Interval T-S Fuzzy Model and Its Application to Identification of Nonlinear Interval Dynamic System based on Interval Data

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Abstract—In this paper, a new fuzzy system model structure—Interval T-S Fuzzy Model (ITSFM) is proposed. Inspired from interval regression analysis, the interval arithmetic is incorporated with classical T-S fuzzy model and the parameters in consequent part of the ITSFM model become to be intervals. Thus, the outputs of the proposed ITSFM are intervals. In addition, we define fuzzy interval set, center membership function and radius membership function for intervals and an arithmetical operation between a constant interval and a general real vector. Then the proposed ITSFM is applied to identification of nonlinear interval dynamic system based on the measured interval data. Experimental results are then presented that indicate the validity and applicability of the proposed ITSFM.

I. INTRODUCTION

Classical fuzzy logic systems (FLS) are able to construct a nonlinear relation for crisp input and output data. It has been used in many fields, such as system identification [1], fuzzy control [12], and fault detection [13]. However, in practical applications, the available information is often uncertain, imprecise and incomplete. Interval-valued data offer a valuable way of representing the available information in complex problems where imprecision or variability must be taken into account. Nevertheless, the classical fuzzy logic (FL) cannot fully handle all the uncertainty present in real-world problems [11]. In order to handle the uncertainty, the type-2 fuzzy logic systems (FLS) were proposed in recent years [6]. It translates all the uncertainties of the measurement into uncertainties about the membership functions, and results in a notation that gives the third dimension to the membership functions of the classical type. Nevertheless, most general type-2 FLS are rather computation intensive. Interval type-2 FLS are later used to simplify computation by identifying the foot print of uncertainty of fuzzy sets [7]. Comparing with classical fuzzy logic systems, the computation of interval type-2 FLS is still large. In [4], a probabilistic fuzzy logic system (PFLS) was proposed for modeling a system with stochastic uncertainties. Different to the ordinary FLS, the PFLS uses the probabilistic fuzzy set instead of the ordinary fuzzy set. Thus it is able to capture the information with stochastic uncertainties. But in practical applications, it is hard to determine probabilistic density function of fuzzy membership grade. Both type-2 FLS and PFLS capture the uncertainties in the system by adding additional information to membership functions of the classical type to form a generalization membership functions. However, the approaches, general type-2 FLS, Interval type-2 FLS and PFLS, only can be used to construct a nonlinear relation for crisp input and output data in the framework of fuzzy systems. In other words, those approaches have incapability of modeling with crisp inputs and interval outputs. Fuzzy regression analysis introduced by Tanaka et al. is another approach to dealing with uncertainty information [5]. Interval regression analysis, where a model with interval coefficients is assumed, is regarded as the simplest version of fuzzy regression analyses [3].

In real world, one source generating dynamical interval-valued data is dynamic systems with uncertain but bounded parameters. The interval-valued data can be obtained by repeated measures of the outputs of the systems for the same input. At present, the most research on interval valued data is about interval time series foresting [16] [17], there few articles about the identification of nonlinear interval dynamic systems based on the input and measured interval output data. In [10], an interval fuzzy modeling method was proposed. It results in lower and upper fuzzy models with a set of lower and upper parameters by the min–max optimization method, the lower and upper fuzzy models are independent, so the approach don’t appropriate for the modeling of interval dynamic systems. In [14], an interval analysis method for dynamic response analysis of nonlinear systems with uncertainty is proposed, but it not refer to the identification of nonlinear systems with uncertainty.

In order to endow the classical T-S fuzzy model with the ability of constructing a model with crisp input and interval output data, in this paper, an extended T-S fuzzy model—Interval T-S Fuzzy Model (ITSFM) is proposed. Different to the existing approaches handling with the uncertainties in the system by adding additional information to membership functions of the classical type to form a generalization membership functions, in our ITSFM model, the interval arithmetic is incorporated with classical T-S fuzzy model and the parameters in consequent part of the ITSFM model become to be interval numbers. Thus, the uncertainty of available information is characterized by the
interval consequent parameters of the ITSFM. In addition, we define fuzzy interval set, center membership function and radius membership function for interval-valued data, and an arithmetical operation between an interval and a general real vector. Finally, the proposed ITSFM is applied to identification of nonlinear interval dynamic system based on the measured interval data. Experimental results are then presented that indicate the validity and applicability of the proposed ITSFM.

This paper is organized as follows: In Section II, we briefly review the theory of interval arithmetic and interval regression analysis; In Section III, the ITSFM model based on interval arithmetic is presented; In Section IV, the proposed ITSFM is applied to identification of nonlinear interval dynamic system; In Section V, a nonlinear interval dynamic system identification example based on ITSFM is given; Finally, the conclusion is given in Section VI.

II. SOME RESULTS IN INTERVAL ARITHMETIC AND INTERVAL REGRESSION ANALYSIS

In this section, we need to briefly look at some important formulations in interval arithmetic and interval regression analysis [9], [18].

A. Interval arithmetic

An interval can be represented by its lower and upper limits as

\[ A = [a_l, a_u] \]

or, equivalently, by its center and radius as

\[ A = (c, r) \]

Let \( A \) and \( B \) be two intervals represented by its lower and upper limits, the operations defined on the intervals are, respectively.

\[ A + B = [a_l + b_l, a_u + b_u] \]
\[ A - B = [a_l - b_u, a_u - b_l] \]
\[ \lambda \cdot A = [\lambda \cdot a_l, \lambda \cdot a_u] \]

Where \( \lambda \) is a real number constant and \( \lambda > 0 \).

If an interval is represented by its center and radius, the operations defined on the intervals are as follows:

\[ A + B = (a_l + b_l, a_u + b_u) \]
\[ A - B = (a_l - b_u, a_u - b_l) \]
\[ \lambda \cdot A = (\lambda \cdot a_l, \lambda \cdot a_u) \]

where \( a_l \), \( a_u \), \( b_l \), and \( b_u \) are the center and radius of interval \( A \) and \( B \), respectively.

B. Interval regression analysis

Fuzzy regression analysis introduced by Tanaka et al. is another approach to dealing with uncertainty information [5]. Interval regression analysis, where a model with interval coefficients is assumed, is regarded as the simplest version of fuzzy regression analysis [18].

An interval linear regression model can be written as:

\[ Y(x) = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha^T x \]

where \( x = (1, x_1, x_2, \cdots, x_n)^T \) is a real input vector, \( \alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n)^T \) is an interval coefficient vector. An interval coefficient \( \alpha_i \) is denoted as \( \alpha_i = (a_i, c_i) \), where \( a_i \) is a center and \( c_i \) is a radius. By interval arithmetic, the regression model can be expressed as:

\[ Y(x) = (a_0, c_0) + (a_1, c_1) x_1 + \cdots + (a_n, c_n) x_n \]

\[ = (a_0 + a_1 x_1 + \cdots + a_n x_n, c_0 + c_1 x_1 + \cdots + c_n x_n) \mid x_n \]

\[ = (a^T x, c^T x) \]

where \( a = (a_0, a_1, \ldots, a_n)^T \), \( c = (c_0, c_1, \ldots, c_n)^T \), \( x = (1, x_1, x_2, \cdots, x_n)^T \), \( |x| = (|1|, |x_1|, |x_2|, \cdots, |x_n|)^T \).

III. INTERVAL T-S FUZZY MODEL

Classical T-S model is able to construct a nonlinear relation for crisp input and output data. However, to the best of our knowledge, there is no fuzzy system model structure being able to construct a nonlinear relation for crisp input and interval output data directly. In order to overcome the above shortcoming of classical fuzzy system model structure, we introduce interval arithmetic to classical T-S fuzzy model. Before introducing the proposed ITSFM, we first review the classical T-S fuzzy model.

A. Classical T-S fuzzy model

T-S fuzzy model deals with a complex system by decomposing input space into several subspaces, each of which is represented by a simple linear model. Generally, its rules take on the following form [10]:

\[ R^i : \text{If } x \text{ is } A_i^1 \text{ and } x_n \text{ is } A_i^n \text{ then } h^i = f_j(x_1, x_2, \cdots, x_n) = \alpha_i^0 + \alpha_i^1 x_1 + \cdots + \alpha_i^n x_n = a_i^T x \]

For \( i = 1, 2, \ldots, r \), where \( r \) denotes the number of rules, \( A_i^j \) is the corresponding fuzzy set, and \( \alpha_i = (\alpha_i^0, \alpha_i^1, \cdots, \alpha_i^n)^T \) is the parameter set in the consequent part. The predicted output of the fuzzy model is inferred as

\[ \hat{y}(x) = \sum_{i=1}^{r} w_i h^i = \sum_{i=1}^{r} w_i \alpha_i^T x \]

where the weight \( w^j = \prod_{j=1}^{r} A_j^j(x_j) \), and \( \bar{w}_i = \frac{w^j}{\sum_i w^j} \).
$R_i$: If $x_1$ is $A_{1i}^i$ and $x_2$ is $A_{2i}^i$, ..., $x_n$ is $A_{ni}^i$
Then $Y^i = \theta_0^i + \theta_1^i x_1 + \cdots + \theta_n^i x_n = \theta^i_0 \mathbf{x}$, \quad (11)

Where $r$ denotes the number of rules, $Y^i$ is the interval-valued output of $i$th rule, $i = 1, 2, \ldots, r$ , $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is the real-valued input vector, $A_{ij}^i$ is the corresponding fuzzy set, $\theta_i = (\theta_0^i, \theta_1^i, \cdots, \theta_n^i)^T$ is the interval parameter vector in the consequent part of the ITSFM, $\theta_j^i = (a_j^i, c_j^i)$ is an interval, $a_j^i$ and $c_j^i$ are the center and radius of $\theta_j^i$, respectively, $j = 0, 1, \ldots, n$ . The predicted interval output of the ITSFM, that is, $\hat{Y}(\mathbf{x})$ is inferred as
\[
\hat{Y}(\mathbf{x}) = \sum_{i=1}^{r} \frac{w^i Y^i}{\sum_{i=1}^{r} w^i} \quad (12)
\]

Where $w^i = \prod_{j=1}^{n} A_{ij}^i(x_j)$

Let $\tilde{w} = \frac{w^i}{\sum_{i=1}^{r} w^i}$, then $\hat{Y}(\mathbf{x})$ can be expressed as
\[
\hat{Y}(\mathbf{x}) = \sum_{i=1}^{r} \tilde{w} \theta^i \mathbf{x}
\]
\[
= \sum_{i=1}^{r} \tilde{w} [(a_0^i, c_0^i) + (a_1^i, c_1^i)x_1 + \cdots + (a_n^i, c_n^i)x_n]
\quad (13)
\]

From formula (4), (6) and $\tilde{w}_i \geq 0$ , one obtains:
\[
\hat{Y}(\mathbf{x}) = \sum_{i=1}^{r} \tilde{w}_i \left[ A^i_1 \mathbf{x}, C^i_1 | \mathbf{x} \right] = \sum_{i=1}^{r} \tilde{w}_i A^i_1 \mathbf{x} + \sum_{i=1}^{r} \tilde{w}_i C^i_1 | \mathbf{x} \right]
\quad (14)
\]

where $A^i_1 = (a_0^i, a_1^i, \ldots, a_n^i), C^i_1 = (c_0^i, c_1^i, \ldots, c_n^i)$ ,
\[
\mathbf{x} = (x_1, x_2, \ldots, x_n)^T, | \mathbf{x} | = (|x_1|, |x_2|, \ldots, |x_n|)^T
\]

Furthermore, $\hat{Y}(\mathbf{x})$ can be expressed as
\[
\hat{Y}(\mathbf{x}) = \left[ A^i_1 \tilde{x}, C^i_1 | \tilde{x} | \right]
\quad (15)
\]

where $A = (A^i_1, A^i_2, \ldots, A^i_m)^T, C = (C^i_1, C^i_2, \cdots, C^i_m)^T$
\[
, \tilde{x} = (\tilde{w}_1 \mathbf{x}, \tilde{w}_2 \mathbf{x}, \cdots, \tilde{w}_r \mathbf{x})^T, | \tilde{x} | = (|\tilde{w}_1 | | \mathbf{x} |, |\tilde{w}_2 | | \mathbf{x} |, \cdots, |\tilde{w}_r | | \mathbf{x} |)^T
\]
\[
A^i_1 \tilde{x} \text{ and } C^i_1 | \tilde{x} | \text{ are the center and radius of the predicted interval output } \hat{Y}(\mathbf{x}), \text{ respectively.}
\]

IV. IDENTIFICATION FOR NONLINEAR INTERVAL DYNAMIC SYSTEM BASED ON ITSFM

Before introduce to identification for nonlinear interval dynamic system based on ITSFM, we define an arithmetical operation between a constant interval data and a general real vector.

Definition 1: The arithmetical operation “$\circ$” between a constant interval data $(a, c)$ and a general real vector $\mathbf{X}$ with dimension $2n$ ($n$ is a positive integer) is defined as:
\[
(a, c) \circ \mathbf{X} = (a, c) \circ (x_{11}, x_{12}, x_{21}, x_{22}, \cdots, x_{n1}, x_{n2})
\quad (16)
\]

For characterize the degree of an interval $Z$ belongs to some fuzzy interval set $A$, we define a membership function for an interval number $Z$ as follows:

Definition 2: Fuzzy interval set $A$ is defined as $A = \{ (Z, \mu_A(Z)) | Z \in U \}$ , $U$ is the set of all intervals, $\mu_A(Z)$ characterize the degree of an interval $Z$ belongs to the fuzzy interval set $A$ , and $\mu_A(Z) = \mu_A^a(a) \land \mu_A^c(c)$, $a$ and $c$ are the center and radius of interval $Z$ , respectively. $A^a$ and $A^c$ are classical fuzzy sets. We define $\mu_A^a(a)$ and $\mu_A^c(c)$ as center membership function and radius membership function, respectively.

In this paper, $\mu_A^a(a)$ and $\mu_A^c(c)$ are obtained by fuzzy c-means clustering algorithm.

A. Identification for nonlinear interval dynamic system based on ITSFM

The considered nonlinear interval dynamic system is as follows:
\[
Y(k+1) = f(Y(k), \ldots, Y(k-n+1), u(k), \ldots, u(k-m+1), \Theta)
\quad (17)
\]

Where $f$ is a nonlinear continuous function, $Y(k+1), Y(k), \ldots, Y(k-n+1)$ are interval–valued outputs, $\Theta$ is an interval–valued parameter vector, $u(k), \ldots, u(k-m+1)$ are general inputs (crisp data).

Diferrent to the classical interval system identification, the available information in the considered interval dynamic system identification problem is mixed type data (interval-valued data and crisp data), the outputs of the constructed model $\hat{Y}$ are also interval-valued data.

The dynamical ITSFM is as follows:
\[
R^i: \text{If } Y(k-1) \text{ is } A_{i1} \text{ and } Y(k-2) \text{ is } A_{i2} \text{ and... and } Y(k-n_y) \text{ is } A_{in_y} \text{ and } u(k-1) \text{ is } B_{i1} \text{ and } u(k-2) \text{ is } B_{i2} \text{ and... and } u(k-n_u) \text{ is } B_{in_u}
\]
\[
\text{Then: } Y_i(k) = p_{0i} + \sum_{j=1}^{n_y} Y_j(k-j) p_{ji} + \sum_{l=1}^{n_u} p_{liu} u(k-l)
\quad (18)
\]
Where $A_{ij}$ is the corresponding fuzzy interval set (see definition 2 for details), $B_{ij}$ are classical fuzzy set, $Y(k-j)$ is interval-valued output, $u(k-l)$ is general input

$$Y(k-j) = (a_j, c_j), \quad p_j^l$$ is a general real vector $[p_j^l]$, $p_j^l = (a_{ij}^0, c_{ij}^0), \quad p_j^l = (a_{ij}^l, c_{ij}^l), \quad i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, n_j$, $r$ is the number of rules, $l = 1, 2, \ldots, n_u$. Based on interval arithmetic and definition 1,

$$Y_i(k) = (a_{0i}, c_{0i}) + \sum_{j=1}^{n_i} (a_{ij}, c_{ij}) \left( \begin{array}{c}
\sum_{i=1}^{n_i} (a_{ij}, c_{ij}) \right) u(k-l)$$

$$= (A_{i}^\prime x(k-l), C_{i}^\prime |x(k-l)|) \quad (19)$$

Where

$$A_i = [a_{i1}^l, p_{i1}^l, p_{i2}^l, \ldots, a_{i1+n_j}^l, a_{i1+n_j+1}^l, \ldots, a_{i1+2n_j}^l], \quad C_i = [c_{i1}^l, p_{i1}^l, p_{i2}^l, \ldots, p_{i1+2n_j}^l]$$

$x(k-l) = [1, a_1, \ldots, a_{n_j}, u(k-1), u(k-2), \ldots, u(k-n_u)]$

Then the predicted output of dynamical ITSFM is

$$\hat{Y}(k) = \sum_{i=1}^{r} \tilde{w}_i Y_i(k)$$

$$= (\sum_{i=1}^{r} \tilde{w}_i A_i^\prime x(k-l), \sum_{i=1}^{r} \tilde{w}_i C_i^\prime |x(k-l)|) \quad (20)$$

Where

$$\tilde{w}_i = \frac{w_i}{\sum w_i}, \quad \tilde{w}_i \geq 0$$

$$w_i = \mu_{A_i}(Y(k-1)) \wedge \mu_{A_i}(Y(k-2)) \wedge \ldots \wedge \mu_{A_i}(Y(k-n_j)) \wedge \mu_{B_i}(u(k-1)) \wedge \ldots \wedge \mu_{B_i}(u(k-n_u))$$

$$= \mu_{A_i\prime}(a_1) \wedge \mu_{A_i\prime}(c_1) \wedge \ldots \wedge \mu_{A_i\prime}(a_{n_j}) \wedge \mu_{A_i\prime}(c_{n_j})$$

$$\mu_{B_i}(u(k-1)) \wedge \ldots \wedge \mu_{B_i}(u(k-n_u))$$

$a_j$ and $c_j$ are the center and radius of $Y(k-j)$, respectively.

Let

$$\tilde{x}(k-l) = [\tilde{w}_1 x(k-l)_1, \tilde{w}_2 x(k-l)_2, \ldots, \tilde{w}_r x(k-l)_r]$$

$$|\tilde{x}(k-l)| = [\tilde{w}_1 |x(k-l)_1|, \tilde{w}_2 |x(k-l)_2|, \ldots, \tilde{w}_r |x(k-l)_r|]$$

$$A = [A_1^\prime, A_2^\prime, \ldots, A_r^\prime], \quad C = [C_1^\prime, C_2^\prime, \ldots, C_r^\prime]$$

Then

$$\hat{Y}(k) = (A^\prime \tilde{x}(k-l), C^\prime |\tilde{x}(k-l)|) \quad (21)$$

### B. The computation of parameters $A$ and $C$

The parameters in (21) can be obtained by Least Squares Method (LSM). That is, $A$ can be got by

$$\min \sum_{k=1}^{N} (a(k) - A^\prime \tilde{x}(k-l))^2, \quad C$$

can be obtained by

$$\min \sum_{k=1}^{N} (c(k) - C^\prime |\tilde{x}(k-l)|)^2, \quad \text{where} \quad a(k) \text{ and } c(k) \text{ are the center and radius of interval-valued output } Y(k), \quad \text{respectively, } N \text{ is the number of measured data.}$$

### V. NUMERICAL EXPERIMENTS

The nonlinear interval dynamic system identification based on the proposed ITSFM is presented for a nonlinear time-invariant system with uncertain physical parameters. These parameters are given as intervals. The observed system is a simplified car dynamics with uncertain parameters of the engine force. The mathematical model that describes the dynamics is [10]

$$u = f_c(u, v) - f_d,$$

$$f_c(u, v) = K_v (1 + a_t u)$$

$$\times (1 + \arctan(A_2 u^2 + a_3 v + a_4)) \quad (22)$$

Where $u$ is the position of the throttle, $m$ stands for the mass of the car and is equal to 1000 kg, $v$ stands for the velocity of the car in m/s, $f_c(u, v)$ is the force of the engine and $f_d$ is the resistance force and is approximated by a constant of 1000 N. The values of the real-valued and interval-valued constants in the model are respectively: $a_t = 3$, $a_3 = -0.35$, $a_4 = 1.2$, $K_v = [600, 900]$ N and $A_2 = [4.2, 7.8]$.

This model, with its uncertain parameters, is used to obtain the interval-valued data set for the identification of the nonlinear interval dynamic system based on the proposed ITSFM. The highest and lowest simulated responses of the car’s dynamics (four combinations of the low and high values of both interval parameters) to the same input signal are obtained and shown in Fig.1. The first half of the data set is used to identification of the considered dynamic system and the second half are used to validate the proposed ITSFM. From Fig.1 we can see that the outputs of nonlinear interval dynamic systems are not crisp data but interval-valued data.

In this example, the dynamical ITSFM is as follows:

$$R^i: \text{If } v(k-l) \text{ is } A_i \text{ then }$$

$$v(k) = p_0^i + v(k-l) \cdot p_1^i + u(k-l) \cdot p_2^i \quad (23)$$

In this example, the clustering number is 4; the center membership function $\mu_{A_i}(a)$ and radius membership function $\mu_{A_i}(c)$ are obtained by FCM algorithm.

The identification results of the considered nonlinear interval dynamic systems on the validation set are given in Fig.2. The local results of enlarged Fig.2 with time from 50 to 100 are given in Fig.3. From Fig.3 we can see clearly that the outputs of the considered nonlinear interval dynamic systems are interval-valued data, the proposed identification algorithm based on ITSFM works well. The higher and lower bound identification errors are given in Fig.4 and Fig.5.
respectively. \( e_1 = Y_h(k) - \hat{Y}_h(k) \), \( e_2 = Y_l(k) - \hat{Y}_l(k) \), \( Y_l(k) \) and \( Y_h(k) \) are the lower limit and upper limit of interval output \( Y(k) \), respectively. \( \hat{Y}_l(k) \) and \( \hat{Y}_h(k) \) are the lower limit and upper limit of predicted interval output \( \hat{Y}(k) \), respectively. \( \hat{Y}_l(k) = a(k) - c(k) \), \( \hat{Y}_h(k) = a(k) + c(k) \), \( a(k) \) and \( c(k) \) are the center and radius of \( \hat{Y}(k) \), respectively. Table I gives root mean square error (RMSE) of the identification results. \( e_l \) represents the RMSE for the lower limit of \( \hat{Y}(k) \) on the validation set, \( e_h \) represents the RMSE of higher limit of \( \hat{Y}(k) \) on the validation set.

\[
e_l = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (Y_l(k) - \hat{Y}_l(k))^2} \tag{24}
\]

\[
e_h = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (Y_h(k) - \hat{Y}_h(k))^2} \tag{25}
\]

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Fig. 1. The set of modelled data (crisp input-interval output data)

Fig. 2. The validation of ITSM where the blue area represents the interval-valued outputs of the considered nonlinear interval dynamic system, the red line represents the predict results given by the ITSM.

Fig. 3. The local results of enlarged Fig. 2 with time from 50 to 100

Fig. 4. The upper bound modeling error
VI. CONCLUSION

In this paper, a new fuzzy system model structure—Interval T-S Fuzzy Model (ITSFM) is proposed. It extends the parameters in consequent part of classical T-S fuzzy model to be intervals, thus the uncertainty of available information is characterized by the interval-valued consequent parameters of the ITSFM. By defining fuzzy interval sets, center membership function and radius membership function for intervals and an arithmetical operation between a constant interval and a general real vector, the proposed ITSFM is applied to identification of nonlinear interval dynamic system based on the measured interval-valued data successfully. The results verify the validity and applicability of the proposed ITSFM. In the future, we believe that the ITSFM will be very promising for many engineering applications, such as system identification, fault detection, robust control [2] and decision making [15], etc., where the available uncertain information is represented by intervals.

REFERENCE


