DETERMINISTIC ASSET LIABILITY MANAGEMENT WITH GEOMETRIC INTERPRETATIONS

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Abstract
We discuss deterministic asset liability management methods for insurance business and consider a few hedging strategies for the interest rate risks by approximating the changes in surplus according to the changes in interest rates. We show that the first and the second derivative matching would be enough for a perfect immunization strategy for the deterministic cash flows of assets and liabilities using the fundamental theorem of plane curves. Also we suggest a simple quadratic programming method which can be used for the deterministic cash flow matching problem with reinvestment and borrowing.

Keywords: asset liability management, interest rate risks, immunization strategy, cash flow matching.

I. Introduction
Financial institutions such as depository institutions, insurance companies, and pension funds have faced the increased volatilities in interest rates over the past two decades. They should understand the characteristics of interest rate risk, make proper decisions on the management of assets and liabilities, and set up a portfolio of assets that matches the liabilities to eliminate or minimize the financial risks from the combined asset and liability portfolio. To provide safe strategies on asset and liability management they need to be well aware of assets and the functioning of financial markets as well as specific liability behaviors. And they have to explore and develop appropriate tools, techniques, and mathematical models that may be useful in managing assets and liabilities to cope with the potential risks. Financial managers should measure the risks they are facing and set up models and management techniques to meet their financial objectives under reasonable assumptions. They should be familiar with assets and liabilities and their interrelationship. Understanding the relationship of the company’s assets and liabilities is important to enhance its profitability as well as to deal with the inherent risks.

Asset and liability management (ALM) can be defined as the ongoing process
of formulating, implementing, monitoring, and revising strategies related to assets and liabilities in attempt to achieve financial objectives for a given set of risk tolerances and constraints. With proper ALM strategies, the portfolios of assets can be composed, updated, expanded, replaced, and redesigned to enhance the performance of assets and to fund the future liabilities. In this paper we consider a few deterministic ALM strategies to hedge interest rate risks in insurance business.

II. Interest Rate Risk

Interest rate risk is called C-3 risk, which is coined by C.L. Trowbridge (when he was Chairman of the Society of Actuaries Committee on Valuation and Related Matters) to denote the risk of losses due to changes in interest: changes in either the level of interest rates or the shape of the yield curves. The letter “C” stands for “contingency”. C-1 risk is the risk of asset defaults and decreases in market values of equity investments. Mortality and morbidity risk is called C-2 risk, it is the risk of losses from increases in claims and from pricing deficiencies, other than those from C-1 and C-3 risks. C-4 risk is accounting, managerial, social and regulatory risks.

For a given block of an insurance business, let us denote \( A_t \) to be the asset cash flow expected during the \( t \)-th year,

\[ A_t = \text{investment income} + \text{capital maturities}, \]

and \( L_t \) to be the liability cash flow expected during the \( t \)-th year,

\[ L_t = \text{policy claims} + \text{policy surrenders} + \text{expenses} - \text{premium income}. \]

Let us denote \( N_t \) to be the net cash flow,

\[ N_t = A_t - L_t. \]

Two kinds of risk may occur when interest rates change:

- Future positive net cash flows may have to be reinvested at a lower interest rate. This risk is called as a reinvestment risk.
- Future negative net cash flows may involve the liquidations of assets at depreciated values because of then higher interest rate. This risk is called as disinvestment risk or price risk.

1. Asset Liability Management Techniques

Asset liability management (ALM) problems are related to many different areas such as immunization with duration matching and convexity or higher order matching, cash-flow matching with optimization theory and computer programming, and risk management. Usually ALM techniques can be classified as immunization (or duration
matching technique) and cash-flow matching (or dedication technique) according to the hedging methodology.

The term “immunization” was first mentioned by a British actuary, Frank M. Redington (1952). Macaulay (1938) used the term “duration”, which Redington rediscovered by “mean term” with the same concept. Immunization means constructing an immunizing (or hedging) asset portfolio whose value sensitivity to changes in interest rates matches liability value sensitivity to interest rate changes.

Cash-flow matching was suggested by Koopmans (1942). Cash-flow matching focuses on constructing the cheapest portfolio of assets such that the accumulated net cash flows are nonnegative. Linear programming or using massive parallel computer systems are important techniques in cash-flow matching.

An ALM problem can be categorized as a deterministic or a stochastic one. For deterministic ALM problems, the liability and the asset cash flows of an insurance business are known to be certain and independent of interest rates. Stochastic ALM problems deal with the uncertainty of asset or liability cash flow streams which may depend on the level of interest rates.

Now we summarize ALM problems by four types; deterministic duration matching technique, deterministic cash-flow matching technique, stochastic duration matching technique, and stochastic cash-flow matching technique. In this paper we discuss only the deterministic ALM methods; deterministic duration matching technique, and deterministic cash-flow matching technique. We will consider the stochastic ALM methods in the future.

2. Deterministic Duration Matching Technique

Duration matching technique is a strategy of constructing an asset portfolio such that the value of assets should be equal to that of liabilities and the duration of the assets matches the duration of the liabilities. In order to consider matching the duration of asset and liability under the deterministic environment, i.e. the yield curves are flat and asset and liability cash flows are independent of interest rates, we first define surplus \( S(i) \) to be the present value of net cash flows, \( \{N_t, t = 1,2,3,...\} \).

For a given interest rate \( i \),

\[
S(i) = \frac{N_1}{1+i} + \frac{N_2}{(1+i)^2} + \frac{N_3}{(1+i)^3} + \ldots \]

\[
= vN_1 + v^2N_2 + v^3N_3 + \ldots
\]
\[ = \sum_t v^t N_t, \]

where \( v = \frac{1}{1 + i}. \)

For a small change in interest rates, \( \Delta i \), using Taylor series, we can express the changes in surplus as below,

\[
S(i + \Delta i) = S(i) + \sum_{k=1}^{\infty} \frac{1}{k!} S^{(k)}(i)\Delta i^k
\]

\[
\approx S(i) + S'(i)\Delta i + \frac{1}{2} S''(i)(\Delta i)^2. \tag{1}
\]

Note that, in the above formula, we only consider the first and the second order derivatives as an approximation. We will justify this approximation in the later section with a geometric interpretation. The first order derivative of surplus function is calculated as

\[
S'(i) = \frac{d}{di} \sum_t v^t N_t = \sum_t \frac{d}{di} (v^t N_t)
\]

\[
= \sum_t \{ (\frac{d}{di} v^t) N_t + v^t (\frac{d}{di} N_t) \}
\]

\[
= \sum_t \{ (tv^{t+1}) N_t + 0 \}
\]

\[
= \sum_t tv^t (L_t - A_t)
\]

\[
= v \left( \sum_t tv^t L_t \sum v^t L_t - \sum v^t A_t \sum v^t A_t \right).
\]

The second order derivative of surplus function is

\[
S''(i) = v^2 \sum t(t+1)v^t (A_t - L_t).
\]

If we do not consider the second order derivative and construct an asset and liability portfolio such that

\[
S'(i) = 0 \tag{2}
\]

then we have

\[
S(i + \Delta i) \approx S(i),
\]

which means the surplus would not change for any small interest rate fluctuations, i.e. it is immunized against interest rate changes.

(2) is equivalent to
\[ \sum_i t v' A_i = \sum_i t v' L_i . \]  

(3)

If we consider the second order derivative and construct an asset and liability portfolio which satisfies (2) and

\[ S^*(i) > 0 \]

or equivalently,

\[ \sum_i t^2 v' A_i > \sum_i t^2 v' L_i , \]

then we have

\[ S(i + \Delta i) > S(i) , \]

which means the surplus increases with small interest rate changes, i.e. there exists a “free lunch”, so called “second derivative profit”. This free lunch comes from the assumption that the yield curves are flat. We state a few problems with deterministic duration matching as follows\(^1\).

- It may be difficult to project the cash flows \( \{ A_i \} \) and \( \{ L_i \} \) accurately.
- Cash flows may not be independent of interest rate fluctuations. There may exist call and put options. When interest rates fall, bonds may be called and mortgages are paid back earlier. When interest rates rise, the customers may surrender their policies and invest their money elsewhere for higher rates of return.
- Yield curves are not flat. If yield curves were always flat, there would be free lunches or second derivative profits.
- Interest rate shock may not be small. In applying the Taylor expansion, \( \Delta i \) is assumed to be small.
- This method is not a buy-and-hold strategy. It requires continuous rebalancing to re-equate the durations and market values of the assets and liabilities. But liquid assets normally do not yield as much as private placement bonds and mortgages. As the market is not frictionless, there will be transaction costs (bid-ask spread).

\(^1\) There are a few research studies to overcome the problems. For more details, we may refer to the papers by Shiu(1988, 1990), Weil (1973), Fisher (1980), Bierwag, Kaufman, and Toevs, (1982), Christensen, Fabozzi, and LoFaso (1995), and the books by Kaufman, Bierwag, and Toevs, editors (1983), Bierwag (1987), and Granito (1984).
• Duration drift may happen, that is durations of assets and liabilities may shorten at different speed as time passes. It may be needed to rebalance even if the interest rate does not change.
• Default risk may exist when different classes of assets are discounted by different yield curves, resulting in a shortfall of asset value to fund liabilities.

3. Deterministic Cash-flow Matching Technique

Deterministic cash-flow matching (or dedication) technique is one of the important and popular strategies in asset liability management (ALM). This strategy is to construct an optimal (or least-cost) investment portfolio of non callable and default-free fixed-income securities such that the accumulated net cash flows are non-negative for all periods in the planning horizon. Such a strategy guarantees that the cash flow of the initial investment portfolio with minimum cost will meet the projected requirements of liabilities regardless of interest rate movements, without any future transactions. This is a buy-and-hold strategy and does not require rebalancing transactions so it is less costly over its lifetime once an optimal portfolio is constructed. Note that higher yield fixed-income securities such as private placement bonds and mortgages are usually illiquid. The decision-maker needs only to know the prices of the fixed-income securities available in the market place and their future cash flows. The decision-maker does not need to worry about the term structure of interest rates, duration, convexity, and so on.

The first step of cash-flow matching technique is to determine the schedule of liabilities to be funded. After liability cash flows are determined the next step is to set portfolio constraints on sector, quality issuer, lot size, and so on. The reinvestment rates for surplus funds and borrowing rates need to be determined. An optimal (or least-cost) portfolio will be identified to meet the fixed liability payment stream by a proper method such as stepwise solutions, linear programming, and integer programming. We categorize the mathematical formulations of this strategy as pure cash-flow matching technique and cash-flow matching with reinvestment and borrowing.

(1) Pure cash-flow matching technique

Pure cash-flow matching is the most conservative portfolio dedication technique. It requires the asset cash flow to be equal to the liability cash flow for each and every period in the planning horizon. The equality constraints of asset and liability cash flows can be relaxed by a set of constraints such that the asset cash flow should be greater than or equal to the liability cash flow for all periods.
Let us denote $x_j$ to be the dollar amount to be invested in the $j$-th security. We want to determine a set \{x\} of investment portfolio with minimum cost such that its cash flow will be greater than or equal to that of liabilities at every time period. We assume that short sales are not allowed i.e. $x_j$ is positive for all $j$. Let $A_{ij}$ be the cash flow in period $t$ from an initial investment of $1$ in the $j$-th security payable at the end of period $t$. The asset cash flow $A_t$ expected during the $t$-th year is,

$$ A_t = \sum_j x_j A_{ij}. $$

The net cash flow $N_t$ is,

$$ N_t = A_t - L_t = \sum_j x_j A_{ij} - L_t, $$

where $L_t$ is the liability outflow due at the end of period $t$.

The cash-flow matching problem is to find the optimal \{x\},

Minimize $\sum_j x_j$ \hspace{1cm}$\text{(5)}$

under the constraints

$$ N_t = \sum_j x_j A_{ij} - L_t \geq 0. \quad \text{(6)} $$

The solution to the above problem can be obtained by a linear programming.

(2) Cash-flow matching with reinvestment and borrowing

An excess surplus cash can be reinvested at a reinvestment rate (or lending rate) $l_t$ and deficit funding amounts should be borrowed at a borrowing rate (or financing rate) $b_t$ during a period $t$. Let us denote $V_t$ to be the cash valance at time $t$, which is the forward accumulation of net cash flows with interest up to time $t$. The model formulation is as follows.

$$ V_1 = N_1. $$

If $V_1 \geq 0$ then $V_2 = (1 + l_2)V_1 + N_2$,

else if $V_1 \leq 0$ then $V_2 = (1 + b_2)V_1 + N_2$.

If $V_2 \geq 0$ then $V_3 = (1 + l_3)V_2 + N_3$,

else if $V_2 \leq 0$ then $V_3 = (1 + b_3)V_2 + N_3$. 

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We consider two types of problem, the first one is to minimize the cost of asset portfolio and the second one is to maximize the value of the final balance at maturity \( m \).

Problem 1.

\[
\text{Minimize } m \sum_{j} x_j
\]

subject to

\[
V_m \geq 0.
\]  

Problem 2.

\[
\text{Maximize } V_m
\]

subject to

\[
\sum_{j} x_j \leq \text{Available Investment Amount}.
\]

Note that these problems are not linear. We can linearize these as follows.

If \( V_{j-1} \geq 0 \) then \( V_j = (1 + l_j) V_{j-1} + N_j \),

else if \( V_{j-1} \leq 0 \) then \( V_j = (1 + b_j) V_{j-1} + N_j \).  

Let us define

\[
V_j^+ = \max(V_j, 0)
\]

and

\[
V_j^- = -\min(V_j, 0),
\]

then we have

\[
V_j = V_j^+ - V_j^-,
\]

with

\[
V_j^+ \geq 0 \text{ and } V_j^- \geq 0.
\]

Now (11) can be written as

\[
V_j = V_j^+ - V_j^-.
\]
\[(1 + l_j) V_{j-1}^+ - (1 + b_j) V_{j-1}^- + N_j.\]

And we have

\[V_1^+ - V_1^- = N_1,\]

\[V_2^+ - V_2^- = (1 + l_2) V_1^+ - (1 + b_2) V_1^- + N_2,\]

\[\vdots\]

\[V_j^+ - V_j^- = (1 + l_j) V_{j-1}^+ - (1 + b_j) V_{j-1}^- + N_j,\]

\[\vdots\]

\[V_{m-1}^+ - V_{m-1}^- = (1 + l_{m-1}) V_{m-2}^+ - (1 + b_{m-1}) V_{m-2}^- + N_{m-1},\]

\[V_m = (1 + l_m) V_{m-1}^+ - (1 + b_m) V_{m-1}^- + N_m.\]

Now we have a linear programming problem as follows.

Problem 1.

Minimize \[\sum_j x_j\] \hspace{1cm} (12)

subject to

\[V_m \geq 0.\] \hspace{1cm} (13)

Problem 2.

Maximize \[V_m\] \hspace{1cm} (14)

subject to

\[\sum_j x_j \leq \text{Available Investment Amount}.\] \hspace{1cm} (15)

Even though the deterministic cash-flow matching (or dedication) technique is one of the important and popular strategies in ALM because of its simplicity with linear program, it has a few problems as follows.\(^2\)

Cash flows of assets or liabilities may not be fixed in advance but vary depending on interest rates of other financial rates, for example, mortgage backed securities (MBS) or single premium deferred annuity (SPDA).

There may be many constraints on the structure of asset portfolios.

It may need re-optimization on a periodic basis according to the realized rate environment. Last year’s optimized portfolio may not be the optimal one any more this year.

It may need a massive computer programming, money manager or dealer firms to construct or re-optimize the portfolio.

It can require relatively high initial funding to meet the constraints on cash flow timing and to take into account of the conservative reinvestment rate.

(3) Quadratic programming

It is true that linear programming is one of the most widely used optimization model. But sometimes non-linear programming is useful or necessary in financial applications. If the objective functions or the constraints are not so complicated to handle then non-linear programming may be a valuable method. For example, we can simplify the algorithm of cash-flow matching with reinvestment and borrowing mentioned above using relatively simple quadratic constraints as follows.

Problem 1.

\[
\text{Minimize} \sum_{j} x_j \\
\text{subject to} \\
V_t = y_t + z_t, \\
y_t \geq 0, \\
z_t \leq 0, \\
y_t, z_t = 0, \\
V_{t+1} = (1 + l_{t+1}) y_t + (1 + b_{t+1}) z_t + N_{t+1}, \\
V_m \geq 0.
\]

Problem 2.

\[
\text{Maximize} V_m \\
\text{subject to}
\]
\[ \sum_{j} x_j \leq \text{Available Investment Amount}, \]
\[ V_t = y_t + z_t, \]
\[ y_t \geq 0, \]
\[ z_t \leq 0, \]
\[ y_t, z_t = 0, \]
\[ V_{t+1} = (1 + l_{t+1}) y_t + (1 + b_{t+1}) z_t + N_{t+1}. \]

III. Geometric Interpretation of the Immunization Theory

1. Basic Concepts of Plane Curves
We state a few basic concepts of plane curves with a number of definitions, lemmas, and theorems.

**Definition 1** Let \( \alpha : (a,b) \rightarrow \mathbb{R}^n \) be a function, where \( (a,b) \) is an open interval in \( \mathbb{R} \) and \( \mathbb{R}^n \) is a Euclidean n-space. We write
\[ \alpha (t) = (a_1(t), \ldots, a_n(t)), \]
where each \( a_j \) is an ordinary real-valued function of a real variable. We say that \( \alpha \) is differentiable if each \( a_j \) is differentiable. Similarly \( \alpha \) is piecewise-differentiable if each \( a_j \) is piecewise-differentiable.

**Definition 2** A curve in \( \mathbb{R}^n \) is a piecewise-differentiable function
\[ \alpha : (a,b) \rightarrow \mathbb{R}^n, \]
where \( (a,b) \) is an open interval in \( \mathbb{R} \). If \( I \) is any subset of \( \mathbb{R} \), we say that
\[ \alpha : I \rightarrow \mathbb{R}^n \]
is a curve if there is an open interval \( (a,b) \) containing \( I \) such that \( \alpha \) can be extended as a piecewise-differentiable function from \( (a,b) \) into \( \mathbb{R}^n. \)

**Definition 3** A curve \( \alpha : (a,b) \rightarrow \mathbb{R}^n \) is said to be regular if it is differentiable and its velocity \( \alpha'(t) \) is everywhere defined and nonzero. If \( \| \alpha'(t) \| = 1 \) then it is said to have unit-speed.

**Definition 4** Let \( \alpha : (a,b) \rightarrow \mathbb{R}^n \) be a curve. A vector field along \( \alpha \) is a function \( Y \)
that assigns to each $t$ with $a < t < b$ a vector $Y(t)$ at the point $\alpha(t)$.

If $n=2$ and $X$ is a vector field along a curve with $X(t) = (x(t), y(t))$, then another vector field $JX$ can be defined by $JX(t) = (-y(t), x(t))$.

**Definition 5** Let $\alpha: (a, b) \to \mathbb{R}^n$ be a regular curve. The curvature $\kappa[\alpha]$ of $\alpha$ is given by the formula

$$\kappa[\alpha](t) = \frac{\alpha''(t) \cdot J\alpha'(t)}{||\alpha'(t)||^3}. \quad (16)$$

**Theorem 1** If $\alpha: (a, b) \to \mathbb{R}^2$ is a regular curve with $\alpha(t) = (x(t), y(t))$, then the curvature $\kappa[\alpha]$ of $\alpha$ is given by

$$\kappa[\alpha](t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'^2(t) + y'^2(t))^{3/2}}. \quad (17)$$

**Proof.** We have $\alpha''(t) = (x''(t), y''(t))$ and $J\alpha'(t) = (-y'(t), x'(t))$, so that

$$\kappa[\alpha](t) = \frac{(x''(t), y''(t)) \cdot (-y'(t), x'(t))}{(x'^2(t) + y'^2(t))^{3/2}}$$

$$= \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'^2(t) + y'^2(t))^{3/2}}. \quad \square$$

It is useful to have some pictures to understand the meaning of positive and negative curvatures for plane curves.

![Positive and negative curvatures](image)

<table>
<thead>
<tr>
<th>$\kappa[\alpha] &gt; 0$</th>
<th>$\kappa[\alpha] &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The parabola $t \to (t, t^2)$</td>
<td>The parabola $t \to (t, -t^2)$</td>
</tr>
</tbody>
</table>

*<Figure 1> Positive and negative curvatures*

Intuitively, it is clear that velocity vector of a highly curved plane curve changes rapidly. This idea can be made precise mathematically. Let us define the turning angle $\theta[\alpha]$ of a curve $\alpha: (a, b) \to \mathbb{R}^2$. We state a lemma without proof.
**Lemma 1** Let $\alpha: (a,b) \to \mathbb{R}^2$ be a regular curve and fix $t_0$ with $a < t_0 < b$. Let $\theta_0$ be a number such that
\[
\frac{\alpha'(t_0)}{||\alpha'(t_0)||} = (\cos \theta_0, \sin \theta_0).
\] (18)

Then there exists a unique differentiable function $\theta[\alpha]: (a,b) \to \mathbb{R}$ such that $\theta[\alpha](t_0) = \theta_0$ and
\[
\frac{\alpha'(t)}{||\alpha'(t)||} = (\cos \theta[\alpha](t), \sin \theta[\alpha](t)),
\] (19)
for $a < t < b$. We call $\theta[\alpha]$ the turning angle. □

Geometrically, the turning angle $\theta[\alpha](t)$ is the angle between the horizontal and $\alpha'(t)$.

![Figure 2> Turning angle](image)

Now let us derive a relationship between the turning angle and the curvature of a plane curve.

**Lemma 2** The turning angle and curvature of a regular curve $\alpha$ in the plane are related by
\[
\theta'[\alpha](t) = ||\alpha'(t)|| \kappa[\alpha](t).
\] (20)

**Proof.** From (19), the derivative of
\[
\frac{\alpha'(t)}{||\alpha'(t)||}
\] is
\[
\frac{\alpha''(t)}{||\alpha'(t)||} + \alpha'(t) \frac{d}{dt} \left( \frac{1}{||\alpha'(t)||} \right). \quad (21)
\]
And using the chain rule, the derivative of \((\cos \theta(\alpha)(t), \sin \theta(\alpha)(t))\) is
\[
\theta'(\alpha)(t) = \frac{\theta'(\alpha)(t) J\alpha'(t)}{\| J\alpha'(t) \|}.
\] (22)

Setting (22) equal to (21) and taking the dot product with \( J\alpha'(t) \), we obtain
\[
\theta'(\alpha)(t) \| \alpha'(t) \| = \frac{\alpha''(t) \cdot J\alpha'(t)}{\| \alpha'(t) \|} = \| \alpha'(t) \|^2 \kappa(\alpha)(t),
\] (23)
divided by \( \| \alpha'(t) \| \), we have
\[
\theta'(\alpha)(t) = \| \alpha'(t) \| \kappa(\alpha)(t).
\]

Now we have the meaning of the curvature of a plane curve geometrically.

**Theorem 2** The turning angle and curvature of a unit-speed curve \( \alpha \) in the plane are related by
\[
\kappa(\alpha)(t) = \frac{d\theta(\alpha)(t)}{dt}.
\] (24)

Thus changes in the turning angle of a curve are measured by its curvature.

Now let us examine how much the curvature determines a plane curve. It is intuitively clear that the image of a plane curve \( \alpha \) under a rotation or a translation of \( \mathbb{R}^2 \) has the same curvature as \( \alpha \). Rotations and translations are examples of Euclidean motions of \( \mathbb{R}^2 \), those maps of \( \mathbb{R}^2 \) into itself which do nor distort distances. We first discuss Euclidean motions of \( \mathbb{R}^2 \) in general. And we state the Fundamental Theorem of Plane Curves: two unit-speed curves in \( \mathbb{R}^2 \) that have the same curvature differ only by a Euclidean motion of \( \mathbb{R}^2 \).

**Definition 6** Let \( A : \mathbb{R}^n \to \mathbb{R}^n \) be a nonsingular linear map.

1. \( A \) is **orientation-preserving** if \( \det(A) \) is positive, or **orientation-reserving** if \( \det(A) \) is negative.
2. \( A \) is called an **orthogonal transformation** if
   \[
   Ap \cdot Aq = p \cdot q
   \]
   for all \( p, q \in \mathbb{R}^n \).
3. A **rotation** of \( \mathbb{R}^n \) is an orientation-preserving orthogonal transformation.
4. An **affine transformation** of \( \mathbb{R}^n \) is a map \( F : \mathbb{R}^n \to \mathbb{R}^n \) of the form
   \[
   F(p) = Ap + q
   \]
   where \( A \) is a linear transformation of \( \mathbb{R}^n \). We call \( A \) the linear part of the affine transformation \( F \). An affine transformation \( F \) is orientation-preserving if \( \det(A) \) is
positive, or orientation-reserving if $\det(A)$ is negative.

(5) A translation of $\mathbb{R}^n$ is an affine map $T_q : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$T_q(p) = p + q.$$ 

(6) A Euclidean motion of $\mathbb{R}^n$ is an affine transformation whose linear part is an orthogonal transformation. □

Now we state the fundamental theorem of plane curves without proof.

**Theorem 3 (Fundamental Theorem of Plane Curves)** Let $\alpha$ and $\gamma$ be unit-speed regular curves in $\mathbb{R}^2$ defined on the same interval $(a,b)$. Assume that $\alpha$ and $\gamma$ have the same curvature. Then there is an orientation-preserving Euclidean motion of $\mathbb{R}^2$ mapping $\alpha$ into $\gamma$. □

2. Immunization Theory and Plane Curves

The purpose of this section is to present geometric interpretations of the immunization theory. Let us consider asset value and liability value as functions of interest rate $i$;

\[
\begin{align*}
\text{Asset value} & = A(i), \\
\text{Liability value} & = L(i), \\
\text{Surplus} & = S(i) = A(i) - L(i).
\end{align*}
\]

Note that $A(i), L(i),$ and $S(i)$ are plane curves with variable $i$.

**Goal:** We want to match the curves $A(i)$ and $L(i)$ as much as possible with the given $L(i)$ curve, and make the curve $S(i)$ as flat as possible.

**Step 1:** The first step is to match the point $A(i_o)$ to the point $L(i_o)$, where $i_o$ is the current interest rate, by transformation of the curve $A(i)$. We call this process as present value matching,

\[
\begin{align*}
A(i_o) & = L(i_o), \\
S(i_o) & = 0.
\end{align*}
\]

**Step 2:** The second step is to match the slope of the tangent lines at the intersection point $(i_o, A(i_o))$. This is done by rotation of the curve $A(i)$. We call this process as duration matching,

\[
\begin{align*}
A'(i_o) & = L'(i_o), \\
S'(i_o) & = 0.
\end{align*}
\]

Value
<Figure 3> Present value matching by transformation of the curve $A(i)$.

<Figure 4> Duration matching by rotation of the curve $A(i)$.
Step 3: The third step is to match the curvatures at the intersection point \((i_0, A(i_0))\).

\[
\kappa[A](i_0) = \kappa[L](i_0), \quad \kappa[S](i_0) = 0.
\]

If we match the curvatures of the curves \(A(i)\) and \(L(i)\) then we match both curves perfectly by the fundamental theorem of plane curves. If we can make \(S'(i_0) = 0\) and \(\kappa[S](i_0) = 0\) then the curve \(S(i)\) is a horizontal line, which means immunization against interest rate changes. For the curve \(S = (i, S(i))\) with \(S'(i_0) = 0\), the curvature is calculated by (17),

\[
\kappa[S](i_0) = \frac{x'(i_0)y''(i_0) - x''(i_0)y'(i_0)}{(x'^2(i_0) + y'^2(i_0))^{3/2}} = S''(i_0).
\]

Given \(S'(i_0) = 0\), the requirement condition \(\kappa[S](i_0) = 0\) is equivalent to \(S''(i_0) = 0\). So if we match the first derivative and the second derivative of the asset curve \(A(i)\) and the liability curve \(L(i)\) then we can have a perfect immunization strategy by the fundamental theorem of plane curves, and we do not need higher order derivative matching any more.

IV. Conclusion

In this paper we discuss the deterministic asset liability management methods such as deterministic duration matching and deterministic cash flow matching techniques which can be used for the interest rate risk managements in insurance business. We approximate the changes in surplus according to the changes in interest rates using Taylor series. Usually the first and the second derivatives are taken for the deterministic ALM methods. Using the fundamental theorem of plane curves we justify that it is enough to consider only the first and the second derivatives for a perfect immunization of interest rate risks in deterministic cash flows of assets and liabilities. Also we suggest a simple quadratic programming method which can be used for the deterministic cash flow matching problem with reinvestment and borrowing.

References
