EXISTENCE AND CONTINUATION THEOREMS OF RIEMANN–LIOUVILLE TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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In this paper we study the existence and continuation of solution to the general fractional differential equation (FDE) with Riemann–Liouville derivative. If no confusion appears, we call FDE for brevity. We firstly establish a new local existence theorem. Then, we derive the continuation theorems for the general FDE, which can be regarded as a generalization of the continuation theorems of the ordinary differential equation (ODE). Such continuation theorems for FDE which are first obtained are different from those for the classical ODE. With the help of continuation theorems derived in this paper, several global existence results for FDE are constructed. Some illustrative examples are also given to verify the theoretical results.

Keywords: Fractional differential equations; Riemann–Liouville derivative; local existence; continuation theorem; global solution.

1. Introduction

Fractional Differential Equations (FDEs) have captured great interest recently due to the active development of the theory of fractional calculus itself as well as its applications. It has been found that fractional differential equations appear in a variety of different areas such as viscoelasticity, electrochemistry, control, porous media and many other branches of science (see Gaul et al., 1991; Hilfer, 2000; Koeller, 1984; Srivastava & Saxena, 2001) and references cited therein). However, the basic theory of FDEs such as existence, uniqueness and stability has been developed slowly. Only very limited qualitative theories of FDEs have been obtained [Hilfer, 2000; Kilbas et al., 2006; Lakshmikantham et al., 2009; Miller & Ross, 1993].

In recent times, the investigation for the existence of solutions to FDEs has attracted much interest [Agarwal et al., 2010; Babakhani & Gejji, 2003; Diethelm & Ford, 2002; Kosmatov, 2009; Lakshmikantham et al., 2009; Muslim et al., 2010; Nieto, 2010; El-Sayed & El-Sayed, 2004; Zhang, 2009]. However, to the best of our knowledge, almost all the existing results on the existence theory for FDEs has focused on developing the existence-uniqueness of solutions on the finite interval \([0, T]\) or \([t_0, t_0 + T]\)
study the global existence of solutions. In this paper, we focus on
studying the global existence theorems of FDEs. Generally speaking, there are two ways to
study the global existence of solutions of differential equations on the half-axis, i.e. the interval \([0, +\infty)\) [Arara et al.,
2010; Baleanu & Mustafa, 2010; Baleanu et al., 2010; Lakshmikantham & Vatsala, 2008], where the
forms of the considered FDEs are somewhat special but the results are interesting. It should be
pointed out that such global existence theorems are fundamental ones in the elementary theory of
FDEs, especially in stability analysis of FDEs, for example, see [Deng et al., 2007a, 2007b; Li et al.,
2009a; Li & Zhang, 2011; Petrás, 2009; Qian et al., 2010; Zhang & Li, 2011; Zhang et al., 2011] and
many references cited therein. In this paper, we focus on
studying the global existence theorems of FDEs. Generally speaking, there are two ways to
study the global existence of solutions by directly using the
theorems for FDEs have not been derived yet. Thus,
we will continue to (see Hypothesis (H) in Sec. 3). In this paper, we start with the study of the
local existence of solution for the following general
initial value problem (IVP) of FDE

\[
\begin{align*}
0 < \alpha < 1, & \quad t \in (0, +\infty), \\
D_0^\alpha x(t)|_{t=0} = x_0, & \quad (\alpha \neq 1), \\
D_0^{1-\alpha} x(t)|_{t=0} = x_0, & \quad \alpha \neq 1.
\end{align*}
\]

(1)

where \(D_0^\alpha\) is the Riemann–Liouville fractional derivative which is introduced in the next section, and \(f : \mathbb{R}^+ \to \mathbb{R}\) admits certain singularity (see Hypothesis (I) in Sec. 3). In this paper, we mainly extend the continuation theorems for ODEs to those of FDEs. Furthermore, we will continue to study the global existence of solutions. The rest of the paper is organized as follows. In
Sec. 2, we present some necessary definitions and previously known results that will be used later on.

Section 3 is devoted to the study of the local existence of solutions, in which we obtain a new local existence theorem for the IVP (1). Furthermore, in
Sec. 4, two continuation theorems for the IVP (1) are given which can be regarded as the generalization of the continuation theorems for ODEs. Finally,
based on the results obtained previously, we present some global existence theorems. Several examples are also included to illustrate the advantages of our results. Conclusions and comments are included in
the last section.

2. Preliminaries

In this section, we present some basic definitions and notations and several lemmas [Kilbas et al.,
2006; Li & Zhao, 2011; Miller & Ross, 1993] which will be used in the following sections.

Let \(C[a,b]\) be the Banach space of all continuous functions mapping \([a, b]\) into \(\mathbb{R}\) with the norm
\(\|x\|_{[a,b]} = \max_{t \in [a,b]} |x(t)|\). Let also \(C_{1-\alpha}[a,b] = \{x(t) : (a, b) \to \mathbb{R} : (t-\alpha)^{1-\alpha}x(t) \in C[a,b]\}\), normed with
\(\|x\|_{C_{1-\alpha}[a,b]} = \max_{t \in [a,b]}|(t-\alpha)^{1-\alpha}x(t)|\), where \(\alpha \in (0, 1]\). It is easy to verify that \(C_{1-\alpha}[a,b]\) is a
Banach space too.

Definition 2.1. The Riemann–Liouville integral of order \(\alpha > 0\) of a function \(f : \mathbb{R}^+ \to \mathbb{R}\) is given by

\[
D_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds
\]

provided that the right-hand side of the above equation is pointwisely defined on \(\mathbb{R}^+\).

Definition 2.2. The Riemann–Liouville derivative with order \(\alpha > 0\) of a function \(f : \mathbb{R}^+ \to \mathbb{R}\) is given by

\[
D_0^\alpha f(t) = D^n D_0^{\gamma-n} f(t)
\]

where \(n = \lfloor \alpha \rfloor + 1, D^n = \frac{d^n}{dt^n}, t > 0\), provided that the right-hand side of the above equation is pointwisely defined on \(\mathbb{R}^+\).

Lemma 2.1 [Kilbas et al., 2006]. Let \(\alpha > 0, 0 \leq \gamma < 1\). If \(\gamma > \alpha\), then the fractional integration operators \(D_0^\gamma\) are bounded from \(C_{1-\alpha}[a,b]\) into \(C_{1-\alpha}[a,b]\). If \(\gamma \leq \alpha\), then the fractional integration operators \(D_0^\gamma\) are bounded from \(C_{1-\alpha}[a,b]\) into \(C_{1-\alpha}[a,b]\).
and only if $x(t)$ is a solution of the Volterra integral equation

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \, ds,$$

with $t \in (0, +\infty)$.

The reader can refer to [Li & Deng, 2007; Li et al., 2009b, 2011; Kilbas et al., 2006; Lakshmikantham et al., 2009; Miller & Ross, 1993] for more details.

**Lemma 2.3.** Let $M$ be a subset of $C_{1-\alpha}[0,T]$. Then $M$ is precompact if and only if the following conditions are satisfied:

(i) $\{t^{1-\alpha}x(t) : x \in M\}$ is uniformly bounded,

(ii) $\{t^{1-\alpha}x(t) : x \in M\}$ is equicontinuous on $[0,T]$.

The proof of this lemma is easy, so is omitted here.

The following lemma and the Schauder’s fixed point theorem play a key role in the proofs of our main results.

**Lemma 2.4** [Kou et al., 2010]. Let $a < b < c$, $0 \leq \gamma < 1$, $x \in C_\gamma[a,b]$, $y \in C[b,c]$ and $x(b) = y(b)$. Define

$$z(t) = \begin{cases} x(t), & t \in (a,b), \\ y(t), & t \in [b,c]. \end{cases}$$

Then $z \in C_\gamma[a,c]$.

**Lemma 2.5** [Granas & Dugundji, 2003] (Schauder Fixed-Point Theorem). Let $U$ be a closed bounded convex subset of a Banach space $X$ and $T : U \to U$ is completely continuous. Then $T$ has a fixed point in $U$.

### 3. Local Existence

In this section, we are concerned with the local existence of solutions for the IVP (1). In the case that the right-hand function $f(t,x)$ admits certain singularity, by applying Schauder fixed-point theorem, a new local existence theorem is obtained.

For convenience, let us list the following hypothesis.

(H) Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ in IVP (1) be a continuous function and there exists a constant $0 \leq \sigma < 1$ such that $(Bx)(t) = t^\sigma f(t, x(t))$ is a continuous bounded map from $C_{1-\alpha}[0,T]$ into $C[0,T]$, where $T$ is a positive constant.

**Remark 3.1.** In fact, let $g : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and $f(t, x) = x^\beta g(t, t^\delta x)$, where $0 \leq \lambda \leq 1$, $\beta \geq 1 - \alpha$, then the operator $B$, defined as above, is a continuous bounded map from $C_{1-\alpha}[0,T]$ into $C[0,T]$.

**Theorem 3.1.** Assume that the condition (H) is satisfied. Then the IVP (1) has at least one solution $x \in C_{1-\alpha}[0,h]$ for some $h > 0(< T)$.

**Proof.** Let

$$E = \left\{ x \in C_{1-\alpha}[0,T] : \left\| x - \frac{x_0}{\Gamma(\alpha)} \right\|_{C_{1-\alpha}[0,T]} \leq b \right\},$$

where $b > 0$ is a constant. Since the operator $B$ is bounded, there exists a constant $M > 0$, such that

$$\sup \{(Bz)(t) : t \in [0,T], x \in E\} \leq M.$$

Again, let

$$D_h = \left\{ x : x \in C_{1-\alpha}[0,h], \sup \left\{ \left\| t^{1-\alpha}x(t) \right\| : 0 \leq t \leq h \right\} \leq b \right\},$$

where $h = \min\left\{ \frac{1}{\Gamma(\alpha + 1 - \sigma)}, T \right\}$. Evidently, $D_h \subseteq C_{1-\alpha}[0,h]$ is a nonempty, bounded and convex subset.

Note that $h \leq T$, we can regard $D_h$ and $C_{1-\alpha}[0,h]$ as the restrictions of $E$ and $C_{1-\alpha}[0,T]$, respectively. Define the operator $A$ as follows:

$$(Ax)(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \, ds,$$

with $t \in (0,h)$.

It follows from (H) and Lemma 2.1 that we have $A(C_{1-\alpha}[0,h]) \subseteq C_{1-\alpha}[0,h]$.

On the other hand, by (2), for any $x \in C_{1-\alpha}[0,h]$, we have

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Noticing that $A D_h \subset D_h$

Next, we show that $A$ is continuous. Let $x_n, x \in D_h, \|x_n - x\|_{C_{[-r, r]}} \to 0$ ($n \to +\infty$). In view of the continuity of $B$, we have

$$\|B x_n - B x\|_{[0, h]} \to 0 \quad \text{as} \quad n \to +\infty.$$ 

Noticing that

$$|t^{1-n}(Ax_n)(t) - t^{1-n}(Ax)(t)|$$

$$\leq \frac{\Gamma(1-\sigma)}{\Gamma(\alpha + 1 - \sigma)} \int_0^t (t-s)^{\alpha-1}s^{-\sigma} ds$$

which means $A D_h \subset D_h$.

Thus $A$ is continuous.

Furthermore, we shall prove that the operator $A D_h$ is equicontinuous. Let $x \in D_h$, and $0 \leq t_1 < t_2 \leq h$. For any given $\varepsilon > 0$, note that

$$\frac{\Gamma(1-\sigma)}{\Gamma(\alpha + 1 - \sigma)} \int_0^t (t-s)^{\alpha-1}s^{-\sigma} ds$$

$$= \frac{\Gamma(1-\sigma)}{\Gamma(\alpha + 1 - \sigma)} t^{1-\sigma} \to 0,$$

as $t \to 0^+$, where $0 \leq \sigma < 1$.

there exists a $(h > \delta_1 > 0)$ such that, for $t \in [0, \delta_1]$,

$$\frac{2M^{1-n}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}s^{-\sigma} ds < \varepsilon.$$ 

In the case with $t_1, t_2 \in [0, h]$, one has

$$\frac{\Gamma(1-\sigma)}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds - \frac{\Gamma(1-\sigma)}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds$$

$$\leq \frac{\Gamma(1-\sigma)}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds + \frac{\Gamma(1-\sigma)}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} s^{-\sigma} ds < \varepsilon.$$ 

In the case with $t_1, t_2 \in [h, h]$, one gets

$$|t_1^{1-n}(Ax_n)(t_1) - t_2^{1-n}(Ax)(t_2)|$$

$$= \frac{\Gamma(1-\sigma)}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds - \frac{\Gamma(1-\sigma)}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds$$

$$\leq \left\{|t_1^{1-n}(t_1-s)^{\alpha-1} - t_2^{1-n}(t_2-s)^{\alpha-1}| f(s, x(s)) ds\right\}$$

On one hand, from the simple fact that if $0 \leq \gamma_1 < \gamma_2 \leq h$, then $\gamma_1^{1-n}(\gamma_1-s)^{\alpha-1} > \gamma_2^{1-n}(\gamma_2-s)^{\alpha-1}$

for $0 \leq s < \gamma_1$, we obtain

$$\frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} (t_2-s)^{\alpha-1} s^{-\sigma} ds$$

$$\leq \frac{M}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{-\sigma} (t_2-s)^{\alpha-1} s^{-\sigma} ds$$

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At last, it follows from (3) and (4) that
\[\left\{ t^{\frac{1}{2}}(x(t) - t^{\frac{1}{2}})(x(t)) \right\} \begin{cases} \frac{M}{1(\alpha)} \left( \int_0^{t_2} (t_2 - s)^{\alpha - 1} f(s, x(s)) ds \right) \\
+ \frac{\delta_1}{2} \frac{M}{1(\alpha)} \left( \int_0^{t_2} (t_2 - s)^{\alpha - 1} ds \right) \\
\leq \int_0^{t_2} (t_2 - s)^{\alpha - 1} ds \ 
\end{cases} \]
implies
\[|t_2^{\frac{1}{2}}(x(t)) - t_2^{\frac{1}{2}}(x(t))| < 2r. \] (4)

At last, it follows from (3) and (4) that \( \{t^{\frac{1}{2}}(x(t)) : x \in D_0 \} \) is equicontinuous.

It is evident that \( \{t^{\frac{1}{2}}(x(t)) : x \in D_0 \} \) is uniformly bounded, due to \( AD_0 \subset D_0 \). By Lemma 2.3, \( AD_0 \) is compact. Therefore \( A \) is completely continuous. By Schauder fixed point theorem and Lemma 2.2, the IVP (1) has a local solution. The proof is completed. \( \blacksquare \)

**Example 3.1.** Consider the following IVP

\[ RL_{0+}^\alpha x(t) = t^{-\frac{1}{2}} \sin(tx^2(t) + 1), \quad t \in (0, +\infty), \]
\[ RL_{0+}^\alpha \dot{x}(t)|_{t=0} = \sqrt{\pi}. \] (5)

It is easy to verify that \( f(t, x) = t^{-\frac{1}{2}} \sin(tx^2 + 1) \) satisfies the condition of Remark 3.1. Thus, by Theorem 3.1, the IVP (5) admits a local solution.

**Remark 3.2.** It is not difficult to see that \( f(t, x) = t^{-\frac{1}{2}} \sin(tx^2 + 1) \) does not satisfy the conditions of Theorem 3.1 in [Lakshmikantham & Vatsala, 2008] and Theorem 3.5 in [Kou et al., 2010]. Thus, Theorem 3.1 is more general than those of [Lakshmikantham & Vatsala, 2008] and [Kou et al., 2010].

**Remark 3.3.** For \( f(t, x) = tx \), \( 0 < \beta < 1 \), we can easily obtain the local existence theorem of solution in this autonomous case. We can see that Theorem 4.5 in [Delbosco & Rodino, 1996] is a special case of Theorem 3.1 of this paper.

### 4. Continuation Theorems

In the present section, we are concerned with the continuation of solutions for IVP (1). We try to extend the continuation theorems for ODEs to those of FDEs. Initially, we give the following definition.

**Definition 4.1.** Let \( x(t) \) on \((0, \beta)\) and \( \tilde{x}(t) \) on \((0, \tilde{\beta})\) both be solutions of (1). If \( \beta < \tilde{\beta} \) and \( x(t) = \tilde{x}(t) \)
for \( t \in (0, \beta) \), we say \( \hat{x}(t) \) is a continuation of \( x(t) \), or \( x(t) \) can be continued to \( (0, \beta) \). A solution \( x(t) \) is noncontinuable if it has no continuation. The existing interval of noncontinuable solution \( x(t) \) is called the maximum existing interval of \( x(t) \).

In this section, we need the following conclusion.

**Lemma 4.1.** Let \( 0 < \alpha \leq 1, \beta > 0, h > 0, 0 \leq \sigma < 1, u \in C_{\sigma}[0, \beta] \) and \( v \in C[\beta, \beta] \). Then
\[
I_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds,
\]
\[
I_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^\beta (t-s)^{\alpha-1}v(s)ds
\]
are continuous on \([\beta, \beta+h]\).

Now, we are in the position to state and prove the following continuation results.

**Theorem 4.1** [Continuation Theorem I for IVP (1)].
Assume that the condition \((H)\) is satisfied. Then \( x = x(t), t \in (0, \beta) \) is noncontinuable if and only if for some \( \tau \in (0, \beta/2) \) and any bounded closed subset \( D \subset [\tau, +\infty) \times \mathbb{R} \), there exists a \( t^* \in [\tau, \beta) \) such that \((t^*, x(t^*)) \notin D \).

**Proof.** Suppose that \( x = x(t) \) can be continued. Then there exists a \( \hat{x}(t) \) of (1) defined on \((0, \beta)(\beta < \beta)\) such that \( x(t) = \hat{x}(t) \) for \( t \in (0, \beta) \), which implies \( \lim_{t \to \beta} x(t) = \hat{x}(\beta) \). Define \( x(t) = \hat{x}(\beta) \). Evidently, \( K = \{(t, x(t)) : t \in [\tau, \beta]\} \) is a compact subset of \([\tau, \beta]\) \times \mathbb{R} \). However, there exists no \( t^* \in [\tau, \beta) \) such that \((t^*, x(t^*)) \notin K \). This contradiction implies that \( x(t) \) is noncontinuable.

The proof will be given in two steps.

**Step 1.** We will show that the \( \lim_{t \to \beta-} x(t) \) exists. Set
\[
H(s,t) = \left| \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} - \frac{x_0}{\Gamma(\alpha)} t_0^{\alpha-1} \right|,
\]
and
\[
J(t) = \int_0^\beta (t-s)^{\alpha-1}s^{-\sigma}ds, \quad t \in [2\tau, \beta].
\]
It is easy to see that \( H(s,t) \) and \( J(t) \) are uniformly continuous on \([2\tau, \beta] \times [2\tau, \beta] \) and \([2\tau, \beta] \) respectively.

For \( \forall t_1, t_2 \in [2\tau, \beta], t_1 < t_2 \), we have
\[
| x(t_1) - x(t_2) | \leq H(t_1, t_2) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s))ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s))ds
\]
\[
\leq H(t_1, t_2) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[ ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}) f(s, x(s))ds \right]
\]
\[
+ \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s))ds
\]
\[
\leq H(t_1, t_2) + \frac{B|x|}{\Gamma(\alpha)} \int_0^{t_2} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})s^{-\sigma}ds + \frac{K}{\Gamma(\alpha + 1)} \int_0^{t_2} (t_2-s)^{\alpha-1}ds
\]
\[
= H(t_1, t_2) + |J(t_1) - J(t_2)| \frac{B|x|}{\Gamma(\alpha)} + \frac{K}{\Gamma(\alpha + 1)} \left( (t_2-t_1)^{\alpha} + (t_1-\tau)^{\alpha} - (t_2-\tau)^{\alpha} \right).
\]
It follows from the uniform continuity of $H(s, t)$ and $J(t)$, together with Cauchy convergence criterion, that $\lim_{n \to \infty} x(t) = x^*$ exists.

**Step 2.** We will show that $x(t)$ is continuably.

Since $D$ is a closed subset, we have $\{\beta, x^*\} \in D$. Define $x(\beta) = x^*$. Then $x(t) \in C_{[-a]}[0, \beta]$. We denote $x_1(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1}$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds, \quad t \in [\beta, \beta + 1]$$

and define the operator $S$ as follows:

$$(Sy)(t) = x_1(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))ds, \quad t \in [\beta, \beta + 1],$$

where $y \in C[\beta, \beta + 1]$. In view of Lemmas 2.1 and 4.1, we have $S(C[\beta, \beta + 1]) \subset C[\beta, \beta + 1]$.

Let $D_0 = \{ (t, y) : \beta \leq t \leq \beta + 1, \quad |y| \leq \max_{0 \leq t \leq \beta + 1} |x_1(t)| + b \}$, $b > 0$.

In view of the continuity of $f$ on $D_0$, we could denote $M = \max_{(t, y) \in D_0} |f(t, y)|$.

Let $D_h = \{ y \in C[\beta, \beta + h] : \max_{0 \leq t \leq \beta + h} |y(t) - x_1(t)| \leq b, \quad y(\beta) = x_1(\beta) \}$.

where $h = \min(1, \frac{1}{[0, \Gamma(\alpha+1)]})$. We can claim that $S$ is completely continuous on $D_h$. First, the operator $S$ is continuous. In fact, let $\{y_n\} \subseteq C[\beta, \beta + h]$. $\|y_n - y\|_{[\beta, \beta + h]} \to 0$ as $n \to \infty$, we have

$$\|\langle S y_n \rangle(t) - \langle S y \rangle(t)\| \\
= \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[ f(s, y_n(s)) - f(s, y(s)) \right] ds \right| \leq \frac{Mh^{\alpha}}{\Gamma(\alpha + 1)} \to 0$$

By virtue of the continuity of $f$ on $D_0$, we have $\|f(s, y_n(s)) - f(s, y(s))\|_{[\beta, \beta + h]} \to 0$ as $n \to \infty$. Therefore

$$\|S y_n - S y\|_{[\beta, \beta + h]} \to 0$$

which implies that $S$ is continuous.

Second, we prove that $SD_h$ is equicontinuous.

For any $y \in D_h$, we have $\langle S y \rangle(\beta) = x_1(\beta)$ and $\|\langle S y \rangle(t) - x_1(t)\| = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds$

$$\leq \frac{M(t - \beta)^{\alpha}}{\Gamma(\alpha + 1)} \leq \frac{Mh^{\alpha}}{\Gamma(\alpha + 1)} \leq h$$

thus $SD_h \subset D_h$.

Set $I(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds$.

By Lemma 4.1, we know that $I(t)$ is continuous on $[\beta, \beta + h]$. For $\forall y \in D_h, \beta \leq t_1 < t_2 \leq \beta + h$, we have

$$\|\langle S y \rangle(t_1) - \langle S y \rangle(t_2)\| \leq H(t_1, t_2) + \frac{1}{\Gamma(\alpha)} \int_0^t (t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} f(s, x(s))ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) f(s, y(s))ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, y(s))ds$$
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\[ \leq H(t_1, t_2) + \frac{1}{\Gamma(\alpha)} \left| \int_{0}^{\beta} ((1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1}) f(s, x(s)) ds \right| \]
\[ + M \frac{1}{\Gamma(\alpha)} \left( \int_{0}^{t_1} (t_1 - s)^{\alpha - 1} + (t_2 - s)^{\alpha - 1} ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} ds \right) \]
\[ = H(t_1, t_2) + \left| (t_1 - t)(1 - \beta)^{\alpha} \right| + \left| (t_2 - t)(t_2 - \beta)^{\alpha} \right|. \quad (6) \]

In view of the uniform continuity of \( I(t) \) on \([\beta, \beta + h]\) and (6), we can conclude that \( (S_{y})(t) : y \in D_h \) is equicontinuous.

Therefore, \( S \) is completely continuous. By Schauder fixed point theorem, \( S \) has a fixed point \( \hat{x}(t) \in D_h \) i.e.
\[ \hat{x}(t) = x_1(t) + \frac{1}{\Gamma(\alpha)} \int_{0}^{\beta} (t - s)^{\alpha - 1} f(s, \hat{x}(s)) ds. \]

where \( \hat{x}(t) = \begin{cases} x(t), & t \in [0, \beta], \\ \hat{x}(t), & t \in [\beta, \beta + h]. \end{cases} \)

It follows from Lemma 2.4 that \( \hat{x} \in C_{\alpha}([0, \beta + h]) \) and
\[ \hat{x}(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha - 1} + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha - 1} f(s, \hat{x}(s)) ds. \]

Therefore, in view of Lemma 2.2, \( \hat{x}(t) \) is a solution of (1) on \([0, \beta + h]\). This yields a contradiction (since \( x(t) \) is a noncontinuable). The proof is thus completed. \( \square \)

**Remark 4.1.** Actually, Theorem 4.1 is a natural generalization of Theorem C in [Driver, 1977, p. 275], which is the continuation theorem for the ODE. To see this, IVP (1) is reduced to an ODE if we set \( \alpha = 1 \).

Now, we give another version of continuation theorem, which is more convenient to be applied.

**Theorem 4.2** (Continuation Theorem II for IVP (1)). Assume that the condition \((H)\) is satisfied. Then \( x = x(t) \), \( t \in (0, \beta) \) is noncontinuable if and only if
\[ \lim_{t \rightarrow \beta} \sup |M(t)| = +\infty, \]
where \( M(t) = (t, x(t)), |M(t)| = (t^2 + x^2(t))^\frac{1}{2} \).

**Proof.** "\( \Rightarrow \)" Suppose that \( x = x(t) \) can be continued. Then there exists a solution \( \hat{x}(t) \) of (1) defined on \((0, \beta)(\beta < \hat{\beta}) \) such that \( x(t) = \hat{x}(t) \) for \( t \in (0, \beta) \), which implies \( \lim_{t \rightarrow \beta} x(t) = \hat{x}(\beta) \). Thus, \( |M(t)| \rightarrow |M(\beta)| \) as \( t \rightarrow \beta^+ \), which yields a contradiction. "\( \Leftarrow \)" Suppose that (7) is not true. Then there exists a sequence \( \{t_k\} \) and a positive constant \( K > 0 \) such that
\[ t_k < t_{k+1}, \quad k \in \mathbb{N}, \]
\[ \lim_{k \rightarrow \infty} t_k = \beta, \quad |M(t_k)| \leq K, \quad (8) \]

i.e.
\[ t_k^2 + x^2(t_k) \leq K^2. \]

By virtue of the boundness of \( \{x(t_k)\}, \{x(t_k)\} \) has a convergent subsequence. Without loss of generality, let
\[ \lim_{k \rightarrow \infty} x(t_k) = x^*. \quad (9) \]

We will show that
\[ \lim_{t \rightarrow \beta} x(t) = x^*, \quad (10) \]

i.e. for any given \( \varepsilon > 0 \), there exists \( T \in (0, \beta) \), such that \( |x(t) - x^*| < \varepsilon, \quad t \in (T, \beta) \).

For sufficiently small \( \tau > 0 \), let
\[ D_1 = \left\{ (t, x) : t \in [\tau, \beta], |x| \leq \sup_{t \in [\tau, \beta]} |x(t)| \right\}. \]

In view of the continuity of \( f \) on \( D_1 \), denote \( M = \max_{(t,y) \in D_1} |f(t, y)|. \)

It follows from (8) and (9) that there exists \( k_0 \) such that \( t_{k_0} > \tau \) and for \( k \geq k_0 \), we have
\[ |x(t_k) - x^*| \leq \frac{\varepsilon}{2}. \]

If (10) is not true, then for \( k \geq k_0 \), there exists \( \eta_k \in (t_k, \beta) \) such that \( |x(\eta_k) - x^*| \geq \varepsilon \) and \( |x(t) - x^*| < \varepsilon, \quad t \in (t_k, \eta_k) \). Thus,
\[
\varepsilon \leq |x(t_k) - x^*| \\
\leq |x(t_k) - x^*| + |x(t_k) - x(t_k)| \\
\leq \frac{\varepsilon}{2} + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_k} (t_k - s)^{\alpha - 1} f(s, x(s)) \, ds \right| \\
\leq \frac{\varepsilon}{2} + \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_k} ((t_k - s)^{\alpha - 1} - (t_k - s)^{\alpha - 1}) f(s, x(s)) \, ds \right| \\
+ \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_k} (t_k - s)^{\alpha - 1} f(s, x(s)) \, ds \right| \\
\leq \frac{\varepsilon}{2} + \frac{\| Bx \|_{\gamma, \alpha}}{\Gamma(\alpha)} |J(t_k) - J(y_k)| + \frac{M}{\Gamma(\alpha + 1)} (2(t_k - t_k)^{\alpha} + (t_k - t)^{\alpha} - (t_k - t)^{\alpha}),
\]

where \( J(t) \) is defined in the proof of Theorem 4.1.

In view of the continuity of \( J(t) \) on \([t_k, \beta] \), for sufficiently large \( k \geq k_0 \), we have
\[
\frac{\| Bx \|_{\gamma, \alpha}}{\Gamma(\alpha)} |J(t_k) - J(y_k)| + \frac{M}{\Gamma(\alpha + 1)} (2(t_k - t_k)^{\alpha} + (t_k - t)^{\alpha} - (t_k - t)^{\alpha}) < \frac{\varepsilon}{2},
\]
which implies
\[
\varepsilon \leq |x(t_k) - x^*| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
This contradiction implies that \( \lim_{t \to \beta^-} x(t) \) exists.

By the similar argument to the proof of Theorem 4.1, we can find a continuation of \( x(t) \). The proof is now complete.

Remark 4.2. If \( f \) in IVP (1) satisfies the global Lipschitz condition with the second variable, then its solution globally exists and it is unique.

5. Global Existence Theorems

In this section, based on the results obtained previously, we study the global existence of solutions for the IVP (1).

Applying Theorem 4.2, we can immediately get the following conclusion about the existence of global solution of IVP (1).

Theorem 5.1. Assume that the condition (H) is satisfied. Let \( x(t) \) be a solution of (1) on \([0, \beta] \). If \( x(t) \) is bounded on \([\tau, \beta] \) for some \( \tau > 0 \), then \( \beta = +\infty \).

Example 5.1. Consider the following IVP
\[
\begin{aligned}
\mathbb{R} L_{a, \beta}^\gamma x(t) &= \frac{t^\gamma \exp(-t_x^2 \sin^2 t)}{1 + t}, \\
\mathbb{R} D_{a, \beta}^\gamma x(t)|_{t=0} &= 1,
\end{aligned}
\]
(11)

By Theorem 3.1, we know that (11) has at least a local solution \( x(t) \) on \([0, b] \) for some \( b > 0 \). In view of Lemma 2.2, \( x(t) \) satisfies the following integral equation:
\[
x(t) = \frac{t^\gamma}{\sqrt{\pi}} + \frac{1}{\Gamma(\alpha)} \left( t - s \right)^{-\frac{\gamma}{2}} \frac{\exp(-s \cdot x_2(s) \sin^2 s)}{1 + s} \left( t - s \right)^{\frac{\gamma}{2}} ds.
\]

Thus
\[
|x(t)| \leq \frac{t^\gamma}{\sqrt{\pi}} + \frac{1}{\Gamma(\alpha)} \left( t - s \right)^{-\frac{\gamma}{2}} \frac{\exp(-s \cdot x_2(s) \sin^2 s)}{1 + s} ds.
\]

Suppose that \([0, \beta] \) \( (\beta < +\infty) \) is the maximum existing interval of \( x(t) \). By simple calculation, we can know that for any \( \tau \in (0, \beta) \), \( x(t) \) is bounded on \([\tau, \beta] \). By virtue of Theorem 5.1, we have \( \beta = +\infty \), i.e. the maximum existing interval of \( x(t) \) is \((0, +\infty)\).

In order to continue our discussion, we first present a useful lemma.

Lemma 5.1 [Ye et al., 2007; Henderson & Quahab, 2010]. Let \( v : [0, b] \rightarrow [0, +\infty) \) be a real function, and \( w(t) \) be a non-negative locally integrable function on \([0, b] \). And there exist \( a > 0 \) and \( 0 < c < 1 \), such that
\[
v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t - s)^c} ds.
\]
Then, there exists a constant \( k = k(\alpha) \) such that for \( t \in [0, b] \),
\[
v(t) \leq w(t) + ka \int_0^t \frac{w(s)}{(t-s)^\alpha} ds.
\]

**Theorem 5.2.** Assume that the condition (H) is satisfied and there exist three non-negative continuous functions \( p(t), w(t), q(t) : [0, +\infty) \rightarrow [0, +\infty) \) such that \([f(t, x)] \leq p(t)w(t^{-\alpha}|x|) + q(t)\), where \( w(r) \leq r \) for \( r \geq 0 \). Then the IVP (1) has one solution in \( C_{1,\alpha}(0, +\infty) \).

\[
|t^{1-\alpha}x(t)| = \frac{x_0}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds
\]
\[
\leq \frac{x_0}{\Gamma(\alpha)} + \frac{\beta-\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (p(s)w(s^{1-\alpha}|x(s)|) + q(s)) ds
\]
\[
\leq \frac{x_0}{\Gamma(\alpha)} + \frac{\beta-\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\beta-\alpha) (s^{1-\alpha}|x(s)|) \int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds.
\]

We take \( v(t) = t^{1-\alpha}|x(t)|, w(t) = \frac{\beta-\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q(s) ds, a = \frac{\beta-\alpha}{\Gamma(\alpha)} \). By Lemma 5.1, we know that \( v(t) = t^{1-\alpha}|x(t)| \) is bounded on \([0, \beta]\). Thus, for any \( r \in (0, \beta) \), \( x(t) \) is bounded on \([r, \beta]\). By Theorem 5.1, the IVP (1) has a solution \( x(t) \) on \((0, +\infty)\).

The following results guarantee the existence and uniqueness of global solution of (1) on \( \mathbb{R}^+ \).

**Theorem 5.3.** Assume that the condition (H) is satisfied and there exists a non-negative continuous function \( p(t) \) defined on \([0, \infty)\) such that \([f(t, x) - f(t, z)] \leq p(t)|x-z|\). Then the IVP (1) has a unique solution in \( C_{1,\alpha}(0, +\infty) \).

The existence of global solution can be obtained by an argument similar to the above. From the Lipschitz-type condition and Lemma 5.1, we can conclude the uniqueness of global solution. The proof is omitted here.

6. Conclusions and Comments

In this paper, we firstly derive a new local existence theorem for the general FDE under a weaker condition. Then we establish two continuation theorems which have never been investigated anywhere before. By using these continuation theorems, we study the global existence theorems. The illustrative examples are also displayed to verify the theoretical results.

It should be pointed out that the global existence and uniqueness of the solution to the FDE have been solved. However, the (asymptotic) stability of the solution has not been derived. As far as we know, the global stability for linear systems and finite-time stability (i.e. continuous dependance) for nonlinear systems are available. The global stability for nonlinear systems still remains open. We hope that the studies on the global stability for the nonlinear fractional differential systems will appear elsewhere.

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**References**