Remarks on fractional derivatives

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Abstract

In this paper, we further discuss the properties of three kinds of fractional derivatives: the Grünwald–Letnikov derivative, the Riemann–Liouville derivative and the Caputo derivative. Especially, we compare the Riemann–Liouville derivative with the Caputo derivative. And sequential property of the Caputo derivative is also derived, which is helpful in translating the higher fractional-order differential systems into lower ones. Besides, we also compare the Riemann–Liouville derivative and the Caputo derivative with the classical derivative.

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1. Introduction

In 1695, Leibniz wrote a letter to L’Hôpital and discussed whether or not the meaning of derivatives with integer orders could be generalized to derivatives with non-integer orders. L’Hôpital was somewhat curious about the problem and asked a simple question in reply: “What if the order will be 1/2?” Leibniz in a re-reply letter dated September 30 of the same year, anticipated: “It will lead to a paradox, from which one day useful consequences will be drawn.” The date September 30, 1695 is regarded as the exact birthday of the fractional calculus. In the following centuries, the theories of fractional calculus (fractional derivatives and fractional integrals) underwent a significant and even heated development, primarily contributed by pure, not applied, mathematicians. Along the way it must have entered the minds of more than one enterprising mathematicians that differential equations in which the derivatives were of fractional order were conceivable. Only in the last few decades, however, did applied scientists and engineers realize that such fractional differential equations provided a natural framework for the discussion of various kinds of questions modelled by fractional equations, such as viscoelastic systems, electrode–electrolyte polarization, etc. For details, see [1–3]; and for minor auxiliary examples, see the introductions of [4,5], and see [6].
In this paper, we study some interesting properties of fractional derivatives which are not presented elsewhere. Generally speaking, three kinds of fractional derivatives are often used, the Grünwald–Letnikov derivative denoted by $GL$, the Riemann–Liouville derivative denoted by $RL$, and the Caputo derivative denoted by $C$. If we study an abstract fractional derivative, we write it as $D$, which is each of the above three fractional derivatives. Unlike integer-order derivative, $D^\alpha D^\beta = D^{\alpha + \beta}$ does not hold for positive constants $\alpha$ and $\beta$. But when $\alpha$ and $\beta$ are positive, $D^{-\alpha} D^{-\beta} = D^{\alpha + \beta}$ holds, here $D^{-\alpha}$ is often called a fractional integral with order $\alpha$. A known fact is that $GL = RL = C$ for $\alpha \in \mathbb{R}^+$. The outline of this paper is arranged as follows. In Section 2 some known definitions and results are listed. Several interesting properties of these definitions are discussed in Section 3.

2. Preliminaries

We first introduced the definition of fractional integral.

$Y_x$, the convolution kernel of order $\alpha \in \mathbb{R}^+$, for fractional integrals, is defined by

$$Y_x(t) \equiv \frac{t^\alpha}{\Gamma(\alpha)} \in L^1_{\text{loc}}(\mathbb{R}^+),$$

where $\Gamma$ is the well-known Euler Gamma function, and

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

**Definition 2.1.** The fractional integral (or, the Riemann–Liouville integral) $D_{0,t}^\alpha$ with fractional order $\alpha \in \mathbb{R}^+$ of function $x(t)$ is defined as:

$$D_{0,t}^\alpha x(t) \equiv Y_x * x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) \, d\tau.$$

In this definition, the initial time is set to zero. The following definitions are also done so. The properties of $Y_x$ and $D_{0,t}^\alpha$ can be found in [2,7].

**Property 2.1**

1. The Laplace transform of $Y_x$ is: $\mathcal{L}[Y_x](s) = s^{-\alpha}$ for $\Re(s) > 0$.
2. The convolution property $Y_x * Y_y = Y_{x+y}$ holds for $\alpha > 0$ and $\beta > 0$, this implies that $D_{0,t}^{\alpha} D_{0,t}^{\beta} = D_{0,t}^{\alpha + \beta}$.
3. Consistency property with the integer-order integral: $\lim_{m \to \infty} D_{0,t}^{-m} x(t) = D_{0,t}^{-m} x(t)$, where $\alpha > 0$, $m \in \mathbb{Z}^+$ and

$$D_{0,t}^{-m} x(t) = \int_0^t \int_0^{t_{m-1}} \cdots \int_0^{t_1} x(\tau) \, d\tau \, dt_1 \cdots dt_{m-1} = \frac{1}{\Gamma(m)} \int_0^t (t-\tau)^{m-1} x(\tau) \, d\tau.$$

**Definition 2.2.** The Grünwald–Letnikov fractional derivative with fractional order $\alpha$ is defined by, if $x(t) \in C^m[0,t]$,

$$GL \alpha x(t) = \sum_{k=0}^{m-1} \frac{x^{(k)}(0)}{\Gamma(-\alpha + k + 1)} t^{-\alpha + k} + \frac{1}{\Gamma(m - \alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau,$$

where $m - 1 \leq \alpha < m \in \mathbb{Z}^+$.

The original definition of the Grünwald–Letnikov fractional derivative is given by a limit, i.e.,

$$GL \alpha x(t) = \lim_{h \to 0, nh = t} h^{-\alpha} \sum_{k=0}^{m} (-1)^k \binom{p}{k} x(t - kh).$$

This limit expression is not convenient for use, so Definition 2.2 is often adopted.
Property 2.2
(1) When \(0 < \alpha < 1\), \(\mathcal{L}[(GLD^\alpha_0,x(t))] (s) = s^\alpha X(s)\), where \(X(s) = \mathcal{L}[x](s)\).
(2) If \(\alpha > 1\), the Laplace transform of the Grünwald–Letnikov fractional derivative does not exist in the classical sense. If we take the sense of finite-part integral, then its Laplace transform is \(\mathcal{L}[(GLD^\alpha_0,x(t))] (s) = s^\alpha X(s)\), where \(X(s) = \mathcal{L}[x](s)\).
(3) \(GLD^\alpha_0,c = ct^{-\alpha} / \Gamma(1-\alpha)\), where \(c\) is a constant.

By Definition 2.2, \(x(t)\) should be in \(C^m[0, t]\). From the pure mathematical point of view such a class of functions is narrow. In order to weaken the conditions on the function \(x(t)\), the Riemann–Liouville definition has to be introduced below.

Definition 2.3. The Riemann–Liouville derivative of fractional order \(\alpha\) of function \(x(t)\) is given as
\[
RLD^\alpha_0,x(t) = \frac{d^m}{dt^m} D^{(m-\alpha)}_{0,t} x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} x(\tau) \, d\tau,
\]
where \(m - 1 \leq \alpha < m \in \mathbb{Z}^+\).

From Definitions 2.2 and 2.3, one can see that \(RLD^\alpha_0,x(t) = GLD^\alpha_0,x(t)\) if \(x(t) \in C^m[0, t]\) which can be verified by using integration by parts.

Property 2.3
(1) \(\mathcal{L}[RLD^\alpha_0,x(t)](s) = s^\alpha X(s) - \sum_{k=0}^{m-1} s^{\alpha-k} [RLD^{k-1}_0,x(t)]_{t=0}\), where \(X(s) = \mathcal{L}[x](s)\), \(m - 1 \leq \alpha < m \in \mathbb{Z}^+\).
(2) \(RLD^\alpha_0,c = ct^{-\alpha} / \Gamma(1-\alpha)\), where \(c\) is a constant.

In Property 2.3(1), fractional-order initial value conditions are needed. Although some interpretations for fractional differentiation were given in [8], not all fractional-order initial value conditions have clear practical meanings so are not easy even impossible to measure. In order to overcome this disadvantage, a new definition was given by Caputo.

Definition 2.4. The Caputo derivative of fractional order \(\alpha\) of function \(x(t)\) is defined as
\[
c^D^\alpha_0,x(t) = D^{(m-\alpha)}_{0,t} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau,
\]
in which \(m - 1 < \alpha < m \in \mathbb{Z}^+\).

Property 2.4
(1) \(\mathcal{L}[cD^\alpha_0,x(t)](s) = s^\alpha X(s) - \sum_{k=0}^{m-1} s^{\alpha-k} x^{(k)}(0)\), where \(X(s) = \mathcal{L}[x](s)\), \(m - 1 < \alpha \leq m \in \mathbb{Z}^+\).
(2) \(cD^\alpha_0,c = 0\), where \(c\) is any constant.

The following definition of generalized fractional derivative is also utilized.

Definition 2.5. \(Y_{-\alpha}\) is the generalized function in the sense of Schwartz, as the unique convolutive inverse of \(Y_{\alpha}\) in the convolution algebra \(D'_0(R)\); with the use of the Dirac distribution, which is the neutral element of convolution, this reads: \(Y_{\alpha} * Y_{-\alpha} = \delta\). With the notation, the generalized fractional derivative of order \(\alpha\) of a casual function or distribution is: \(GD^\alpha_0,x(t) \triangleq Y_{-\alpha} * x(t)\).

Property 2.5
(1) The Laplace transform \(Y_{-\alpha}\) is: \(\mathcal{L}[Y_{-\alpha}](s) = s^\alpha \) for \(\Re(s) > 0\).
(2) From the definition of \(Y_{-\alpha}\), we have the convolution property \(Y_{\alpha} * Y_{\beta} = Y_{\alpha+\beta}\) for any real numbers \(\alpha, \beta\), which translates into a sequential property: \(GD^\alpha_0,cGD^\beta_0,x = GD^{\alpha+\beta}_0,x\).
(3) Definition 2.2 does not always coincide with the Riemann–Liouville derivative of fractional order \( \alpha \), it depends on the regularity of the function \( x(t) \) at the origin. For Riemann–Liouville derivative, the fundamental sequentiality property can be violated, moreover it is only a left-inverse of the fractional integral.

(4) For \( 0 < \alpha < 1 \), \( x'(t) \in L^1_{\text{loc}}(R^+) \),

\[
cD_0^\alpha x(t) \triangleq \mathcal{G}D_0^\alpha x(t) - x(0)Y_{1-\alpha} = \int_0^t Y_{1-\alpha}(t - \tau)x'(\tau) \, d\tau.
\]

The above definitions and properties for fractional derivatives can be found in any book on fractional calculus. In the following section, some further properties and comparisons among them are discussed.

3. Further discussions on fractional derivatives

In this section, we discuss the properties of Gru¨nwald–Letnikov derivative, Riemann–Liouville derivative and Caputo derivative. If we assume that function \( x(t) \) is smooth enough, then the Gru¨nwald–Letnikov derivative is equivalent to the Riemann–Liouville derivative. Since we restrict our studies in a class of smooth functions in this paper, we just need to compare Riemann–Liouville derivative with Caputo derivative.

**Theorem 3.1.** If \( x(t) \in C^m[0, T] \) for \( T > 0 \) and \( m - 1 < \alpha < m \in Z^+ \). Then \( cD_0^\alpha x(0) = 0 \).

**Proof.** By the definition of Caputo derivative, one has

\[
cD_0^\alpha x(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - \tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau, \quad t < T.
\]

Put

\[ M = \max_{t \in [0, T]} |x^{(m)}(t)|, \] where \( M \) is a positive constant,

then,

\[
|cD_0^\alpha x(t)| \leq \frac{M}{\Gamma(m - \alpha)} \int_0^t (t - \tau)^{m-\alpha-1} \, d\tau = \frac{M}{\Gamma(m - \alpha + 1)} t^{m-\alpha},
\]

which follows that \( cD_0^\alpha x(0) = 0 \). \( \square \)

**Remark 3.1**

(1) If \( x(t) \in C^0[0, T] \) for \( T > 0 \) and \( \alpha > 0 \), then

\[ D^\alpha_0 x(0) = 0, \quad \text{or} \quad \lim_{t \to 0} \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} x(\tau) \, d\tau = 0. \]

(2) **Theorem 3.1** does not hold for the Riemann–Liouville derivative.

**Theorem 3.2.** If \( x(t) \in C^m[0, \infty) \) and \( m - 1 < \alpha < m \in Z^+ \), then

1. \( cD_0^\alpha x(t) = \mathcal{RL}D^\alpha_0 x(t) = x(t) - \sum_{k=0}^{m-1} \frac{\alpha^k}{k!} x^{(k)}(0) \);
2. \( cD_0^\alpha cD_0^\alpha x(t) = \mathcal{RL}D^\alpha_0 D^\alpha_0 x(t) = x(t) \) holds for \( m = 1 \);
3. \( D^\alpha_0 cD_0^\alpha x(t) = x(t) - \sum_{k=0}^{m-1} \frac{\alpha^k}{k!} x^{(k)}(0) \);
4. \( D^\alpha_0 \mathcal{RL}D_0^\alpha x(t) = x(t) - \sum_{k=1}^{m} \left[ \mathcal{RL}D^\alpha_0 x(t) \right] t^{\alpha-k} \frac{\alpha^{\alpha-k}}{\Gamma(\alpha-k+1)} \);
5. \( \mathcal{RL}D_0^m D^\alpha_0 x(t) = x(t), \ D^m_0 \mathcal{RL}D_0^\alpha x(t) = x(t) - \sum_{k=0}^{m-1} \frac{\alpha^k}{k!} x^{(k)}(0) \).
Proof

(1) Integration by parts and differentiation complete the proof.

(2) The proof of $\mathcal{RL}_t^\alpha D_0^\alpha x(t) = x(t)$ was given in [2], this equation is true for an arbitrary $x \in \mathbb{R}^+$. If $m = 1$, then $D_0^\alpha x(0) = 0$ due to Remark 3.1. Using Theorem 3.2(1), we get $cD_0^\alpha D_0^\alpha x(t) = x(t)$.

The proofs of (3)–(5) can be referred to [2]. The proof is completed.

Next, we discuss the lower-terminal and upper-terminal properties of the fractional order of the Riemann–Liouville derivative and Caputo derivative.

For the Riemann–Liouville derivative with order $\alpha$, where $m - 1 \leq \alpha < m \in \mathbb{Z}^+$,

$$\mathcal{RL}_t^\alpha D_0^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} x(\tau) \, d\tau,$$

one has

$$\lim_{\alpha \rightarrow (m-1)^+} \mathcal{RL}_t^\alpha D_0^\alpha x(t) = \lim_{\alpha \rightarrow (m-1)^+} \left( \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} x(\tau) \, d\tau \right)$$

$$= \lim_{\alpha \rightarrow (m-1)^+} \left( \frac{\sum_{k=0}^{m-1} x^{(k)}(0) \Gamma(-\alpha + k + 1)}{\Gamma(-\alpha + k + 1)} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau \right)$$

$$= x^{(m-1)}(0) + \int_0^t x^{(m)}(\tau) \, d\tau = \frac{d^{m-1}x(t)}{dt^{m-1}},$$

and

$$\lim_{\alpha \rightarrow m^-} \mathcal{RL}_t^\alpha D_0^\alpha x(t) = \lim_{\alpha \rightarrow m^-} \left( \frac{\sum_{k=0}^{m-1} x^{(k)}(0) \Gamma(-\alpha + k + 1)}{\Gamma(-\alpha + k + 1)} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau \right)$$

$$= x^{(m)}(0) + \int_0^t x^{(m+1)}(\tau) \, d\tau = \frac{d^m x(t)}{dt^m},$$

in which $\Gamma(1) = 1$, $\Gamma(0) = \infty$ and $\Gamma(-k) = \infty$ for $k \in \mathbb{Z}^+$ are used.

From the above discussions, $\mathcal{RL}_t^\alpha D_0^\alpha$ is a “bridge” between $d^{m-1}/dt^{m-1}$ and $d^m/dt^m$. So the Riemann–Liouville derivative is a reasonable generalization of the classical derivative. If $x(t)$ is defined in a suitable function space $X$, then the dual space of $X$ is defined by $X^+ = \{ \mathcal{RL}_t^\alpha D_0^\alpha : x \in \mathbb{R}^+ \} \cup \{ \text{the unity } 1 \}$, then $\mathcal{RL}_t^\alpha D_0^\alpha$ is a continuous linear operator with respect to $x$. We call this property fractional-order consistency of the Riemann–Liouville derivative. See Fig. 1.

For the Caputo derivative $cD_0^\alpha D_0^\alpha$ with order $\alpha$, where $m - 1 \leq \alpha < m \in \mathbb{Z}^+$,

$$cD_0^\alpha D_0^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau,$$

one gets

$$\lim_{\alpha \rightarrow (m-1)^+} cD_0^\alpha D_0^\alpha x(t) = \lim_{\alpha \rightarrow (m-1)^+} \left( \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) \, d\tau \right) = \int_0^t x^{(m)}(\tau) \, d\tau = x^{(m-1)}(t) - x^{(m-1)}(0),$$

with order $\alpha$, where $m - 1 \leq \alpha < m \in \mathbb{Z}^+$,
For the second equation, \( x(t) \) should have \((m + 1)\)th order derivative; the property regarding this equation is called fractional-order upper consistency of the Caputo derivative for \( m - 1 < \alpha < m \in \mathbb{Z}^+ \). Obviously, \( C D_{0+}^\alpha \) does not have fractional-order lower consistency of the Caputo derivative for \( m - 1 < \alpha < m \in \mathbb{Z}^+ \). See Fig. 1.

From Fig. 1, if \( \alpha < 0 \), then \( RL D_{0+}^\alpha = C D_{0+}^\alpha \).

In the following we discuss the sequential properties of the Caputo derivative. In general, for \( \alpha, \beta \in \mathbb{R}^+ \), each of

\[
RL D_{0+}^{\alpha} RL D_{0+}^{\beta} = RL D_{0+}^{\alpha+\beta}, \quad RL D_{0+}^{\alpha} RL D_{0+}^{\beta} = RL D_{0+}^{\alpha+\beta} = RL D_{0+}^{\alpha+\beta}, \quad RL D_{0+}^{\alpha} RL D_{0+}^{\beta} = RL D_{0+}^{\alpha+\beta}
\]

is not true. This can be seen from a simple example. Choose a constant \( c \neq 0 \), one can easily get

\[
RL D_{0+}^{1/2} RL D_{0+}^{1/2} = 0, \quad RL D_{0+}^{1/2} RL D_{0+}^{1/2} = c, \quad RL D_{0+}^{3/2} RL D_{0+}^{3/2} = c
\]

Under suitable assumptions of fractional orders, there have existed some studies, see [2,9–13].

In the following, we only consider the Caputo derivative.

**Theorem 3.3.** If \( x(t) \in C^1[0, T] \) for some \( T > 0 \), then

\[
c D_{0+}^{\alpha_1} c D_{0+}^{\alpha_2} x(t) = c D_{0+}^{\alpha_1} x(t) = c D_{0+}^{\alpha_1+\alpha_2} x(t), \quad t \in [0, T],
\]

where \( \alpha_1, \alpha_2 \in \mathbb{R}^+ \) and \( \alpha_1 + \alpha_2 \leq 1 \).
Proof. When $x_1 + x_2 < 1$, using Property 2.5(4) and Theorem 3.1 leads to
\[ cD_{0,t}^{x_1}cD_{0,t}^{x_2}x(t) = cD_{0,t}^{x_2}(cD_{0,t}^{x_1}x(t) - x(0)Y_{1-x_1}) = cD_{0,t}^{x_2}(cD_{0,t}^{x_1}x(t) - x(0)Y_{1-x_1}) - cD_{0,t}^{x_1}x(0)Y_{1-x_2} = cD_{0,t}^{x_1+x_2}x(t) - x(0)Y_{1-x_1-x_2} = cD_{0,t}^{x_1+x_2}x(t). \]

When $x_1 + x_2 = 1$, by using Theorem 3.2(2), we have
\[ cD_{0,t}^{x_1}cD_{0,t}^{x_2}x(t) = cD_{0,t}^{x_1}cD_{0,t}^{x_2}x'(t) = x'(t) = cD_{0,t}^{x_1+x_2}x(t). \]

The proof is completed. \(\square\)

The special case of Theorem 3.3, i.e., \(0 < x_1 = x_2 < 0.5\), was indirectly proved and applied, see [14]. In this theorem, the condition "\(x_1 + x_2 \leq 1\)" cannot be removed. See a simple counterexample below:
\[ cD_{0,t}^{0.6}cD_{0,t}^{0.5}t = \frac{1}{T(0.9)}t^{-0.1}, \quad cD_{0,t}^{1.1}t = 0. \]

Theorem 3.4. If \(x(t) \in C^m[0, T]\) for \(T > 0\), then
\[ cD_{0,t}^{x}x(t) = cD_{0,t}^{x_1} \cdots cD_{0,t}^{x_n}cD_{0,t}^{x_{n+1}}x(t), \quad t \in [0, T], \]
where \(x = \sum_{i=1}^{n} x_i, x_i \in (0, 1], m - 1 \leq x < m \in Z^+\) and there exist \(i_k < n\) s.t. \(\sum_{k=1}^{i_k} x_i = k\), and \(k = 1, 2, \ldots, m - 1\).

Proof. Applying Theorem 3.3 and \(x_n + x_{n-1} + \cdots + x_{n-1+1} < 1\), one obtains
\[ cD_{0,t}^{x_1} \cdots cD_{0,t}^{x_1}x(t) = x'(t), \]
\[ cD_{0,t}^{x_2} \cdots cD_{0,t}^{x_{n+1-1}}x'(t) = x'(t), \]
\[ \cdots \]
\[ cD_{0,t}^{2} \cdots cD_{0,t}^{2}x^{(m-2)}(t) = x^{(m-1)}(t), \]
\[ cD_{0,t}^{x} \cdots cD_{0,t}^{x}x^{(m-1)}(t) = cD_{0,t}^{x}x^{(m-1)}(t) = D_{0,t}^{x-1}x^{(m)}(t) = cD_{0,t}^{x}x(t). \]

This ends the proof. \(\square\)

According to Theorem 3.4, higher fractional-order differential systems can be reduced to lower fractional-order differential systems.

At last, we compare two initial value problems.
\[
\left\{ \begin{array}{ll}
\text{RL}D_{0,t}^{\alpha}x(t) = f(x(t), t), & m - 1 < \alpha < m \in Z^+, \ t > 0, \\
\left[ \text{RL}D_{0,t}^{\alpha-k}x(t) \right]_{t=0} = x_0^k, & k = 1, 2, \ldots, m,
\end{array} \right. \tag{3.1}
\]

and
\[
\left\{ \begin{array}{ll}
cD_{0,t}^{\alpha}x(t) = f(x(t), t), & m - 1 < \alpha < m \in Z^+, \ t > 0, \\
x^{(k)}(0) = x_0^k, & k = 0, 1, \ldots, m - 1.
\end{array} \right. \tag{3.2}
\]

A feature of classical/fractional differential equations is that one needs to specify initial value conditions to make sure that the solution is unique [15]. In many situations these initial value conditions describe certain properties of the solution at the beginning of the process. If we study the initial value problem (3.2), we need to determine the initial values \(x(0), x'(0), \ldots, x^{(m-1)}(0)\). These initial values in application have clear physical meanings. For example, if \(x(t)\) indicates displacement, then \(x'(t)\) stands for speed and \(x''(t)\) expresses acceleration. It should be noted that initial value problem (3.1) is mathematically rigorous and elegant. But in application, it is often difficult even impossible to measure the fractional-order initial values if such dynamical process does not have clear physical background or has no physical meaning. So the Caputo fractional-order systems are often used in modelling and analysis [4–6,15–29].
In [30], Podlubny gave some explanations for fractional derivatives. In [8], Heymans and Podlubny further interpreted fractional-order initial values. If suitable fractional-order initial values can be reasonably expounded and numerically determined in application, to study problem (3.1) is quite useful. So we can simultaneously study models (3.1) and (3.2) in order to understand the actual model as accurately and genuinely as possible.

References