Mixed spline function method for reaction–subdiffusion equations

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Abstract

In this paper, we develop two classes of finite difference schemes for the reaction–subdiffusion equations by using a mixed spline function in space direction, forward and backward difference in time direction, respectively. It has been shown that some of the previous known difference schemes can be derived from our schemes if we suitably choose the spline parameters. By Fourier method, we prove that one class of difference scheme is unconditionally stable and convergent, the other is conditionally stable and convergent. Finally, some numerical results are provided to demonstrate the effectiveness of the proposed difference schemes.

1. Introduction

The research of fractional differential equations has attracted increasing interests in recent years, due to the applications in modeling many physical and chemical processes [16,18,22]. Such equations with fractional derivatives are often nonlinear, it is not easy even impossible to obtain their analytical solutions. So to find their numerical solutions becomes more and more urgent and important.

Up to now, some different methods for solving the fractional derivatives and integrals, fractional ordinary differential equations, space, time and space–time fractional partial differential equations have been proposed. Yuste and Acedo [25] proposed an explicit finite difference scheme for the subdiffusion equation. Chen et al. [2] presented a Fourier method for the fractional diffusion equation which also means subdiffusion. Some implicit, explicit finite difference methods for the reaction–subdiffusion equations were given in Chen et al. [1]. Garrappa et al. [6–8] proposed the explicit Adams multistep methods for fractional ordinary differential equations. On the other hand, a new numerical method was introduced for the modified subdiffusion equation with a nonlinear source term [15]. Cui [3,4] derived the compact finite difference schemes for the generalized one-dimensional sine–Gordon equation and the fractional diffusion equation. Sousa [20] constructed a numerical method to approximate the fractional advection diffusion problem, in [21], she also gave a numerical method for the fractional diffusion equations by spline function method. Li et al. [11] derived some other numerical approaches to fractional calculus and fractional ordinary differential equation. In [12], they obtained the approximation solution of the nonlinear fractional partial differential equations with subdiffusion and superdiffusion. Li and Xu [14] investigated initial boundary value problems of the space–time fractional diffusion equation and constructed a numerical method for solving it.
Meng et al. [17] used the orthogonal spline collocation method for the semi-discretization scheme of the one-dimensional coupled nonlinear Schrödinger equations. Recently, Li and Zeng [13] obtained some numerical solutions for the fractional ordinary differential equation by using three different finite difference schemes.

In this paper, we study the following reaction–subdiffusion equations:

$$\frac{\partial u(x,t)}{\partial t} = RL D^\gamma_0 t [K \frac{\partial^2 u(x,t)}{\partial x^2} - Cu(x,t)] + f(x,t), \quad 0 \leq t \leq T, \quad 0 < x < L,$$

subject to the initial, boundary conditions

$$u(x,0) = \phi(x), \quad 0 < x < L,$$

$$u(0,t) = \phi_1(t), \quad 0 \leq t \leq T,$$

$$u(L,t) = \phi_2(t), \quad 0 \leq t \leq T,$$

where functions $f(x,t), \phi(x), \phi_1(t)$ and $\phi_2(t)$ are sufficiently smooth. $K_1 > 0$ is the generalized diffusion coefficient, $C > 0$ is the rate constant for the bimolecular reaction. As usual, the fractional Riemann–Liouville derivative operator $RL D^\gamma_0 t w(x,t)$ indicates subdiffusion due to $1 - \gamma \in [0,1)$, which is defined as follows

$$RL D^\gamma_0 t w(x,t) = \frac{1}{\Gamma(\gamma)} \frac{\partial}{\partial t} \int_0^t w(s) \frac{1}{(t-s)^{1-\gamma}} ds,$$

in which $\Gamma(\cdot)$ is the Gamma function [18,22].

The process is called subdiffusion because it is characterized by a mean square displacement that varies sublinearly with time [26],

$$\langle x^2(t) \rangle \sim \frac{2K}{\Gamma(1+\gamma)} t^{\gamma}, \quad t \to \infty,$$

where $0 < \gamma < 1$ is the anomalous diffusion exponent. The diffusion is anomalously slower (subdiffusion) compared to the normal diffusion behavior, $\langle x^2(t) \rangle \propto t$. Subdiffusion motion is particularly important in the context of complex systems such as glassy and disordered materials, in which pathways are constrained for geometric or energetic reasons. It is also particularly germane to the way in which experiments in low dimensions have to be carried out [26].

As far as we know, there are a few finite difference methods for the above reaction–subdiffusion equation and we have not seen a difference scheme whose order of convergence is more than one in time and more than two in space, respectively. The aim of this paper is to construct higher order of difference schemes to solve the reaction–subdiffusion equation by using a mixed spline function in spatial derivative. We find out that some of the already existed difference schemes for solving Eq. (1) are the special cases of our schemes.

The outline of this paper is organized as follows. In Section 2, we derive a mixed parameters spline function and get two numerical methods for the reaction–subdiffusion equation, meanwhile, we also study the solvability conditions of the difference schemes. In the following section, we investigate the stability by the Fourier method and prove that one difference scheme is unconditionally stable and the other is conditionally stable. Furthermore, we show the stability condition. In Section 4, we study the convergence of the difference schemes. In Section 5, some numerical results are presented to demonstrate the effectiveness of our proposed finite difference schemes. The paper concludes with a summary in the last section.

2. Mixed spline function and numerical schemes

So far, there exist some polynomial and non-polynomial parameters spline functions [5,10,23,24]. In this section, we construct a spline function with new parameters, named the mixed spline function, and gain different accuracy by choosing different parameters.

Let $x_i = ih$ ($i = 0, 1, \ldots, n$) and $t_k = k\tau$ ($k = 0, 1, \ldots, m$), where $h = \frac{L}{n}$ and $\tau = \frac{T}{m}$ are space and time step lengths, respectively.

We consider a uniform mesh $\Omega$ with nodal points $x_i$ on $[0,L]$, for each segment $[x_i,x_{i+1}]$ ($i = 0, 1, \ldots, n-1$) and point $t_k$ ($k = 0, 1, \ldots, m$), we define the following mixed spline function $S_{kl}(x,t_k)$:

$$S_{kl}(x,t_k) = a_i e^{\rho(x-x_i)} + b_i \cos \rho (x-x_i) + c_i (x-x_i) + d_i, \quad x \in [x_i,x_{i+1}], \quad i = 0, 1, \ldots, n, \quad k = 0, 1, \ldots, m,$$

where $a_i, b_i, c_i$ and $d_i$ are constants and $\rho$ is an arbitrary parameter, and $S_{kl}(x,t_k)$ is termed as a mixed spline function.

Suppose that the spline function passes through the points $(x_i,u(x_i,t_k))$ and $(x_{i+1},u(x_{i+1},t_k))$. We first denote

$$S_{l0}(x_i,t_k) = u(x_i,t_k), \quad S_{l0}(x_{i+1},t_k) = u(x_{i+1},t_k),$$

$$S_{l1}(x_i,t_k) = M(x_i,t_k), \quad S_{l1}(x_{i+1},t_k) = M(x_{i+1},t_k).$$

From a series of algebraic computation, we get:
Theorem 1.

(i) If \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \) and \( \lambda_1 = \lambda_2 \) in (8), then \( p = 2 \);
(ii) If \( \lambda_1 = \lambda_3 = \frac{1}{2} \), \( \lambda_2 = \frac{5}{6} \) in (8), then \( p = 4 \).

Let us suppose that the function \( u(x, t) \) is \((v - 1)\)-times continuously differentiable in the interval \([0, T]\) with respect to \( t \) and that \( \frac{\partial^{v-1} u}{\partial x^{v-1}} \) is integrable in \([0, T]\) with respect to \( t \). Then for every \( 1 - \gamma \in (0, 1) \), the Riemann–Liouville derivative \( c_\gamma D_{0+}^{\gamma} u(x, t) \) exists and coincides with the Grünwald-Letnikov derivative \( c_\gamma D_{0+}^{\gamma} u(x, t) \) which is defined in the following form [22],

\[
c_\gamma D_{0+}^{\gamma} u(x, t) = \frac{1}{\Gamma(1 - \gamma)} \sum_{j=0}^{[\frac{t}{\Delta}]} (t - j \Delta)^{-\gamma} c_\gamma^{j+1, \gamma} u(x, t - j \Delta) + O(\Delta^\delta),
\]
such a formula is not “unique” because there are many different choices of “generating” functions for $\mathbf{s}_{q,j}^{(i-\gamma)}$ that leads to different approximation order $\eta$, where we always assume that $u(x,t)$ is smooth enough.

Generally speaking, the Riemann–Liouville derivative $\mathbf{RLD}_{0,t}^{1-\gamma} u(x,t)$ can be approximated as

$$\mathbf{RLD}_{0,t}^{1-\gamma} u(x,t) = \mathbf{CLD}_{0,t}^{1-\gamma} u(x,t) = \frac{1}{\tau^{1-\gamma}} \sum_{j=0}^{[\frac{1}{\tau}]} \mathbf{s}_{q,j}^{(i-\gamma)} u(x,t - j\tau) + O(\tau^\eta).$$

Here, we take $q = 1$, then the above formula reduces to

$$\mathbf{RLD}_{0,t}^{1-\gamma} u(x,t) = \mathbf{CLD}_{0,t}^{1-\gamma} u(x,t) = \frac{1}{\tau^{1-\gamma}} \sum_{j=0}^{[\frac{1}{\tau}]} \mathbf{s}_{q,j}^{(i-\gamma)} u(x,t - j\tau) + O(\tau),$$

where

$$\mathbf{s}_{1,0}^{(i-\gamma)} = 1, \quad \mathbf{s}_{1,j}^{(i-\gamma)} = (-1)^j \left(1 - \gamma \right)^j = (-1)^j \frac{\Gamma(2 - \gamma)}{\Gamma(1 + j) \Gamma(2 - \gamma - j)}, \quad j \geq 1,$$

which can be evaluated by following equalities,

$$\mathbf{s}_{1,j}^{(i-\gamma)} = 1 = (-1)^i \left(1 - \gamma \right)^i = \frac{\Gamma(2 - \gamma)}{\Gamma(1 + i) \Gamma(2 - \gamma - i)}, \quad i \geq 1.$$

At the points $(x_i, t_k)$ $(i = 0, 1, \ldots, n, k = 0, 1, \ldots, m)$, from Eq. (1) and (9), we have

$$\frac{\partial u(x_i, t_k)}{\partial t} = \mathbf{RLD}_{1,t}^{1-\gamma} \left[ K \frac{\partial^2 u(x_i, t_k)}{\partial x^2} - Cu(x_i, t_k) \right] + f(x_i, t_k) = \frac{1}{\tau^{1-\gamma}} \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \left[ K \frac{\partial^2 u(x_i, t_{k-j})}{\partial x^2} - Cu(x_i, t_{k-j}) \right] + f(x_i, t_k) + O(\tau),$$

where $\mu_1 = K \tau^{-1}, \mu_2 = C \tau^{-1}$.

Accordingly, we get

$$\sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \frac{\partial^2 u(x_{i-1,j}, t_{k-j})}{\partial x^2} = \frac{1}{\mu_1} \left[ \frac{\partial u(x_{i-1,j}, t_{k-j})}{\partial t} + \mu_2 \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} u(x_{i-1,j}, t_{k-j}) - f(x_{i-1,j}, t_{k-j}) \right] + O(\tau^{2-\gamma}), \quad i = 0, \ldots, n, \quad k = 0, \ldots, m. \quad (10)$$

Replacing $i$ with $i - 1$ and $i + 1$ in (10) respectively yields

$$\sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \frac{\partial^2 u(x_{i+1,j}, t_{k-j})}{\partial x^2} = \frac{1}{\mu_1} \left[ \frac{\partial u(x_{i+1,j}, t_{k-j})}{\partial t} + \mu_2 \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} u(x_{i+1,j}, t_{k-j}) - f(x_{i+1,j}, t_{k-j}) \right] + O(\tau^{2-\gamma}), \quad i = 1, \ldots, n, \quad k = 0, \ldots, m \quad (11)$$

and

$$\sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \frac{\partial^2 u(x_{i-1,j}, t_{k-j})}{\partial x^2} = \frac{1}{\mu_1} \left[ \frac{\partial u(x_{i-1,j}, t_{k-j})}{\partial t} + \mu_2 \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} u(x_{i-1,j}, t_{k-j}) - f(x_{i-1,j}, t_{k-j}) \right] + O(\tau^{2-\gamma}),$$

\[ i = 0, \ldots, n - 1, \quad k = 0, 1, \ldots, m. \quad (12) \]

From (8), we can get

$$h^2 \left[ \lambda_1 \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \frac{\partial^2 u(x_{i-1,j}, t_{k-j})}{\partial x^2} + \lambda_2 \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \frac{\partial^2 u(x_{i-1,j}, t_{k-j})}{\partial x^2} + \lambda_2 \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \frac{\partial^2 u(x_{i+1,j}, t_{k-j})}{\partial x^2} \right]$$

$$= \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \left[ u(x_{i-1,j}, t_{k-j}) - 2u(x_{i,j}, t_{k-j}) + u(x_{i+1,j}, t_{k-j}) \right] + \left( \sum_{j=0}^{k} \mathbf{s}_{1,j}^{(i-\gamma)} \right) O(h^{p+2}), \quad i = 1, \ldots, n - 1, \quad k = 0, 1, \ldots, m. \quad (13)$$

Inserting (10)–(12) into (13), and carrying on the algebra computation gives
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} \left[ \frac{\partial u(x_{i-1}, t_k)}{\partial t} + \frac{\partial u(x_i, t_k)}{\partial t} + \frac{\partial u(x_{i+1}, t_k)}{\partial t} \right] &= \mu_1 \sum_{j=0}^{k} \sigma_{ij}^{(-\gamma)} \times [u(x_{i-1}, t_{k-j}) - 2u(x_i, t_{k-j}) + u(x_{i+1}, t_{k-j})] \\
&\quad - \mu_2 \frac{h^2}{\tau} \sum_{j=0}^{k} \sigma_{ij}^{(-\gamma)} \times \left[ \frac{\partial}{\partial x} u(x_{i-1}, t_{k-j}) + \frac{\partial}{\partial x} u(x_i, t_{k-j}) + \frac{\partial}{\partial x} u(x_{i+1}, t_{k-j}) \right] \\
&\quad + \frac{h^2}{\tau} \left[ \frac{\partial}{\partial t} f_i(x_i, t_k) + \frac{\partial}{\partial t} f_i(x_i, t_k) \right] + O\left( \tau h^2 \right).
\end{align*}
\]

For \( \frac{\partial u(x, t_k)}{\partial t} \), \((s = i - 1, i, i + 1)\), we use the following forward and backward difference schemes, respectively,

\[
\begin{align*}
\frac{\partial u(x, t_{k+1})}{\partial t} &= \frac{u(x, t_{k+1}) - u(x, t_k)}{\tau} + O(\tau) \\
\frac{\partial u(x, t_{k-1})}{\partial t} &= \frac{u(x, t_{k-1}) - u(x, t_k)}{\tau} + O(\tau).
\end{align*}
\]

Substituting (15) and (16) into (14), respectively, omitting the higher order term, and letting \( u^n_i \) be the numerical solution, then one obtains the following two classes of finite difference schemes for Eq. (1).

\[
\begin{align*}
\mu_3 \left[ \frac{\partial u^n_{i-1}}{\partial t} + \frac{\partial u^n_i}{\partial t} + \frac{\partial u^n_{i+1}}{\partial t} \right] &= \mu_3 \left[ \frac{\partial u^n_{i-1}}{\partial t} + \frac{\partial u^n_i}{\partial t} + \frac{\partial u^n_{i+1}}{\partial t} \right] = \mu_3 \left[ \frac{\partial u^n_{i-1}}{\partial t} + \frac{\partial u^n_i}{\partial t} + \frac{\partial u^n_{i+1}}{\partial t} \right] \\
&\quad - \mu_2 \frac{h^2}{\tau} \sum_{j=0}^{k} \sigma_{ij}^{(-\gamma)} \left[ \frac{\partial}{\partial x} u^n_{i-1} + \frac{\partial}{\partial x} u^n_i + \frac{\partial}{\partial x} u^n_{i+1} \right] + \frac{h^2}{\tau} \left[ \frac{\partial}{\partial t} f_i(x_i, t_k) + \frac{\partial}{\partial t} f_i(x_i, t_k) \right] + O\left( \tau h^2 \right).
\end{align*}
\]
Proof. Firstly, the Eq. (7) can be rewritten as
\[
\frac{\partial^2 u(x_i, t_k)}{\partial x^2} = \frac{1}{h^2} \left( \frac{\partial^2 u(x_i, t_k)}{\partial x^2} \right) + O(h^p),
\]
where the differences \( \Delta_x, \nabla_x \) and \( \partial_x^2 \) are first-order forward, backward and second-order center difference operators with respect to \( x \).

According to (1), (14), (15), Lemma 1 and above formula, we define the local truncation error \( R^{k+1}_l \) of the difference scheme (17) as follows:
\[
R^{k+1}_l = \frac{u(x_i, t_{k+1}) - u(x_i, t_k)}{\tau} - \frac{K_f}{h^2} \sum_{j=-1}^{k} \sigma_{1j}^{1-\gamma} \frac{\partial^2 u(x_i, t_{k-j})}{\partial x^2} + \frac{C}{\tau^{1-\gamma}} \sum_{j=0}^{k} \sigma_{1j}^{1-\gamma} u(x_i, t_{k-j}) - f(x_i, t_k)
\]
\[
= \left[ \frac{u(x_i, t_{k+1}) - u(x_i, t_k)}{\tau} - \frac{\partial u(x_i, t_k)}{\partial t} \right] + K_f \left[ \frac{D h}{\partial_t} \frac{\partial^2 u(x_i, t_k)}{\partial x^2} - \frac{1}{\tau^{1-\gamma}} \sum_{j=0}^{k} \sigma_{1j}^{1-\gamma} \frac{\partial^2 u(x_i, t_{k-j})}{\partial x^2} \right]
\]
\[
+ C \left[ \frac{D h}{\partial_t} u(x_i, t_k) - \frac{1}{\tau^{1-\gamma}} \sum_{j=0}^{k} \sigma_{1j}^{1-\gamma} u(x_i, t_{k-j}) \right]
\]
\[
= O(\tau) + \frac{K_f}{\tau^{1-\gamma}} \left( \sum_{j=0}^{k} \sigma_{1j}^{1-\gamma} \right) O(h^p) = O(\tau + h^p).
\]

Using again Theorem 1, we can obtain above conclusions. \( \square \)

For the local truncation error \( R^l \) of the difference scheme (21), we also get the same results by almost the similar method. This finishes the proof of Theorem 2.

Remark 1.

(i) If \( \lambda_1 = \lambda_3 = 0, \lambda_2 = 1 \), then the difference schemes (17) and (21) reduces to the difference schemes (17) and (12) of [1], respectively;
(ii) When \( K_f = 1, C = 0 \), and \( \lambda_1 = \lambda_3 = 0, \lambda_2 = 1 \), then the difference scheme (21) reduces to the difference scheme (7) of [2];
(iii) When \( C = 0, \) and \( \lambda_1 = \lambda_3 = \frac{1}{2}, \lambda_2 = \frac{1}{3} \), then the difference scheme (21) reduces to the difference scheme (11) of [3].

Next we analyze the solvability of difference schemes (17) and (21).

Firstly, denote
\[

u^0 = [\phi(x_1), \phi(x_2), \ldots, \phi(x_{n-1})]^T,
\]
\[
u^k = [u^k_1, u^k_2, \ldots, u^k_{n-1}]^T, \quad k = 1, \ldots, m,
\]
\[
f^k = [f^k_1, f^k_2, \ldots, f^k_{n-1}]^T, \quad k = 0, 1, \ldots, m,
\]
\[
\Lambda_1 = \begin{pmatrix}
\lambda_2 & \lambda_3 \\
\lambda_1 & \lambda_2 & \lambda_3 \\
& \ddots & \ddots & \ddots \\
& & \lambda_1 & \lambda_2 & \lambda_3 \\
& & & \lambda_1 & \lambda_2
\end{pmatrix},
\]
\[
\Lambda_2 = \begin{pmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & 1 & -2 & 1 \\
& & & 1 & -2
\end{pmatrix}.
\]

Then the difference scheme (17) can be written in a matrix form:
\[
\bar{A}u^{k+1} = \sum_{j=0}^{k} \bar{B}_j u^{k-j} + \bar{F}^k + \bar{C}, \quad k = 0, 1, \ldots, m - 1,
\]

where
\[
\bar{A} = \mu_3 \Lambda_1, \quad \bar{B}_0 = \mu_3 \Lambda_1 + \mu_4 \sigma_{00}^{1-\gamma} \Lambda_2 - \mu_2 h^2 \sigma_{00}^{1-\gamma} \Lambda_1,
\]
\[
\bar{F} = h^2 \Lambda_1, \quad \bar{B}_j = \mu_4 \sigma_{jj}^{1-\gamma} \Lambda_2 - \mu_2 h^2 \sigma_{jj}^{1-\gamma} \Lambda_1, \quad j = 1, 2, \ldots, k.
\]
Proof. From Lemma 2, we know that the eigenvalues of the tridiagonal matrix are positive due to Theorem 3. Under the conditions of \( \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1 = \lambda_3 \) and \( \lambda_1 < \frac{1}{2} \), the difference scheme (17) is uniquely solvable.

Similarly, we can rewrite the difference scheme (21) as a following matrix form:

\[
\begin{pmatrix}
\tilde{a} & \cdots & \cdots & \cdots & \cdots \\
\tilde{b} & a & b & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \tilde{c}
\end{pmatrix}
\]

Lemma 2 ([19]). The tridiagonal matrix of order \( n - 1 \)

\[
T = \begin{pmatrix}
b & a & 0 & \cdots & \cdots & \cdots \\
c & b & a & \cdots & \cdots & \cdots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & a \\
\end{pmatrix}
\]

has following eigenvalues

\[
\zeta_i = b + 2a \sqrt{-\frac{c}{a}} \cos \left( \frac{i\pi}{n} \right), \quad i = 1, \ldots, n - 1.
\]

Theorem 3. Under the conditions of \( \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1 = \lambda_3 \) and \( \lambda_1 < \frac{1}{2} \), the difference scheme (17) is uniquely solvable.

Proof. From Lemma 2, we know that the eigenvalues of the tridiagonal matrix \( \lambda_1 \)

\[
\zeta_i = \lambda_2 + 2\lambda_3 \sqrt{\frac{\lambda_1}{\lambda_3}} \cos \left( \frac{i\pi}{n} \right) = 1 - 4\lambda_1 \sin^2 \left( \frac{i\pi}{2n} \right), \quad i = 1, \ldots, n - 1
\]

are positive due to \( \lambda_1 < \frac{1}{2} \).

From \( \tilde{A} = \mu_3 \Lambda_1 \), we get

\[
\det (\tilde{A}) = \mu_3^{n-1} \det (\Lambda_1) = \mu_3^{n-1} \prod_{i=1}^{n-1} \zeta_i > 0.
\]

It immediately follows that the difference scheme (17) is uniquely solvable.

Similarly, we can rewrite the difference scheme (21) as a following matrix form:

\[
\tilde{A}u^k = \sum_{j=1}^{k} \tilde{B}_j u^{k-j} + \tilde{F} u^k + \tilde{G}, \quad k = 1, \ldots, m,
\]

where

\[
\tilde{A} = \mu_3 \Lambda_1 - \mu_1 \sigma_{1,0}^{(1-\gamma)} \Lambda_2 + \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \Lambda_1,
\]

\[
\tilde{B}_1 = \mu_3 \Lambda_1 + \mu_1 \sigma_{1,0}^{(1-\gamma)} \Lambda_2 - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \Lambda_1,
\]

\[
\tilde{F} = h^2 \Lambda_1,
\]

\[
\tilde{G} = \begin{pmatrix}
-\mu_2 \lambda_1 [\varphi_1(t_k) + \varphi_1(t_{k-1})] + (\mu_1 - \mu_2 \lambda_1 h^2) \sum_{j=0}^{k} \sigma_{1,j}^{(1-\gamma)} \varphi_1(t_{k-j}) + h^2 \lambda_1 f(x_0, t_k) & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \ddots \\
-\mu_3 \lambda_3 [\varphi_2(t_k) + \varphi_2(t_{k-1})] + (\mu_1 - \mu_2 \lambda_3 h^2) \sum_{j=0}^{k} \sigma_{1,j}^{(1-\gamma)} \varphi_2(t_{k-j}) + h^2 \lambda_3 f(x_n, t_k)
\end{pmatrix}.
\]
Theorem 4. Under the conditions of \( \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1 = \lambda_2 \) and \( \lambda_1 < \frac{1}{N} \), the difference scheme (21) is uniquely solvable.

Proof. Obviously, the eigenvalues of the \( \tilde{A} \) are

\[
\tilde{\zeta}_i = \left( \lambda_2 \mu_3 + 2 \mu_1 + \lambda_3 \mu_2 \right) + 2 \left( \lambda_3 \mu_3 - \mu_1 + \lambda_3 \mu_2 \right) \cos \left( \frac{i \pi}{n} \right)
\]

\[
= \left( \mu_3 + \mu_2 \right) \left[ 1 - 4 \lambda_i \sin^2 \left( \frac{i \pi}{2n} \right) + 4 \mu_1 \sin^2 \left( \frac{i \pi}{2n} \right) \right], \quad i = 1, \ldots, n - 1.
\]

From \( \lambda_1 < \frac{1}{N} \), we know \( \tilde{\zeta}_i > 0 \), accordingly,

\[
\det (\tilde{A}) = \prod_{i=1}^{n-1} \tilde{\zeta}_i > 0.
\]

So the difference scheme (21) is also uniquely solvable. \( \square \)

3. Stability analysis of the difference schemes

In this section, we use the Fourier method to study the stability of the difference schemes (17) and (21) as follows [19]:

Lemma 3 ([1,2]). The coefficients \( \sigma_{1,j}^{(1-\gamma)} \) \( (j = 0, 1, \ldots) \) satisfy

- (i) \( \sigma_{1,0}^{(1-\gamma)} = 1, \sigma_{1,1}^{(1-\gamma)} = \gamma - 1, \sigma_{1,j}^{(1-\gamma)} < 0, j = 1, \ldots; \)
- (ii) \( \sum_{j=0}^{\infty} \sigma_{1,j}^{(1-\gamma)} = 0; \forall k \in \mathbb{N}^+, -\sum_{j=1}^{k} \sigma_{1,j}^{(1-\gamma)} < 1. \)

Let \( U_k^i \) be the approximate solution of (17). Define

\[
r_k^i = U_k^i - U_k^i, \quad i = 1, \ldots, n-1, \quad k = 0, 1, \ldots, m
\]

and

\[
r_k = [r_k^1, r_k^2, \ldots, r_k^{n-1}]^T.
\]

So, we easily get the following roundoff error equation of (17)

\[
\mu_3 \left[ \lambda_1 r_{k-1}^i + \lambda_2 r_k^i + \lambda_3 r_{k+1}^i \right] = \mu_3 \left[ \lambda_1 r_{k-1}^i + \lambda_2 r_k^i + \lambda_3 r_{k+1}^i \right] + \mu_1 \sigma_{1,0}^{(1-\gamma)} \left[ r_{k-1}^i - 2 r_k^i + r_{k+1}^i \right]
\]

\[
- \mu_2 h^2 \sigma_{1,1}^{(1-\gamma)} \left[ \lambda_1 r_{k-1}^i + \lambda_2 r_k^i + \lambda_3 r_{k+1}^i \right] + \mu_1 \sum_{j=1}^{k} \sigma_{1,j}^{(1-\gamma)} \left[ r_{k-1}^j - 2 r_k^j + r_{k+1}^j \right]
\]

\[
- \mu_2 h^2 \sum_{j=1}^{k} \sigma_{1,j}^{(1-\gamma)} \left[ \lambda_1 r_{k-1}^j + \lambda_2 r_k^j + \lambda_3 r_{k+1}^j \right], \quad i = 1, \ldots, n-1; \quad k = 0, 1, \ldots, m-1.
\]  

By using the Fourier method, we define the grid functions below

\[
r_k^i(x) = \begin{cases} 
  r_k^i, & \text{when } x_i - \frac{h}{2} < x \leq x_i + \frac{h}{2}, \quad i = 1, \ldots, n-1, \\
  0, & \text{when } 0 \leq x \leq h/2 \quad \text{or} \quad L - h/2 < x \leq L.
\end{cases}
\]

Expand the function \( r_k^i(x) \) into a Fourier series

\[
r_k^i(x) = \sum_{l=-\infty}^{\infty} \eta_k(l) e^{2\pi i l x/L},
\]

where

\[
\eta_k(l) = \frac{1}{L} \int_0^L r_k^i(x) e^{-2\pi i l x/L} dx, \quad l^2 = -1.
\]

Denoting

\[
\|r_k^i\|_2 = \left[ \int_0^L |r_k^i(x)|^2 dx \right]^{1/2}
\]

and using the Parseval equality
\[
\int_0^L |r^k(x)|^2 \, dx = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2,
\]

one gets
\[
||r^k||_2^2 = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2.
\]  

(26)

According to above analysis, we suppose that system (25) has the following solution:
\[
t_r^k = \eta_ke^{jlb},
\]

(27)

where \(b = 2\pi l/L\).

Substitution (27) in (25) gives
\[
\mu_3\lambda\eta_{k+1} = \left[ \mu_3\lambda - 2\mu_1\omega_{1,0}^{(1-\gamma)}(1 - \cos \beta h) - \mu_2\hbar^2\omega_{1,0}^{(1-\gamma)}\lambda \right] \eta_k - \left[ 2\mu_1(1 - \cos \beta h) + \mu_2\hbar^2\lambda \right] \sum_{j=1}^{\text{deg}} \omega_{1,j}^{(1-\gamma)} \eta_{k-j},
\]

(28)

where \(\lambda = \lambda_2 + 2\lambda_1 \cos \beta h\).

**Lemma 4.** Under the conditions of
\[
\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_1 = \lambda_3, \quad \lambda_1 < \frac{1}{4}
\]

and
\[
\tau' \leq \frac{(1 - 4\lambda_1)\hbar^2}{4K_y + C(1 - 4\lambda_1)\hbar^2},
\]

we have
\[
0 < \frac{\mu_3\lambda - 2\mu_1\omega_{1,0}^{(1-\gamma)}(1 - \cos \beta h) - \mu_2\hbar^2\omega_{1,0}^{(1-\gamma)}\lambda}{\mu_3\lambda} < 1.
\]

**Proof.** At first, it follows from \(\lambda_1 < \frac{1}{4}\) that
\[
\lambda = \lambda_2 + 2\lambda_1 \cos \beta h = 1 - 4\lambda_1 \sin^2 \left( \frac{1}{2} \beta h \right) > 0.
\]

For \(\mu_1, \mu_2, \mu_3 > 0\), it is obvious that
\[
\frac{\mu_3\lambda - 2\mu_1\omega_{1,0}^{(1-\gamma)}(1 - \cos \beta h) - \mu_2\hbar^2\omega_{1,0}^{(1-\gamma)}\lambda}{\mu_3\lambda} < 1.
\]

(29)

Secondly, from \(\tau' \leq \frac{(1 - 4\lambda_1)\hbar^2}{4K_y + C(1 - 4\lambda_1)\hbar^2}\), we have
\[
(1 - 4\lambda_1)(\tau' - C)\hbar^2 - 4K_y \geq 0.
\]

(30)

Furthermore, from (30), we get
\[
(1 - 4\lambda_1)\mu_3 - 4\mu_1\omega_{1,0}^{(1-\gamma)}(1 - 4\lambda_1)\mu_2\hbar^2\omega_{1,0}^{(1-\gamma)} \geq 0.
\]

(31)

On the other hand, from \(\tau' \leq \frac{(1 - 4\lambda_1)\hbar^2}{4K_y + C(1 - 4\lambda_1)\hbar^2}\), we get
\[
\tau' \leq \frac{1}{C'},
\]

that is,
\[
\mu_3 - \mu_2\hbar^2\omega_{1,0}^{(1-\gamma)} \geq 0.
\]

(32)

According to (31) and (32), we obtain
\[
\mu_3 - \mu_2\hbar^2\omega_{1,0}^{(1-\gamma)} \geq 4 \left( \lambda_1\mu_3 + \mu_1\omega_{1,0}^{(1-\gamma)} - \lambda_1\mu_2\hbar^2\omega_{1,0}^{(1-\gamma)} \right) \sin^2 \left( \frac{1}{2} \beta h \right),
\]

Furthermore,
\[ \mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda \geq 0, \]
i.e.,
\[ \frac{\mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda}{\mu_3 \lambda} \geq 0. \] (33)

From (29) and (33), we know that
\[ 0 \leq \frac{\mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda}{\mu_3 \lambda} < 1. \]

This accomplishes the proof of Lemma 3. □

**Lemma 5.** Suppose that \( \eta_{k+1} \) \((k = 0, 1, \ldots, m - 1)\) is the solution of Eq. (28). If
\[ \lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_1 = \lambda_3, \quad \lambda_1 < \frac{1}{4} \] (34)
and
\[ \tau \leq \frac{(1 - 4 \lambda_1) h^2}{4K_7 + C(1 - 4 \lambda_1) h^2}, \] (35)
then
\[ |\eta_{k+1}| \leq |\eta_0|, \quad k = 0, 1, \ldots, m - 1. \]

**Proof.** For \( k = 0 \), from the Eq. (28), we get
\[ |\eta_1| = \frac{\left| \mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda \right| |\eta_0|}{\mu_3 \lambda}. \]

From Lemma 4, it is very obvious that
\[ |\eta_1| \leq |\eta_0|. \]

Now, we suppose that
\[ |\eta_\ell| \leq |\eta_0|, \quad (\ell = 1, \ldots, k). \]

For \( k > 0 \), from Eq. (28) one has
\[ |\eta_{k+1}| = \frac{\left| \mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda \right| |\eta_k|}{\mu_3 \lambda} \leq \frac{2 \mu_1 (1 - \cos \beta \beta) + \mu_2 h^2 \lambda}{\mu_3 \lambda} \sum_{j=1}^{k} \sigma_{1,j}^{(1-\gamma)} |\eta_{k-j}| + \frac{2 \mu_1 (1 - \cos \beta \beta) + \mu_2 h^2 \lambda}{\mu_3 \lambda} \sum_{j=1}^{k} \sigma_{1,j}^{(1-\gamma)} |\eta_{k-j}| \leq \left\{ \mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda \right\} |\eta_k| + \frac{2 \mu_1 (1 - \cos \beta \beta) + \mu_2 h^2 \lambda}{\mu_3 \lambda} \sum_{j=1}^{k} \left( - \sigma_{1,j}^{(1-\gamma)} \right) |\eta_{k-j}| \leq \left\{ \mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda + \frac{2 \mu_1 (1 - \cos \beta \beta) + \mu_2 h^2 \lambda}{\mu_3 \lambda} \sum_{j=1}^{k} (-\sigma_{1,j}^{(1-\gamma)}) \right\} |\eta_0| \leq \left\{ \mu_3 \lambda - 2 \mu_1 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \beta) - \mu_2 h^2 \sigma_{1,0}^{(1-\gamma)} \lambda + \frac{2 \mu_1 (1 - \cos \beta \beta) + \mu_2 h^2 \lambda}{\mu_3 \lambda} \right\} |\eta_0| = |\eta_0|. \]

This finishes the proof. □

**Theorem 5.** If the parameters \( \lambda_1, \lambda_2, \lambda_3 \), space step length \( h \) and time step length \( \tau \) satisfy (34) and (35), then the difference scheme (17) is stable.

**Proof.** According to Lemma 4 and (26), we obtain
\[\lVert r^{k+1}\rVert_2 \leq \lVert r^k\rVert_2, \quad k = 0, 1, \ldots, m - 1,\]

which means that the difference scheme (17) is stable under conditions (34) and (35). The proof is finished.

Now, we use the similar method to analyze the stability of the difference scheme (21). From (21), we can get the roundoff error equation as follows:

\[\mu_3 \left[ \lambda_1 r_{i-1}^k + \lambda_2 r_i^k + \lambda_3 r_{i+1}^k \right] - \mu_1 \sigma_{i,0}^{(1-\gamma)} [r_i^k - 2r_i^j + r_{i+1}^j] + \mu_3 h^2 \sigma_{i,1}^{(1-\gamma)} [\lambda_1 r_i^k + \lambda_2 r_i^k + \lambda_3 r_{i+1}^k] = \mu_3 \left[ \lambda_1 r_{i-1}^k + \lambda_2 u_i^k - \lambda_3 r_{i+1}^k \right] + \mu_1 \sigma_{i,1}^{(1-\gamma)} [r_i^k - 2r_i^j + r_{i+1}^j - r_i^{k-1}] - \mu_3 h^2 \sigma_{i,1}^{1-\gamma} [\lambda_1 r_i^{k-1} + \lambda_2 r_i^{k-1} + \lambda_3 r_{i+1}^{k-1}] + \mu_2 \sum_{j=2}^{k} \sigma_{i,j}^{(1-\gamma)} [r_{i-1}^j - 2r_i^{j-1} + r_{i+1}^{j-1}] - h^2 \mu_2 \sum_{j=2}^{k} \sigma_{i,j}^{(1-\gamma)} [\lambda_1 r_{i-1}^j + \lambda_2 r_i^j + \lambda_3 r_{i+1}^j]. \quad i = 1, \ldots, n - 1; \quad k = 1, \ldots, m. \quad (36)\]

We suppose that the solution of Eq. (36) has the form

\[r_i^k = \eta_k e^{i\beta h}. \quad (37)\]

Substituting (37) into (38) gives

\[\mu_3 \lambda + \mu_3 \sigma_{i,0}^{(1-\gamma)} (1 - \cos \beta h) + \mu_3 h^2 \sigma_{i,1}^{(1-\gamma)} \lambda \eta_k = \mu_3 \lambda - 2 \mu_1 \sigma_{i,1}^{(1-\gamma)} (1 - \cos \beta h) - \mu_3 h^2 \sigma_{i,1}^{(1-\gamma)} \lambda \eta_{k-1} - 2 \mu_1 (1 - \cos \beta h) + \mu_3 h^2 \lambda \sum_{j=2}^{k} \sigma_{i,j}^{(1-\gamma)} \eta_{k-j}. \quad (38)\]

\[\Box\]

**Lemma 6.** Suppose that \(\eta_k (k = 1, \ldots, m)\) is the solution of Eq. (38). If

\[\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_1 = \lambda_3, \quad \lambda_1 < \frac{1}{4}, \quad (39)\]

then

\[|\eta_k| \leq |\eta_0|, \quad (k = 1, \ldots, m). \]

**Proof.** For \(k = 1, \) from (39), we get

\[\eta_1 = \frac{\mu_3 \lambda - 2 \mu_1 \sigma_{i,1}^{(1-\gamma)} (1 - \cos \beta h) - \mu_3 h^2 \sigma_{i,1}^{(1-\gamma)} \lambda}{\mu_3 \lambda + 2 \mu_1 \mu_3 h^2 \sigma_{i,1}^{(1-\gamma)} \lambda}, \quad \eta_0 = \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 (1 - \gamma) \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda}. \]

For \(\mu_1, \mu_2, \mu_3 > 0\) and \(0 < \gamma < 1,\) so

\[|\eta_1| = \left| \frac{\mu_3 \lambda - 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 (1 - \gamma) \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda} \right| |\eta_0| \leq |\eta_0|.

Now, we suppose that

\[|\eta_{k-1}| \leq |\eta_0|, \quad (\ell = 1, \ldots, k - 1). \]

According to (38) and **Lemma 2**, one has

\[|\eta_k| = \left| \frac{\mu_3 \lambda - 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 (1 - \gamma) \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda} \eta_{k-1} + \frac{2 \mu_1 (1 - \cos \beta h) + \mu_3 h^2 \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda} \sum_{j=2}^{k} \sigma_{i,j}^{(1-\gamma)} \eta_{k-j} \right| \leq \left| \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 (1 - \gamma) \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda} \eta_{k-1} + \frac{2 \mu_1 (1 - \cos \beta h) + \mu_3 h^2 \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda} \sum_{j=2}^{k} \sigma_{i,j}^{(1-\gamma)} \eta_{k-j} \right| \leq \left( \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 (1 - \gamma) \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_3 h^2 \lambda} \right) |\eta_0| \leq |\eta_0|.

This ends the proof. \[\Box\]

**Theorem 6.** If the parameters \(\lambda_1, \lambda_2\) and \(\lambda_3\) satisfy (39), then the difference scheme (21) is unconditionally stable.
\section*{4. Convergence analysis of the difference schemes}

In this section, we study the convergence of the difference schemes (17) and (21).

First of all, let
\[ \varepsilon^k_i = u(x_i, t_k) - u_i^k, \quad i = 1, \ldots, n - 1, \quad k = 1, \ldots, m \]
and denote
\[ \varepsilon^k = (\varepsilon^k_1, \varepsilon^k_2, \ldots, \varepsilon^k_{n-1})^T, \quad k = 1, \ldots, m, \]
\[ R^k = (R^k_1, R^k_2, \ldots, R^k_{n-1})^T, \quad k = 1, \ldots, m. \]

From Eq. (17) and the definition of the local truncation error $R^k_{i+1}$, we obtain
\begin{equation}
\mu_3 [\lambda_2 \varepsilon_{i+1}^k + \lambda_2 \varepsilon^k_{i+1} + \lambda_3 \varepsilon^k_{i+1}] = \mu_3 [\lambda_2 \varepsilon^k_{i-1} + \lambda_2 \varepsilon^k_i + \lambda_3 \varepsilon^k_i] + \mu_1 \sigma_{10}^{(1-\gamma)} [\varepsilon^k_{i-1} - 2\varepsilon^k_i + \varepsilon^k_{i+1}]
\end{equation}

\[ - \mu_2 h^2 \sigma_{10}^{(1-\gamma)} [\lambda_1 \varepsilon^k_{i-1} + \lambda_2 \varepsilon^k_i + \lambda_3 \varepsilon^k_i] + \mu_1 \sum_{j=1}^{k} \sigma_{ij}^{(1-\gamma)} [\varepsilon^k_{i-1} - 2\varepsilon^k_i + \varepsilon^k_{i+1}]
\]

\[ - \mu_2 h^2 \sum_{j=1}^{k} \sigma_{ij}^{(1-\gamma)} [\lambda_1 \varepsilon^k_{i-1} + \lambda_2 \varepsilon^k_i + \lambda_3 \varepsilon^k_i] + h^2 [\lambda_1 R^k_{i-1} + \lambda_2 R^k_i + \lambda_3 R^k_{i+1}] . \]  

Using the same idea of the stability analysis as before, we define the following grid functions:
\[ \varepsilon^k(x) = \begin{cases} 
\varepsilon^k_i, & \text{when } x_i - \frac{h}{2} < x \leq x_i + \frac{h}{2}, i = 1, 2, \ldots, n-1, \\
0, & \text{when } 0 \leq x < \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leq L
\end{cases} \]

and
\[ R^k(x) = \begin{cases} 
R^k_i, & \text{when } x_i - \frac{h}{2} < x \leq x_i + \frac{h}{2}, i = 1, 2, \ldots, n-1, \\
0, & \text{when } 0 \leq x < \frac{h}{2} \text{ or } L - \frac{h}{2} < x \leq L
\end{cases} \]

respectively.

Then, $\varepsilon^k(x)$ and $R^k(x)$ have the following Fourier series expansions, respectively,
\[ \varepsilon^k(x) = \sum_{|l|=-\infty}^{\infty} \zeta_k(l) e^{2\pi i l x / L}, \quad R^k(x) = \sum_{|l|=-\infty}^{\infty} \zeta_k(l) e^{2\pi i l x / L}, \]

where
\[ \zeta_k(l) = \frac{1}{L} \int_0^L \varepsilon^k(x) e^{-2\pi i l x / L} dx, \quad \zeta_k(l) = \frac{1}{L} \int_0^L R^k(x) e^{-2\pi i l x / L} dx. \]

Next,
\begin{equation}
\|\varepsilon^k\|_2 = \left( \sum_{i=1}^{n-1} h |\varepsilon^k_i|^2 \right)^{\frac{1}{2}} = \left( \sum_{|l|=-\infty}^{\infty} |\zeta_k(l)|^2 \right)^{\frac{1}{2}} \tag{41}
\end{equation}

and
\begin{equation}
\|R^k\|_2 = \left( \sum_{i=1}^{n-1} h |R^k_i|^2 \right)^{\frac{1}{2}} = \left( \sum_{|l|=-\infty}^{\infty} |\zeta_k(l)|^2 \right)^{\frac{1}{2}}. \tag{42}
\end{equation}

Now, we suppose that
\[ \varepsilon^k_i = \bar{\zeta}_k e^{i\phi}, \quad R^k_i = \bar{\zeta}_k e^{i\phi}. \]

Inserting these two formula into (40), we get
\[ \mu_3 \dot{\zeta}_{k+1} = \left[ \mu_3 \ddot{\zeta} - 2 \mu_4 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \hat{h}) - \mu_2 \hat{h}^2 \sigma_{1,0}^{(1-\gamma)} \dot{\zeta} \right]_k - \left[ 2 \mu_1 (1 - \cos \beta \hat{h}) + \mu_2 \hat{h}^2 \dot{\zeta} \right] \sum_{j=1}^{k} \sigma_{ij}^{(1-\gamma)} \zeta_{k-j} + \lambda \dot{\zeta}_{k+1}, \]  

where \( k = 0, 1, \ldots, m - 1 \).

**Lemma 7.** Suppose that \( \zeta_{k+1} \) (\( k = 0, 1, \ldots, m - 1 \)) is the solution of Eq. (43). If the parameters \( \lambda_1, \lambda_2, \lambda_3 \), space step length \( h \) and time step length \( \tau \) satisfy (34) and (35), then there exists a positive constant \( c_2 \), such that

\[ |\zeta_{k+1}| \leq c_2 (k + 1) \tau|\zeta_1|, \quad (k = 0, 1, \ldots, m - 1). \]

**Proof.** Because \( \zeta_0 = 0 \), we know

\[ \zeta_0 = \zeta_0(0) = 0. \]

Meanwhile, in view of Theorem 2 and the left-hand equality of (42), there exists a positive constant \( c_1 \), such that

\[ \left| R_{k+1} \right| = O(\tau + h^p) \leq c_1 (\tau + h^p) \]

and

\[ \| R_{k+1} \| \leq c_1 \sqrt{(n - 1) h (\tau + h^p)} \leq c_1 \sqrt{L (\tau + h^p)}. \]

Using again the convergence of the series in the right-hand side of (42) yields

\[ |\zeta_{k+1}| = |\zeta_{k+1}(0)| \leq c_2 |\zeta_1| = c_2 |\zeta_1(t)|. \]

From (43), when \( k = 0 \), we know that

\[ \zeta_1 = \frac{\mu_3 \dot{\zeta} - 2 \mu_4 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \hat{h}) - \mu_2 \hat{h}^2 \sigma_{1,0}^{(1-\gamma)} \dot{\zeta}}{\mu_3 \ddot{\zeta}} \]

Noticing (34) and (45), one gets

\[ |\zeta_1| = \tau |\zeta_1| \leq c_2 \tau |\zeta_1|. \]

Suppose that

\[ |\zeta_1| \leq c_2 \tau |\zeta_1|, \quad \ell = 1, \ldots, k. \]

For \( k > 0 \), from (43), (34) and (35), we have

\[ |\zeta_{k+1}| \leq \left| \zeta_k + \frac{\mu_3 \ddot{\zeta} - 2 \mu_4 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \hat{h}) - \mu_2 \hat{h}^2 \sigma_{1,0}^{(1-\gamma)} \dot{\zeta}}{\mu_3 \ddot{\zeta}} \right| \left( \left| \zeta_k \right| + \sum_{j=1}^{k} \left| \sigma_{ij}^{(1-\gamma)} \right| \left| \zeta_{k-j} \right| + \tau |\zeta_{k+1}| \right) \]

\[ \leq c_2 \tau \left\{ \left| \zeta_k \right| + \left| \mu_3 \ddot{\zeta} - 2 \mu_4 \sigma_{1,0}^{(1-\gamma)} (1 - \cos \beta \hat{h}) - \mu_2 \hat{h}^2 \sigma_{1,0}^{(1-\gamma)} \dot{\zeta} \right| \right\} \]

\[ \leq c_2 \tau \left\{ c_2 k |\zeta_1| + \tau |\zeta_{k+1}| \right\} \leq c_2 k |\zeta_1| + c_2 \tau |\zeta_1| = c_2 (k + 1) \tau |\zeta_1|. \]

All this completes the proof. \( \square \)
Theorem 7.

(i) When the parameters $\lambda_1, \lambda_2, \lambda_3$, space step length $h$ and time step length $\tau$ satisfy (34) and (35), the difference scheme (17) is $L_2$-convergent, and the order of convergence is $O(\tau + h^3)$.

(ii) When the parameters $\lambda_1 = \lambda_3 = \frac{1}{12}, \lambda_2 = \frac{3}{4}$, space step length $h$ and time step length $\tau$ satisfy (35), the difference scheme (17) is $L_2$-convergent, and the order of convergence is $O(\tau + h^4)$.

Proof. Using (35), (41), (42), (44) and Lemma 7, we get

$$\left\|k^{k+1}\right\|_2 \leq c_2(k + 1)\tau \left\|R^1\right\|_2 \leq c_1 c_2 \sqrt{L}(k + 1)\tau(\tau + h^p).$$

Because of $k \leq m - 1$, then

$$(k + 1)\tau \leq T,$$

so,

$$\left\|k^{k+1}\right\|_2 \leq c(\tau + h^p),$$

where $c = c_1 c_2 T \sqrt{L}$.

Applying Theorem 1, we can get above conclusions. This ends the proof.\[\square\]

Next, we will analyze the convergence of the difference scheme (21) by using similar method. From Eq. (21) and the definition of the local truncation error $R_k$, we have

$$\mu_3 \left[\lambda_1 \frac{\partial u}{\partial x}^{k-1} + \lambda_2 \frac{\partial u}{\partial x}^{k-1} + \lambda_3 \frac{\partial u}{\partial x}^{k-1} - \mu_1 \frac{\partial^2 u}{\partial x^2}^{k-1} \right] - \mu_2 h^2 \frac{\partial^2 u}{\partial x^2}^{k-1} + \mu_3 \frac{\partial^2 u}{\partial x^2}^{k-1}$$

$$= \mu_3 \left[k_{i+1}^{k-1} + \lambda_2 k_{i-1}^{k-1} + \lambda_3 k_{i-1}^{k-1} - \mu_1 k_{i+1}^{k-1} - \mu_2 h^2 k_{i+1}^{k-1} - \mu_3 h^2 k_{i+1}^{k-1} + \lambda_2 k_{i-1}^{k-1} + \lambda_3 k_{i-1}^{k-1} + \lambda_3 k_{i+1}^{k-1}\right]$$

$$+ \mu_4 \sum_{j=2}^k \left[\frac{\partial^2 u}{\partial x^2}^{k-1} \left[k_{i+1}^{k-1} + k_{i-1}^{k-1} + \lambda_3 k_{i+1}^{k-1} + \lambda_2 k_{i-1}^{k-1}\right] + h^2 \left[k_{i+1}^{k-1} + k_{i-1}^{k-1} + \lambda_3 k_{i+1}^{k-1} + \lambda_2 k_{i-1}^{k-1}\right]\right].$$

Substituting $\frac{\partial u}{\partial x} = \zeta e^{ith}$ and $R_k = \zeta e^{ith}$ into (46) yields

$$\left[\mu_3 \lambda_1 + 2 \mu_1 \frac{\partial^2 u}{\partial x^2}^{k-1}(1 - \cos \beta h) + \mu_2 h^2 \frac{\partial^2 u}{\partial x^2}^{k-1}\right] \zeta_k = \left[\mu_3 \lambda_1 - 2 \mu_1 \frac{\partial^2 u}{\partial x^2}^{k-1}(1 - \cos \beta h) - \mu_2 h^2 \frac{\partial^2 u}{\partial x^2}^{k-1}\right] \zeta_{k-1}$$

$$- 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda \sum_{j=2}^k \frac{\partial^2 u}{\partial x^2}^{k-1} \zeta_{k-j} + \lambda h^2 \zeta_k,$$

where $k = 1, \ldots, m$.

Lemma 8. Suppose that $\zeta_k (k = 1, \ldots, m)$ is the solution of Eq. (47). If the parameters $\lambda_1, \lambda_2, \lambda_3$ satisfy (34), then there exists a positive constant $\tilde{c}_1$, such that

$$\left|\zeta_k\right| \leq \tilde{c}_1 k \left|\zeta_1\right|, \quad (k = 1, \ldots, m).$$

Proof. Obviously, in view of Theorem 2 and the left-hand equality of (42), there exists a positive constant $\tilde{c}_1$, such that

$$\left|R_k\right| = O(\tau + h^3) \leq \tilde{c}_1(\tau + h^3)$$

and

$$\left|R^1\right| \leq \tilde{c}_1 \sqrt{n-1} h(\tau + h^3) \leq \tilde{c}_1 \sqrt{L}(\tau + h^3).$$

On the other hand, we also know that there is a positive constant $\tilde{c}_2$, such that

$$\left|\zeta_k\right| = \left|\zeta_k(l)\right| \leq \tilde{c}_2 \left|\zeta_1(l)\right|.$$

For $k = 1$, from (47), we have

$$\zeta_1 = \frac{\mu_3 \lambda_1 + 2 \mu_1 (1 - \gamma)(1 - \cos \beta h) + \mu_2 h^2 (1 - \gamma) \lambda_1}{\mu_3 \lambda_1 + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda_1} \zeta_0 + \frac{\lambda h^2}{\mu_3 \lambda_1 + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda_1} \zeta_1 \leq \tau \zeta_1.$$

Noticing (49), one gets
Table 1
The maximum errors, temporal and spatial convergence orders of difference scheme (17) for \( \lambda_3 = \lambda_1 = \frac{1}{4}, \lambda_2 = \frac{1}{8} \).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>The maximum errors</th>
<th>Temporal convergence orders</th>
<th>Spatial convergence orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>2.372488e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>5.851496e-002</td>
<td>1.0098</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>1.457898e-002</td>
<td>1.0025</td>
</tr>
<tr>
<td>0.4</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>2.00014e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>5.934761e-002</td>
<td>1.0079</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>1.479581e-002</td>
<td>1.0020</td>
</tr>
<tr>
<td>0.6</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>2.423927e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>6.009182e-002</td>
<td>1.0061</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>1.499085e-002</td>
<td>1.0015</td>
</tr>
<tr>
<td>0.8</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>2.443819e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>6.072781e-002</td>
<td>1.0044</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>1.515869e-002</td>
<td>1.0011</td>
</tr>
</tbody>
</table>

Table 2
The maximum errors, temporal and spatial convergence orders of difference scheme (17) for \( \lambda_3 = \lambda_1 = \frac{1}{4}, \lambda_2 = \frac{1}{8} \).

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>The maximum errors</th>
<th>Temporal convergence orders</th>
<th>Spatial convergence orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>4.829255e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>2.919103e-002</td>
<td>1.0121</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>2.351233e-003</td>
<td>0.9085</td>
</tr>
<tr>
<td>0.4</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>4.868143e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>2.961896e-002</td>
<td>1.0097</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>2.547204e-003</td>
<td>0.8849</td>
</tr>
<tr>
<td>0.6</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>4.900263e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>3.000310e-002</td>
<td>1.0074</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>2.461454e-003</td>
<td>0.9019</td>
</tr>
<tr>
<td>0.8</td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{4} )</td>
<td>4.928560e-001</td>
<td>–</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{2}, \tau = \frac{1}{16} )</td>
<td>3.033289e-002</td>
<td>1.0054</td>
</tr>
<tr>
<td></td>
<td>( h = \frac{1}{32}, \tau = \frac{1}{16} )</td>
<td>2.213167e-003</td>
<td>0.9442</td>
</tr>
</tbody>
</table>

\[ |\xi_1| \leq |\tau| |\xi_1| \leq \bar{c}_2 |\tau| |\xi_1| . \]

Now, we suppose that
\[ |\xi_1| \leq \bar{c}_2 \ell |\xi_1|, \quad \ell = 1, \ldots, k - 1. \]

Then when \( k > 1 \), from (47), we get
\[
|\xi_k| = \left| \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_2 h^2 (1 - \gamma) \alpha}{\mu_3 \lambda + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda} \right| |\xi_{k-1}| + \frac{2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda} \sum_{j=2}^{k} \sigma_{1,-j} |\xi_j| + \tau |\xi_k| \]
\[
\leq \bar{c}_2 (k-1) \tau \left\{ \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_2 h^2 (1 - \gamma) \alpha}{\mu_3 \lambda + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda} |\xi_{k-1}| + \frac{2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda}{\mu_3 \lambda + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda} \sum_{j=2}^{k} \sigma_{1,-j} |\xi_j| + \tau |\xi_k| \right\} \]
\[
\leq \bar{c}_2 (k-1) \tau \left\{ \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_2 h^2 (1 - \gamma) \alpha}{\mu_3 \lambda + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda} \left( \sigma_{1,-1} + 2 \sum_{j=2}^{k} \sigma_{1,-j} \right) |\xi_{k-1}| + \bar{c}_2 |\tau| |\xi_1| \right\} \]
\[
\leq \bar{c}_2 (k-1) \tau \left\{ \frac{\mu_3 \lambda + 2 \mu_1 (1 - \gamma) (1 - \cos \beta h) + \mu_2 h^2 (1 - \gamma) \alpha}{\mu_3 \lambda + 2 \mu_1 (1 - \cos \beta h) + \mu_2 h^2 \lambda} \left( \sigma_{1,-1} + 2 \sum_{j=2}^{k} \sigma_{1,-j} \right) |\xi_{k-1}| + \bar{c}_2 k |\tau| |\xi_1| \right\}.
\]

This completes the proof. \( \square \)

Accordingly, we obtain the following Theorem.

**Theorem 8.**

(i) When the parameters \( \lambda_1, \lambda_2, \lambda_3 \) satisfy (34), the difference scheme (21) is \( L_2 \)-convergent, and the order of convergence is \( O(\tau + h^2) \).
Table 3
The maximum errors, temporal and spatial convergence orders of difference scheme (21) for $\lambda_1 = \lambda_3 = \frac{1}{6}, \lambda_2 = \frac{2}{6}$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>The maximum errors</th>
<th>Temporal convergence orders</th>
<th>Spatial convergence orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{6}$</td>
<td>$2.481369e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{12}$</td>
<td>$6.203175e-002$</td>
<td>$1.0000$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{24}$</td>
<td>$1.550731e-002$</td>
<td>$1.0000$</td>
</tr>
<tr>
<td>0.4</td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{6}$</td>
<td>$2.464630e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{12}$</td>
<td>$6.164778e-002$</td>
<td>$0.9996$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{24}$</td>
<td>$1.541374e-002$</td>
<td>$0.9999$</td>
</tr>
<tr>
<td>0.6</td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{6}$</td>
<td>$2.450901e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{12}$</td>
<td>$6.136358e-002$</td>
<td>$0.9989$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{24}$</td>
<td>$1.534668e-002$</td>
<td>$0.9997$</td>
</tr>
<tr>
<td>0.8</td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{6}$</td>
<td>$2.440656e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{12}$</td>
<td>$6.117995e-002$</td>
<td>$0.9980$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{3}, \tau = \frac{1}{24}$</td>
<td>$1.530543e-002$</td>
<td>$0.9995$</td>
</tr>
</tbody>
</table>

Table 4
The maximum errors, temporal and spatial convergence orders of difference scheme (21) for $\lambda_1 = \lambda_3 = \frac{1}{12}, \lambda_2 = \frac{5}{6}$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>The maximum errors</th>
<th>Temporal convergence orders</th>
<th>Spatial convergence orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{8}$</td>
<td>$4.962545e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{16}$</td>
<td>$3.101518e-002$</td>
<td>$1.0000$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{32}$</td>
<td>$1.444916e-003$</td>
<td>$1.1060$</td>
</tr>
<tr>
<td>0.4</td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{8}$</td>
<td>$4.925788e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{16}$</td>
<td>$3.082639e-002$</td>
<td>$0.9995$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{32}$</td>
<td>$1.359560e-003$</td>
<td>$1.1257$</td>
</tr>
<tr>
<td>0.6</td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{8}$</td>
<td>$4.892552e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{16}$</td>
<td>$3.068955e-002$</td>
<td>$0.9987$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{32}$</td>
<td>$1.402166e-003$</td>
<td>$1.1130$</td>
</tr>
<tr>
<td>0.8</td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{8}$</td>
<td>$4.864525e-001$</td>
<td>$- $</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{16}$</td>
<td>$3.060390e-002$</td>
<td>$0.9976$</td>
</tr>
<tr>
<td></td>
<td>$h = \frac{1}{2}, \tau = \frac{1}{32}$</td>
<td>$1.587398e-003$</td>
<td>$1.0672$</td>
</tr>
</tbody>
</table>

Fig. 5.1. The comparison of numerical solutions by difference scheme (17) with $\lambda_1 = \lambda_3 = \frac{1}{6}, \lambda_2 = \frac{5}{6}$ at $t = 0.5$ ($\frac{1}{2}, \tau = \frac{1}{32}$).
When the parameters $k_1 = \frac{1}{12}$, $k_2 = \frac{5}{6}$, the difference scheme (21) is $L^2$-convergent, and the order of convergence is $O(s + h^4)$.

Proof. The proof is almost the same as that of Theorem 7 so omit here.

Remark 2. From (17), (21) and Theorems 5–8, we find out that difference scheme (17) is explicit, conditionally stable and convergent, but difference scheme (21) is implicit, unconditionally stable and convergent.

5. Numerical examples

In this section, we implement our difference schemes by the following numerical examples.

Example 1. Consider the equation below
\[ \frac{\partial u(x,t)}{\partial t} = \frac{RLD}{C_0} \left[ \sin(\pi \gamma) \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{\pi^2} u(x,t) \right] + f(x,t), \quad 0 < x < 1, 0 < t < 1, \]

where \( f(x,t) = \sin(\pi x) \left[ 2t + \frac{2(\sin(\pi \gamma) + \frac{1}{\pi^2})}{\sin(\pi \gamma)} \right] \). The exact solution of this equation is \( u(x,t) = t^2 \sin(\pi x) \). The initial condition and the boundary conditions are satisfied with the exact solution \( u(x,t) \) given above.

We define the maximum error of the exact solution and the numerical solution by the following form:

\[ E_\infty = \max_{0 < k < m} \max_{0 < t < h} \left| u_k^h - u(x_k, t_h) \right| \]

Firstly, we use difference scheme (17) to solve the above equation. The maximum errors, temporal and spatial convergence orders are tabulated in Tables 1 and 2 for different parameters \( \lambda_1, \lambda_2, \lambda_3 \) and \( \gamma \). Next, we use difference scheme (21) to solve the above equation. The maximum errors, temporal and spatial convergence orders are listed in Tables 3 and 4 for different parameters \( \lambda_1, \lambda_2, \lambda_3 \), and \( \gamma \). From Tables 1–4, it can be seen that we can get different spatial convergence orders by choose different parameters \( \lambda_1, \lambda_2, \lambda_3 \), which support the theoretical analysis.
Example 2. Consider the equation below
\[ \frac{\partial u}{\partial t} = RLD \left[ \Gamma(\gamma + 1) \left( \frac{\partial^2 u}{\partial x^2} - u(x, t) \right) \right] + f(x, t), \quad 0 < x < 1, \quad 0 < t < 1, \]

with the following initial and boundary conditions
\[ u(x, 0) = 0, \quad 0 < x < 1, \]
\[ u(0, t) = 0, \quad 0 \leq t \leq 1, \]
\[ u(1, t) = 0, \quad 0 \leq t \leq 1, \]

where \( f(x, t) = x^{3-\gamma}(1-x)^{2+\gamma} \left[ 1 + \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 2)} \right] - \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 2)} \left( 3 - \gamma \right)(2 - \gamma)x^{3-\gamma}(1-x)^{2+\gamma} - 2(3 - \gamma)(2 + \gamma)x^{2+\gamma}(1-x)^{1+\gamma} \).

Firstly, we choose \( \lambda_1, \lambda_2, \lambda_3, \tau \) and \( h \) to meet (34) and (35), then we use difference scheme (17) to solve the above equation. Figs. 5.1 and 5.2 show the numerical solutions with \( \gamma = 0.2, 0.4, 0.6, 0.8, 1 \) at \( t = 0.5 \) and \( t = 0.8 \) for different parameters \( \lambda_1, \lambda_2 \) and \( \lambda_3 \), respectively. Figs. 5.3 and 5.4 show the solution surface of the above equation when \( \gamma = 0.7 \) for different parameters \( \lambda_1, \lambda_2 \) and \( \lambda_3 \).
Secondly, we choose $k_1, k_2, k_3$ to meet (34), then we use difference scheme (21) to solve the above equation. Figs. 5.5 and 5.6 show the numerical solutions with $c = 0.2, 0.4, 0.6, 0.8, 1$ at $t = 0.4$ and $t = 0.6$ for different parameters $k_1, k_2$ and $k_3$, respectively. In Figs. 5.7 and 5.8, we display the solution surface of the above equation when $c = 0.3$ for different parameters $k_1, k_2$ and $k_3$.

From Figs. 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, we find that with the increase of $c$, the solution increases more slowly which is similar to the results reported in Yuste et al. [26,27].

6. Conclusion

In this paper, we propose two classes of difference schemes for solving the reaction–subdiffusion equations based on a new mixed spline function. And we show that difference scheme (21) is unconditionally stable and difference scheme (17) is conditionally stable by using the Fourier method. The truncation errors of these two difference schemes are $O(s + h^2)$ and $O(s + h^4)$. In the end, the numerical results support the theoretical analysis.

References