -v + \text{h.o.t.}, \\
    \frac{dv}{dt} &= (v + 2)Q(u + 1, v + 2) \\
    &= u + \text{h.o.t.},
\end{align*}
\]
in which the expressions of \(P, Q\) are the same as model \((1)\).
Clearly the equilibrium \((1, 2)\) of model \((1)\) becomes the origin of \((15)\) correspondingly. System \((15)\) belongs to the class of system \((3).\)

Under the transformation
\[
V_3 = \frac{\pi}{1982}(-22 + 88a + 82A_{10} + 304aA_{01} + 104A_{01}^2 + 92A_{10} - 16aA_{10} + 128A_{01}A_{10} - 16A_{10}^2 - 141A_{11} + 432aA_{11} + 48A_{01}A_{11} + 168A_{10}A_{11} - 162A_{11}^2),
\]
and if \(V_3 = 0\), then
\[
\begin{align*}
    V_5 &= \frac{\pi}{124416}(11 + 38A_{01} - 2A_{10} + 54A_{11})^3b_2, \\
    V_7 &= \frac{\pi}{5159780352}(11 + 38A_{01} - 2A_{10} + 54A_{11})^3b_4, \\
    V_9 &= \frac{\pi}{8024904034040}(11 + 38A_{01} - 2A_{10} + 54A_{11})^3b_6, \\
    V_{11} &= \frac{\pi}{7487812485248748480}(11 + 38A_{01} - 2A_{10} + 54A_{11})^3b_8.
\end{align*}
\]
The above expressions of \(b_k, k = 2, 3, 4, 5\) are functions of \(A_{01}, A_{10}, A_{11}\).

3.2. Limit cycles bifurcation of the positive equilibrium point \((1, 2)\)

With the help of a computer, we can obtain the following theorem.

**Theorem 3.2.** The origin of \((15)\) can become a fine focus of fifth order if \(V_3 = V_5 = V_7 = V_9 = V_{11} = 0\) and \(2 + 2A_{10} + 3A_{11}(−9 + 4A_{01} + 6A_{11})\times (−1 + 6A_{01} + 9A_{11})(−1 + 12A_{01} + 18A_{11})\times (8 + 18A_{01} + 27A_{11})(12A_{01} + 12A_{01}^2 + 10A_{11})\times (15 + 36A_{01} + 27A_{11}^2 + 6A_{01} + 24A_{01}^2 + 11A_{11} + 72A_{01}A_{11} + 54A_{01}^2)^3(198 - 282A_{01} + 24A_{01}^2 + 1247A_{11} + 360A_{01}A_{11} + 1350A_{11}^2)A_{11} \neq 0.

**Proof.** By analyzing the expressions of \(V_k, k = 3, 5, 7, 9, 11\) in Theorem 3.1, we will try to find a group of values about \(a, A_{01}, A_{10}, A_{11}\) such that \(V_3 = V_5 = V_7 = V_9 = V_{11} = 0\). If \(V_5 = V_7 = 0\) will deduce that the resultant of \(V_5, V_7\) about \(A_{10}\) vanishes. \(V_5 = V_7 = 0\) will deduce that the resultant of \(V_5, V_7\) about \(A_{10}\) vanishes. \(V_5 = V_7 = 0\) will deduce that the resultant of
$V_5, V_{11}$ about $A_{10}$ vanishes. Let

\[ r_{23} = \text{Resultant}[V_5, V_7, A_{10}], \]
\[ r_{24} = \text{Resultant}[V_5, V_9, A_{10}], \]
\[ r_{25} = \text{Resultant}[V_5, V_{11}, A_{10}]. \]

These expressions can be easily obtained with a personal computer. While $V_5 = V_7 = V_9 = V_{11} = 0$ will deduce $r_{23} = r_{24} = r_{25} = 0$. By using the computer software Mathematica 7.0, we obtain the highest common factor on $r_{23}, r_{24}, r_{25}$, as the following expression

\[
m_1 = (2 + 2A_{10}^2 + 3A_{11})^4(-9 + 4A_{10} + 6A_{11})(-1 + 6A_{10} + 9A_{11})^3(-1 + 12A_{10} + 18A_{11})^3 \]
\[ \times (8 + 18A_{10} + 27A_{11})(12A_{10} + 12A_{11}^2 + 10A_{11} + 36A_{10}A_{11} + 27A_{11}^2)(6A_{10} + 24A_{11}^3 + 11A_{11}) \]
\[ + 72A_{10}A_{11} + 54A_{11}^2)^15(198 - 282A_{10} + 24A_{11}^2 + 1247A_{11} + 360A_{10}A_{11} + 1350A_{11}^2)A_{11}. \]

It is clear that $m_1 = 0$ will be deduced from $V_5 = V_7 = V_9 = V_{11} = 0$. Let

\[ r_1 = \text{Resultant}[r_{23}, r_{24}, A_{10}], \quad r_2 = \text{Resultant}[r_{23}, r_{25}, A_{10}]. \]

At the same time, we can give the resultant of $r_1, r_2$ about $A_{11}$ by a personal computer as follows.

\[ r = \text{Resultant}[r_1, r_2, A_{11}] = 37761926701046268966426668989157443143429305626 \cdots 68268831683587 \]
\[ \neq 0. \]

Through the above analysis, if $m_1 \neq 0$ when $V_5 = V_7 = V_9 = V_{11} \neq 0$, then the conclusion of Theorem 3.2 holds. This completes the proof. ■

In fact, we can find 22 groups of real number solutions on $a, A_{01}, A_{10}, A_{11}$ such that $V_5 = V_7 = V_9 = V_{11} \neq 0$, for example:

\begin{align*}
(1) & \quad a \approx -7.49138, \\
& \quad A_{01} \approx 27.40994928038, \\
& \quad A_{10} \approx 7.737970130965, \\
& \quad A_{11} \approx -19.202292100; \\
(2) & \quad a \approx -2.73 \times 10^{15}, \\
& \quad A_{01} \approx 4.5619604862, \\
& \quad A_{10} \approx 2.0619604862, \\
& \quad A_{11} \approx -3.3736032871; \\
(3) & \quad a \approx -5.71478, \\
& \quad A_{01} \approx 6.30985650705, \\
& \quad A_{10} \approx 3.714784706580, \\
& \quad A_{11} \approx -2.706571004702; \\
(4) & \quad a \approx 1.26 \times 10^{-15}, \\
& \quad A_{01} \approx 2.523637475, \\
& \quad A_{10} \approx 1.27363747493. \\
\end{align*}

According to Theorem 2.1, it is clear that the following theorem holds.

**Theorem 3.3.** Suppose that the origin of (15) (or the positive equilibrium point (1, 2) of (1)) is a focus of fifth order, then under a certain parameter perturbation condition, the positive equilibrium point (1, 2) of (1) can bifurcate five limit cycles, in which three limit cycles are stable if $V_{11} < 0$, two limit cycles are stable if $V_{11} > 0$.

In fact, $V_{11} \approx -5.11395 \times 10^4 < 0$ if the parameter group $(a, A_{01}, A_{10}, A_{11})$ satisfies (18), $V_{11} \approx -2.10952 \times 10^{15} < 0$ if parameter group $(a, A_{01}, A_{10}, A_{11})$ satisfies (19), $V_{11} \approx 0.00024656 > 0$ if parameter group $(a, A_{01}, A_{10}, A_{11})$ satisfies (20), $V_{11} \approx 0.0208931 > 0$ if parameter group $(a, A_{01}, A_{10}, A_{11})$ satisfies (21), hence the conclusions of Theorem 3.3 can come true.

From the symmetrical system quality, positive singular points (1, 2) and (2, 1) have the same
topological structure, therefore they have the same bifurcation behavior. Hence (2, 1) of system (1) becomes a fine focus of fifth order if (1, 2) is a fine focus of fifth order. Moreover, in terms of limit cycles problem, the following theorem holds.

**Theorem 3.4.** Suppose that the origin of (15) is a fine focus of fifth order, then under a certain parameter perturbation condition, each one of the positive equilibrium points (1, 2) and (2, 1) of (1) can bifurcate five limit cycles at the same step perturbation, in sum ten limit cycles can occur in the disturbed model (1), in which six limit cycles will be stable.

**Proof.** Suppose that the origin of (15) is a fine focus of fifth order, according to Theorem 2.1 and the invariance of translation, it is clear that the positive equilibrium point (1, 2) of (1) can bifurcate five limit cycles under a certain parameter perturbation condition, in which three limit cycles are stable if \( \lambda_1 < 0 \), two limit cycle is stable if \( \lambda_1 > 0 \). At the same time, the system lies in a symmetrical vector field with regard to \( y = x \), then (1, 2) and (2, 1) have the same topological structure, hence the positive equilibrium point (2, 1) of (1) will also bifurcate five limit cycles (or three stable limit cycles) when the positive equilibrium point (1, 2) of (1) bifurcates five limit cycles (or three stable limit cycles). The conclusions of Theorem 3.4 hold.

**Remark.** It can be seen from the many published papers that the study on the Hilbert Number of Kolmogorov model is a hot topic. In terms of the Hilbert Number of quartic Kolmogorov model, our results are new. But the following problems on the bifurcation of limit cycles of the Kolmogorov model will continue to be hot topics:

1. Whether the Hilbert Number of quartic Kolmogorov model will be more than 107
2. In Kolmogorov model \( \frac{dx}{dt} = f(x, y) \), \( \frac{dy}{dt} = g(x, y) \), the degrees of \( f(x, y) \) and \( g(x, y) \) can perhaps be higher because the relation between two populations (prey and predator) will perhaps be more complex and affected by many factors. Hence, the bifurcation behavior of Kolmogorov model of higher degrees is also worth investigating.
3. In Kolmogorov model \( \frac{dx}{dt} = f(x, y) \), \( \frac{dy}{dt} = g(x, y) \), if \( f(x, y) \) and \( g(x, y) \) are not polynomials, then its bifurcation behavior is also worth studying.

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