Mixed Finite Elements for Electromagnetic Analysis

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Abstract

The occurrence of spurious solutions is a well-known limitation of the standard nodal finite element method when applied to electromagnetic problems. The two commonly used remedies that are used to address this problem are (i) The addition of a penalty term with the penalty factor based on the local dielectric constant, and which reduces to a Helmholtz form on homogeneous domains (regularized formulation); (ii) A formulation based on a vector and a scalar potential. Both these strategies have some shortcomings. The penalty method does not completely get rid of the spurious modes, and both methods are incapable of predicting singular eigenvalues in non-convex domains. Some non-zero spurious eigenvalues are also predicted by these methods on non-convex domains. In this work, we develop mixed finite element formulations which predict the eigenfrequencies (including their multiplicities) accurately, even for nonconvex domains. The main feature of the proposed mixed finite element formulation is that no ad-hoc terms are added to the formulation as in the penalty formulation, and the improvement is achieved purely by an appropriate choice of finite element spaces for the different variables. We show that the formulation works even for inhomogeneous domains where ‘double noding’ is used to enforce the appropriate continuity requirements at an interface. For two-dimensional problems, the shape of the domain can be arbitrary, while for the three-dimensional ones, with our current formulation, only regular domains (which can be nonconvex) can be modeled. Since eigenfrequencies are modeled accurately, these elements also yield accurate results for driven problems.

Keywords: Maxwell Equations, Nodal finite elements, Spurious Modes,
Mixed finite element formulation

1. Introduction

One of the major problems in computational electromagnetics using conventional nodal finite elements has been the occurrence of spurious solutions. In structural mechanics, mesh refinement helps to reduce the errors; however, in the electromagnetics setting, it merely increases the number of spurious modes. Edge elements [1, 2, 3, 4, 5], which use basis function associated with each edge are widely used to circumvent this difficulty. These elements ensure tangential continuity of the field along an element edge and model the null space of curl operator accurately. They can model both, singularities and inhomogeneous domains. However, they also have their limitations [6, 7]. Since the normal component is discontinuous across element faces even for homogeneous domains, the efficiency is reduced. Another disadvantage is that coupling with structural or thermal variables (where nodal finite elements are used) in multiphysics problems could be difficult.

In order to deal with spurious modes within the framework of the nodal finite element method, a penalty function or regularization method [8, 9, 10, 11, 12, 13, 14] is used, where a term involving the divergence of the electric field is added to the original variational formulation. However, this method does not eliminate spurious modes, and in fact just pushes them towards the higher part of the spectrum; the problems with this method are particularly severe when non-convex and/or inhomogeneous domains are involved. Costabel and Dauge [15] and Otin [11] have proposed a weighted penalty method where a weight is appended to the penalty parameter which tends to zero at a field singularity. An alternative approach is the use of potentials [16, 17, 18] along with the choice of a suitable gauge. Even in this method, a penalty-type term has to be added. The method is quite robust even for inhomogeneous domains. However, because of the presence of the penalty term, it cannot predict the singular eigenvalues on nonconvex domains.

Recent works include using a combination of cubic Hermite splines and quadratic Lagrange interpolations [19], and a least squares finite element method [20]. The former method can be applied only to regular geometries, while the latter uses nonstandard elements with stabilizing face bubble functions.
In this work, we develop nodal-based mixed finite element formulations for two and three-dimensional problems. The two-dimensional elements yields very accurate approximations of the eigenvalues, including the correct multiplicities, and including singular eigenvalues for non-convex domains. Non-homogeneous and curved domains can also be modeled with these elements. The three-dimensional elements can currently be applied only to Cartesian geometries (including domains with singularities and inhomogeneous domains); further work is required to extend their capabilities to non-regular geometries. The main feature of the proposed mixed formulation is that no additional terms, such as the penalty terms that are used in regularized or potential-based formulations, are added to the variational formulation, and in addition, there are no parameters that have to be chosen by the user; alternative formulations which a-priori eliminate spurious modes are presented in References [13, 21]. The improvement is achieved simply by using appropriately chosen interpolations for the various fields. The other main feature is that standard $C^0$ Lagrange interpolations are used for all the fields, and standard Gaussian quadrature is used to compute all the matrices.

The outline of the remainder of the article is as follows. We first briefly review the variational and finite element formulations for the potential method. Next we discuss the variational formulation and interpolations for the proposed mixed finite elements. Finally, we show the high accuracy and robustness of the proposed formulation by comparing the solutions obtained against either analytical or benchmark solutions obtained using existing numerical strategies.

2. Mathematical Formulation

2.1. Maxwell Equations in Electromagnetics

The strong form of the Maxwell equations is [22]

\[
\begin{align*}
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \quad (1a) \\
\nabla \cdot B &= 0, \quad (1b) \\
\frac{\partial D}{\partial t} - \nabla \times H &= -j, \quad (1c) \\
\nabla \cdot D &= \rho, \quad (1d)
\end{align*}
\]

where $E$ and $H$ are the electric and magnetic fields, $D$ is the electric displacement (electric flux), $B$ is the magnetic induction (magnetic flux), $\rho$ is
the charge density and $j$ is the current density. The above governing equations are supplemented by the constitutive relations

$$D = \varepsilon E,$$  \hspace{1cm} (2a)

$$B = \mu H,$$ \hspace{1cm} (2b)

where $\varepsilon$ and $\mu$ are the electric permittivity and magnetic permeability, respectively. Substituting the constitutive relations into Eqns. (1a), (1c) and (1d), and assuming that $\varepsilon$ and $\mu$ are independent of time, we get

$$\frac{\partial H}{\partial t} + \frac{1}{\mu} \nabla \times E = 0,$$ \hspace{1cm} (3)

$$\varepsilon \frac{\partial E}{\partial t} - \nabla \times H = -j,$$ \hspace{1cm} (4)

$$\nabla \cdot (\varepsilon E) = \rho.$$ \hspace{1cm} (5)

From Eqns. (1d), (2a) and (4), we get the compatibility condition

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.$$  (6)

Eliminating $H$ from Eqns. (3) and (4), we get the governing equation for $E$ as

$$\frac{\varepsilon}{c^2} \frac{\partial^2 E}{\partial t^2} + \frac{\partial j}{\partial t} + \nabla \times \left( \frac{1}{\mu} \nabla \times E \right) = 0.$$ \hspace{1cm} (7)

The boundary conditions are that $E \times n$ is specified over part of the surface $\Gamma_e$, and $H \times n$ is specified over the remaining part $\Gamma_h$. Assuming absence of impressed surface currents, at a material discontinuity, both $E \times n$ and $H \times n$ should be continuous across the material interface.

Introducing the relative permittivity and relative permeability $\varepsilon_r := \varepsilon / \varepsilon_0$ and $\mu_r = \mu / \mu_0$, where $\varepsilon_0$ and $\mu_0$ are the permittivity and permeability for vacuum, Eqn. (6) can be written as

$$\frac{\varepsilon_r}{c^2} \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial j}{\partial t} + \nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) = 0,$$ \hspace{1cm} (8)

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light.

In the frequency domain, Eqn. (7) can be written as

$$\nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) - k_0^2 \varepsilon_r E = -i\omega \mu_0 j.$$ \hspace{1cm} (9)
where \( i = \sqrt{-1} \) and \( k_0 = \omega^2 / c^2 \) is the wave number. Assuming \( j \) to be zero, the above equation reduces to

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) = k_0^2 \epsilon_r E.
\]

The above equation is used to solve the eigenvalue problem.

2.2. Variational Formulation

2.2.1. Conventional Potential Methods

Regularized formulations [9] yield wrong multiplicities of eigenvalues even for homogeneous domains, and spurious values on inhomogeneous ones. Hence, we focus on the potential formulation presented by Bardi et al. [18], which is quite robust in that it yields correct results for both homogeneous and inhomogeneous problems provided the domain is convex. If the domain is non-convex, then due to the presence of a penalty-type term in its formulation, it fails to find the singular eigenvalues. Since the potentials are continuous across a material interface, a standard finite element formulation for the potentials can be used.

The governing differential equation given by Eqn. (8) is modified by replacing \( E \) by \( A + \nabla \phi \) as

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times A \right) - k_0^2 \epsilon_r A - k_0^2 \epsilon_r \nabla \phi = -i \omega \mu_0 j,
\]

where \( A \) and \( \phi \) are vector and scalar potentials. The variational formulation is obtained by taking the dot product of the above equation with the variation \( A_\delta \), and carrying out an appropriate integration by parts. A penalty term is added (see [18]) to remove the spurious modes. Thus the final variational form is given by

\[
\int_\Omega \frac{1}{\mu_r} (\nabla \times A_\delta) \cdot (\nabla \times A) d\Omega - k_0^2 \int_\Omega \epsilon_r A_\delta \cdot A d\Omega - k_0^2 \int_\Omega \epsilon_r A_\delta \cdot \nabla \phi + \\
\int_\Omega \frac{1}{\epsilon_r \mu_r} (\nabla \cdot A_\delta) [\nabla \cdot (\epsilon_r A)] d\Omega = i \omega \mu_0 \int_{\Gamma_h} [A_\delta \times n] \cdot \mathbf{H} d\Gamma \\
- i \omega \mu_0 \int_\Omega A_\delta \cdot j d\Omega,
\]

Multiplying Eqn. (5) by the variation \( \phi_\delta \) and replacing \( E \) by \( A + \nabla \phi \), we get

\[
\int_\Omega \nabla \phi_\delta \cdot (\epsilon_r A) d\Omega + \int_\Omega \nabla \phi_\delta \cdot (\epsilon_r \nabla \phi) d\Omega = - \int_\Omega \frac{\phi_\delta \rho}{\epsilon_0} d\Omega.
\]
For eigenanalysis Eqns. (11) and (12) reduce to
\[
\int_{\Omega} \frac{1}{\mu_r} (\nabla \times A_\delta) \cdot (\nabla \times A) \, d\Omega + \int_{\Omega} \frac{1}{\epsilon_r \mu_r} (\nabla \cdot (\epsilon_r A_\delta)) \nabla \cdot (\epsilon_r A) \, d\Omega = k_0^2 \int_{\Omega} \epsilon_r A_\delta \cdot A \, d\Omega + k_0^2 \int_{\Omega} \epsilon_r A_\delta \cdot \nabla \phi, \tag{13}
\]
\[
k_0^2 \int_{\Omega} \nabla \phi_\delta \cdot (\epsilon_r A) \, d\Omega + k_0^2 \int_{\Omega} \nabla \phi_\delta \cdot (\epsilon_r \nabla \phi) \, d\Omega = 0, \tag{14}
\]
where the latter equation has been multiplied by $k_0^2$ to obtain a symmetric stiffness matrix. On parts of the boundary where $E \times n = 0$, we prescribe $A \times n$ and $\phi$ to be zero.

Although, due to the presence of the penalty term, there are no zero eigenvalues corresponding to $A$, there are zero eigenvalues corresponding to $\phi$ as mentioned in Reference [18], and the number of zero eigenvalues is equal to the number of unsuppressed $\phi$ degrees of freedom. This method is used to generate solutions (for problems where analytical solutions are not available) against which we compare the proposed mixed finite element formulation described in the following subsection.

2.2.2. Proposed Mixed Formulation

We now present mixed finite element formulations that predict both, the null space of $\nabla \times E$ and also the nonzero eigenvalues with the correct multiplicity. Inhomogeneous domains are handled by using ‘double noding’ at the interface along with Lagrange multipliers to enforce the continuity requirements on $E \times n$; the implementation is similar to the implementation of $E \times n = 0$ for perfectly conducting surfaces as discussed in Section 2.3.3. Due to the absence of any penalty term in the formulation, the mixed formulation can predict the singular eigenvalues for nonconvex domains accurately.

We start by introducing the new variable
\[
h := -i \mu_0 \omega H = \frac{1}{\mu_r} \nabla \times E. \tag{15}
\]

The above relation is implemented in a weak sense within the context of our mixed finite element formulation in addition to the variational statement of the governing equation, which we rewrite in terms of $h$. Thus, the variational
The statements are given by

\[
\int_{\Omega} (\nabla \times E_{\delta}) \cdot h \, d\Omega - k_0^2 \int_{\Omega} \epsilon_r E_{\delta} \cdot E \, d\Omega = i\omega \mu_0 \int_{r_0} [E_{\delta} \times n] \cdot \bar{H} \, d\Gamma \\
- i\omega \mu_0 \int_{\Omega} E_{\delta} \cdot j \, d\Omega,
\]

(16)

\[
\int_{\Omega} h_{\delta} \cdot [\nabla \times E - \mu_r h] \, d\Omega = 0.
\]

(17)

For eigenanalysis we set the load terms to zero, so that Eqn. (16) becomes

\[
\int_{\Omega} (\nabla \times E_{\delta}) \cdot h \, d\Omega = k_0^2 \int_{\Omega} \epsilon_r E_{\delta} \cdot E \, d\Omega,
\]

(18)

2.3. Finite Element Formulation

For both the conventional and the proposed mixed formulations, let the fields \(E\) and \(E_{\delta}\) be discretized as

\[
E = N \hat{E}, \\
E_{\delta} = N \hat{E}_{\delta},
\]

where \(N\) is the matrix of standard Lagrange interpolations. Using the above interpolations, and assuming the material to be homogeneous for simplicity (we will relax this constraint later), we get

\[
\nabla \times E = B \hat{E}, \\
\nabla \cdot E = B_p \hat{E}, \\
\nabla \times E_{\delta} = B \hat{E}_{\delta}, \\
\nabla \cdot E_{\delta} = B_p \hat{E}_{\delta},
\]

where

\[
B = \begin{bmatrix}
0 & -\frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & 0 & -\frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial y} & \cdots \\
\frac{\partial N_1}{\partial z} & 0 & -\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & 0 & -\frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & -\frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & 0 & \cdots \\
\end{bmatrix},
\]

(19)

\[
B_p = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial z} & 0 & \cdots \\
\frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_2}{\partial x} & \cdots \\
\end{bmatrix}.
\]
If one assumes a transverse electric (TE) field \((E_z = 0)\), then only \((\nabla \times E)_z = \partial E_y / \partial x - \partial E_x / \partial y\) and \(E = (E_x, E_y)\) needs to be considered in the variational formulations. Thus, in place of Eqns. (19) we have

\[
B = \begin{bmatrix}
-\frac{\partial N_1}{\partial y} & -\frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\end{bmatrix},
\]

\(B_p = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\end{bmatrix}.
\]

2.3.1. Conventional Potential Formulation

For implementing the potential-based method, the vector potential \(A\) and its variation \(A_\delta\) are discretized as

\[
A = N \hat{A},\]

\[
A_\delta = N \hat{A}_\delta,
\]

leading to

\[
\nabla \times A = B \hat{A},\]

\[
\nabla \cdot A = B_p \hat{A},\]

\[
\nabla \times A_\delta = B \hat{A}_\delta,\]

\[
\nabla \cdot A_\delta = B_p \hat{A}_\delta.
\]

Similarly, for the scalar potential \(\phi\), we have

\[
\phi = N \hat{\phi},\]

\[
\phi_\delta = N \hat{\phi}_\delta,
\]

where

\[
B_\phi = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \cdots \\
\end{bmatrix}.
\]

The discretized forms of Eqns. (13) and (14) are given by

\[
\begin{bmatrix}
K_{AA} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{A} \\
\hat{\phi}
\end{bmatrix} = k_0^2
\begin{bmatrix}
M_{AA} & M_{A\phi} \\
M_{\phi A} & M_{\phi\phi}
\end{bmatrix}
\begin{bmatrix}
\hat{A} \\
\hat{\phi}
\end{bmatrix},
\]

where

\[
2K_{AA} = k_0^2
\begin{bmatrix}
M_{AA} & M_{A\phi} \\
M_{\phi A} & M_{\phi\phi}
\end{bmatrix}
\begin{bmatrix}
\hat{A} \\
\hat{\phi}
\end{bmatrix}.
\]
where

\[
K_{AA} = \int_{\Omega} \frac{1}{\mu_r} \left[ B^T B + B^T_p B_p \right] d\Omega, \quad (23a)
\]

\[
M_{AA} = \int_{\Omega} \epsilon_r N^T N d\Omega, \quad (23b)
\]

\[
M_{A\phi} = \int_{\Omega} \epsilon_r N^T B_\phi d\Omega, \quad (23c)
\]

\[
M_{\phi A} = \int_{\Omega} \epsilon_r B^T_\phi N d\Omega, \quad (23d)
\]

\[
M_{\phi\phi} = \int_{\Omega} \epsilon_r B^T_\phi B_\phi d\Omega, \quad (23e)
\]

Similarly, for driven problems, from Eqns. (11) and (12), we have

\[
\left( \begin{bmatrix} K_{AA} & 0 \\ K_{\phi A} & K_{\phi\phi} \end{bmatrix} - k_0^2 \begin{bmatrix} M_{AA} & M_{A\phi} \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{A} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} F_A \\ F_\phi \end{bmatrix}, \quad (24)
\]

where \(K_{\phi A}\) and \(K_{\phi\phi}\) are given by the same expressions as those for \(M_{\phi A}\) and \(M_{\phi\phi}\) in Eqns. (23d) and (23e), and

\[
F_A = -i\mu_0 \omega \int_{\Omega} N^T j d\Omega, \quad (25a)
\]

\[
F_\phi = -\int_{\Omega} N^T_\phi \rho \epsilon_0 d\Omega. \quad (25b)
\]

2.3.2. Mixed Formulation

We interpolate \(h\) and \(h_\delta\) as

\[
h = P\beta, \\
h_\delta = P\beta_\delta,
\]

where \(P\) are the interpolation functions for \(h\), and \(\beta\) are unknown parameters to be determined. Note that the interpolations are in some cases continuous and in some case discontinuous across element boundaries. If they are discontinuous, then the parameters \(\beta\) can be condensed out at an element level resulting in the same size of \(K\) as in the conventional nodal formulation as we now show.
Substituting the interpolations for $h$ and $h_{\delta}$ into Eqns. (18) and (17) and using the arbitrariness of $\hat{E}_{\delta}$ and $\beta_{\delta}$, we get

$$G^T \beta = k_0^2 M \hat{E},$$  
(26a)

$$G \hat{E} - H \beta = 0,$$  
(26b)

where

$$G = \int_{\Omega} P^T B \, d\Omega,$$

$$H = \int_{\Omega} \mu P^T P \, d\Omega.$$

From Eqns. (26b), we have $\beta = H^{-1} G \hat{E}$, which when substituted into Eqn. (26a) yields

$$K_h \hat{E} = k_0^2 M \hat{E},$$  
(27)

where

$$K_h = G^T H^{-1} G.$$  
(28)

From the above expression, it is evident that a small matrix $H$ needs to be inverted at an element level in order to form the element level ‘stiffness matrix’ $K_h$. The computational cost of this inversion is negligible compared to the computational cost of solving the global set of equations.

In case one assumes a continuous interpolation for $h$ (which is typically one order lower than the interpolation for $E$), then the $\beta$ degrees of freedom cannot be condensed out at an element level, and one solves the system given by Eqns. (26a) and (26b), namely

$$\begin{bmatrix} 0 & G^T \\ G & -H \end{bmatrix} \begin{bmatrix} \hat{E} \\ \beta \end{bmatrix} = k_0^2 \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{E} \\ \beta \end{bmatrix}.$$  
(29)

Even in this case, the number of $\beta$ degrees of freedom are much smaller compared to the $\hat{E}$ degrees of freedom (because of the lower-order interpolation being used for $h$). An additional advantage is that one obtains nodal values of a physically important variable, namely $h$ (which is proportional to $H$) in case a continuous interpolation for $h$ is used—in case of a discontinuous interpolation, averaging across elements sharing a node is performed to obtain a nodal value.

In the above mixed formulation that has been presented, note the complete absence of either any factors that need to be adjusted by the user or
terms that are added in an ad-hoc way to the variational formulation (such as penalty terms).

For driven problems given by Eqn. (16), the mixed finite element formulation with a discontinuous interpolation for $h$ is given by

$$(K_h - k_0^2M)\hat{E} = f$$

(30)

where $K_h$ is given by Eqn. (28) and

$$f = -i\omega\mu_0 \int_\Omega N^T j d\Omega.$$

Elements based on a discontinuous interpolation for $h$:

The stiffness matrix $K_h$ for these elements is formulated using Eqn. (28). The planar 4-node quadrilateral, 9-node quadrilateral and 7-node triangular elements, denoted by A4, A9 and T7 are used for TE fields, while the three-dimensional 8-node and 27-node hexahedral elements denoted by S8 and S27 are used for vector problems. Full integration, namely, $2 \times 2$, $3 \times 3$, 12-point$^1$, $2 \times 2 \times 2$ and $3 \times 3 \times 3$ integration rules are used for these elements. The natural coordinates are denoted by $(\xi, \eta)$ for the planar elements and by $(\xi, \eta, \zeta)$ in the case of three-dimensional elements. The Jacobian matrix is denoted by $J$. The interpolations used for $h$ are as follows:

4-node quadrilateral $Q_1$-$P_0$ (A4)


7-node triangular element (T7) The 7-node triangular element is obtained by adding a midnode to the conventional 6-node triangular element. The shape functions for the electric field are denoted by $N_i, i = 1, 2, \ldots, 7$:

$$\begin{align*}
N_1 &= \bar{N}_1 + 3N_b, & \bar{N}_1 &= \xi(2\xi - 1), \\
N_2 &= \bar{N}_2 + 3N_b, & \bar{N}_2 &= \eta(2\eta - 1), \\
N_3 &= \bar{N}_3 + 3N_b, & \bar{N}_3 &= \rho(2\rho - 1), \\
N_4 &= \bar{N}_4 - 12N_b & \bar{N}_4 &= 4\xi\eta, \\
N_5 &= \bar{N}_5 - 12N_b & \bar{N}_5 &= 4\rho\eta, \\
N_6 &= \bar{N}_6 - 12N_b & \bar{N}_6 &= 4\xi\rho, \\
N_7 &= 27N_b,
\end{align*}$$

$^1$A 7-point rule also yields almost identical results.
with the bubble mode \( N_b = \xi \eta (1 - \xi - \eta) \). The interpolation for \( h = h_x \) is

\[
P = \begin{bmatrix} 1 & \xi & \eta \end{bmatrix}.
\]

8-node hexahedral element (S8)

\[
P = \text{cof} \, J^T \begin{bmatrix} 1 & \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \zeta \end{bmatrix},
\]

where \( \text{cof} \, J^T = (\det J) J^{-1} \) is the transformation matrix that transforms the interpolation functions from natural coordinates to physical ones.

27-node hexahedral element (S27)

\[
P = \text{cof} \, J^T \begin{bmatrix} P_L \xi^2 & \xi^2 \eta & \xi^2 \zeta & \xi \eta \xi \zeta & P_L \eta^2 & \eta^2 \xi & \eta^2 \zeta & \eta \zeta \eta \zeta \\ 0_{1 \times 8} & 0 & 0 & 0 & 0_{1 \times 8} & 0 & 0 & 0 \\ 0_{1 \times 8} & 0 & 0 & 0 & 0_{1 \times 8} & 0 & 0 & 0 \end{bmatrix} 3 \times 36,
\]

where

\[
P_L = \begin{bmatrix} 1 & \xi & \eta & \zeta & \xi \eta & \xi \zeta & \xi \zeta \end{bmatrix}.
\]

A continuous interpolation was also tried for both for these elements, and also for a tetrahedral element, but yields erroneous results. In their current form, the three-dimensional S8 and S27 elements work only for regular geometries (where the Jacobian within an element is a constant) but otherwise yield correct results even for nonconvex and inhomogeneous domains. The two-dimensional mixed finite elements have no such restriction, however, and yield good results for arbitrary (convex/nonconvex and homogeneous/inhomogeneous) domains.

A 9-node quadrilateral with a discontinuous interpolation

\[
P = \begin{bmatrix} 1 & \xi & \eta \end{bmatrix},
\]

was also tried. Similar to the A4 element which yields a wrong multiplicity of the eigenvalue 18 on a square domain (see Table (5.6) in Reference [23]), this element is also mildly unstable, in the sense that it yields a wrong multiplicity of the eigenvalue 49 on a fine mesh for the square domain problem discussed in Section 3.1.1; in spite of this small drawback, it might be suitable for getting a good engineering approximation. Hence, we have preferred to use a continuous interpolation for \( h \) in the case of the 9-node element as described.
in the following section. This modification gets rid of this mild instability yielding a stable element.

Elements based on a continuous interpolation for \( h \):

For the 6-noded triangular element \( P_2-P_1 \) (T6), we use the formulation given by Eqn. (29) with \( P \) now given by the usual interpolation functions (i.e., \((\xi, \eta, 1 - \xi - \eta)\)) for a 3-noded triangle. The mid-side nodes 4, 5 and 6 only have the \((E_x, E_y)\) degrees of freedom associated with them, while the corner nodes 1, 2 and 3 have three degrees of freedom, namely, \((E_x, E_y, h)\) at each node where \( h = h_z \) (thus, there are 15 degrees of freedom per element). A 6-point integration rule is used for all the matrices.

Similarly, for the 9-node quadrilateral element \( Q_2-Q_1 \), we use the four corner nodes and the standard bilinear interpolation functions associated with them for interpolating \( h \) (thus, there are 22 degrees of freedom per element). A standard \( 3 \times 3 \) quadrature rule is used for computing all the matrices.

2.3.3. Implementation of \( E \times n = 0 \) for curved boundaries

The constraint \( E \times n = 0 \) is implemented in a weak sense using Lagrange multipliers in our formulation. This is especially convenient when the boundary is curved. For example, the set of equations given by Eqn. (27) for elements adjacent to the boundary (for elements not adjacent to the boundary, the Lagrange multiplier degrees of freedom are suppressed) for a planar quadrilateral element gets modified to

\[
\left[ \begin{array}{c}
K_h \\
\int_{-1}^{1} \int_{-1}^{1} N^T t N d\xi d\eta
\end{array} \right]
\left[ \begin{array}{c}
\int_{-1}^{1} N^T t N \lambda d\xi \\
0
\end{array} \right]
\left[ \begin{array}{c}
\dot{E} \\
\dot{\lambda}
\end{array} \right] = k_0^2
\left[ \begin{array}{cc}
M & 0 \\
0 & 0
\end{array} \right]
\left[ \begin{array}{c}
\dot{E} \\
\dot{\lambda}
\end{array} \right]
\]

where \( \xi \) is the natural coordinate that parametrizes the boundary of the element, and

\[
t = \begin{bmatrix}
\frac{\partial x}{\partial \xi} \\
\frac{\partial y}{\partial \xi}
\end{bmatrix},
\]

\[
N_\lambda = [N_1 \quad N_2 \quad \ldots].
\]

For three-dimensional hexahedral elements, it gets modified to

\[
\left[ \begin{array}{c}
K_h \\
\int_{-1}^{1} \int_{-1}^{1} N^T t N d\xi d\eta
\end{array} \right]
\left[ \begin{array}{c}
\int_{-1}^{1} \int_{-1}^{1} N^T t N \lambda d\xi d\eta \\
0
\end{array} \right]
\left[ \begin{array}{c}
\dot{E} \\
\dot{\lambda}
\end{array} \right] = k_0^2
\left[ \begin{array}{cc}
M & 0 \\
0 & 0
\end{array} \right]
\left[ \begin{array}{c}
\dot{E} \\
\dot{\lambda}
\end{array} \right]
\]
where \( t \) is a \( 2 \times 3 \) matrix containing along its rows two linearly independent vectors \( t_1 \) and \( t_2 \) that are both perpendicular to the normal \( n \), \((\xi, \eta)\) are the natural coordinates that parametrize the surface of the element, and

\[
\begin{align*}
    n &= \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \frac{\partial x}{\partial \xi} \times \frac{\partial x}{\partial \eta}, \\
    N_\lambda &= \begin{bmatrix} N_1 & 0 & N_2 & 0 & \ldots \\ 0 & N_1 & 0 & N_2 & \ldots \end{bmatrix}.
\end{align*}
\]

Since the Lagrange multiplier technique cannot handle abrupt changes in slope of the continuum (e.g. corner of a square, edges of a cube), the entire \( E \) vector along corner or edges with a discontinuous normal \( n \) is set to zero explicitly, and the corresponding Lagrange multipliers are also suppressed.

The above discussion for two-dimensional elements has been restricted to finding the TE modes. In order to get the transverse magnetic (TM) modes we have to consider the formulation based on \( H \). Then instead of Eqn. (7), we get

\[
\frac{\mu_r}{\epsilon^2} \frac{\partial^2 H}{\partial t^2} + \nabla \times \left( \frac{1}{\epsilon_r} \nabla \times H \right) = \epsilon_0 \nabla \times j. \tag{31}
\]

By comparing Eqns. (7) and (31), we see that the TM modes can be obtained simply by interchanging \( \mu_r \) and \( \epsilon_r \) in the finite element formulation (now in terms of \( H \)), and by not imposing any boundary condition on the boundary since \( E \times n \) and hence \((\nabla \times H) \times n\), is zero on the boundary (this natural boundary condition is thus implemented in a weak sense).

3. Numerical Examples

In the following examples we use SI units. B9 and B6 denote the (potential-based) conventional 9-node quadrilateral and 6-node triangular elements respectively, whereas A9, T6 and T7 denote the proposed mixed 9-node rectangular and 6- and 7-node triangular elements respectively. In the three-dimensional case, B8 and B27 denote the conventional 8-node and 27-node hexahedral elements, while S8 and S27 stands for the corresponding mixed elements. For all the homogeneous problems considered in this work, we have assumed \( \epsilon_r = \mu_r = 1.0 \). Since we do not penalize the divergence of \( E \) explicitly (penalization suppresses the singular eigenvalues in the case of nonconvex domains as in the case of the potential-based formulation), we
get zero eigenvalues corresponding to the gradient of some scalar field, and which is mesh and element-type dependent. The zero eigenvalues reported in the tables are generally of the order $10^{-10}$ or smaller; thus, we see that the null space is computed extremely accurately by the proposed method.

3.1. Eigenvalue Problems

3.1.1. Square domain with perfectly conducting surfaces

In this example, we find the eigenvalues for a square domain of dimension $\pi$ with perfectly conducting surfaces. Tables 1 and 2 list the results obtained using various elements. Uniform $8 \times 8$ and $16 \times 16$ meshes of A9 and B9 elements are used. By subdividing each 9-node element into two 6-node triangular elements, we obtain an equivalent mesh of T6 elements. To obtain the mesh of 7-node triangular elements, we simply add a midnode to each T6 element. The edge element results are taken from Reference [1]. The convergence of eigenfrequencies with mesh refinement can be observed for the proposed formulation. As already mentioned, the zero eigenfrequencies (the null space of the curl operator) are also approximated very accurately, and number of these frequencies are also listed in the tables.

Although in this example the conventional formulation (B9 elements) works well, we see in the following example that its performance degrades sharply for nonconvex geometries.

3.1.2. L-shaped domain with perfectly conducting surfaces

The L-shaped domain is obtained by deleting one quadrant from the square domain problem considered in the previous example. It is known that this nonconvex domain has one singular eigenvalue. Table 3 presents the results for various elements. Each of the 3 quadrants of the L-shaped domain is meshed using uniform $8 \times 8$ A9 and B9 meshes. The T6 and T7 triangular element meshes are generated from these meshes in the same way as in the previous example. The edge element results are again taken from Reference [1]. It can be seen from Table 3 that the conventional formulation using B9 elements cannot predict the singular eigenvalue as mentioned earlier. Furthermore, it also predicts a spurious eigenvalue (1.604). The proposed mixed elements do not have these drawbacks, and predict even the singular eigenvalue accurately.

Now consider the same L-shaped domain as above, but with inhomogeneous properties as shown in Fig. (1) of Reference [1]. Table 4 presents the
<table>
<thead>
<tr>
<th>Analytical</th>
<th>Existing Strategies</th>
<th>Mixed FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Edge [1] B9</td>
<td>A9 T6 T7</td>
</tr>
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<td>0.999830 1.000033</td>
<td>1.000537 1.000824 1.000199</td>
</tr>
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<td>1.000537 1.000833 4.003755</td>
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<td>17.292131 17.254970 16.071232</td>
</tr>
<tr>
<td>16</td>
<td>- 16.120357</td>
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<td>18.575545 19.860792 18.308402</td>
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<td>21.322424 22.590069 20.326378</td>
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<td>21.322424 23.347671 20.329678</td>
</tr>
<tr>
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<td>26.579879 27.735667 25.188429</td>
</tr>
<tr>
<td>25</td>
<td>- 25.432691</td>
<td>28.730671 30.114254 25.945208</td>
</tr>
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<tr>
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</tr>
<tr>
<td>32</td>
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Number of Computed Zeros

|         | 166 | 225 | 320 | 320 | 17 |

Table 1: Values of $k_0^2$ for the square domain problem (coarse mesh).
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<tr>
<th>Analytical</th>
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<td>Existing Strategies</td>
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<td>Mixed FEM</td>
</tr>
<tr>
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</tr>
<tr>
<td>T6</td>
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<td>T7</td>
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<tr>
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<tr>
<th>Number of Computed Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>166</td>
</tr>
</tbody>
</table>

Table 2: Values of $k^2_0$ for the square domain problem (fine mesh).
<table>
<thead>
<tr>
<th>Existing Strategies</th>
<th>Mixed FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge [1]</td>
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</tr>
<tr>
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<td>1.467369</td>
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<td>-</td>
<td>1.604001</td>
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<td>4.000131</td>
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<td>4.005540</td>
<td>4.000132</td>
</tr>
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<td>5.067330</td>
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<tr>
<td>11.426100</td>
<td>12.296406</td>
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<table>
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<th>Number of Computed Zeros</th>
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</thead>
<tbody>
<tr>
<td>267</td>
</tr>
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</table>

Table 3: Values of $k_0^2$ for the L-shaped domain problem.
results for different elements with the same meshes used as in the homogeneous domain case. In the case of our mixed formulation, the material discontinuity is handled using ‘double noding’ at the interface, and Lagrange multipliers to enforce the continuity requirements. With the conventional formulation using B9 elements, the singular eigenvalue is again missing, and a spurious eigenvalue is also present just as in the homogeneous domain case. The proposed mixed elements A9, T6 and T7 are free of these shortcomings.

3.1.3. Curved L-shaped domain with perfectly conducting surfaces

This example taken from [24] is challenging due to the presence of both mesh distortion and a singular eigenvalue. We use a $8 \times 4$ mesh (in the $r$ and $\theta$ directions) of B9/A9 elements for discretizing each of the three curved quadrants of the domain. The meshes for T6 and T7 elements are constructed from these meshes as in the previous examples. Table 5 presents the results for the proposed mixed finite elements, and shows that the proposed method can handle both mesh distortion and singular eigenvalues. Note again that these values are obtained without having to adjust any parameter in the

<table>
<thead>
<tr>
<th>Existing Strategies</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Edge [1] B9</td>
<td>A9 T6 T7</td>
</tr>
<tr>
<td>0.175980 -</td>
<td>0.175999 0.176021 0.175664</td>
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<tr>
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<td>0.397368 0.397368 0.397354</td>
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<tr>
<td>- 0.407228 - - -</td>
<td>- - - -</td>
</tr>
<tr>
<td>0.964840 0.969815</td>
<td>0.966481 0.967389 0.966528</td>
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<tr>
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<td>0.981002 0.982087 0.980428</td>
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<tr>
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Number of Computed Zeros

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Table 4: Values of $k_0^2$ for the inhomogeneous L-shaped domain problem.
Existing Strategies

<table>
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<tr>
<th>Benchmark [24]</th>
<th>B9</th>
<th>Mixed FEM</th>
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<th></th>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>10.183637</td>
<td>10.205428</td>
<td>10.120844</td>
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</tr>
</tbody>
</table>

Number of Computed Zeros

|              | 337 | 644 | 644 | 305 |

Table 5: Values of $k_0^2$ for the curved L-shaped domain problem.

3.1.4. Circular domain with perfectly conducting surfaces

Consider a circular domain of radius unity with perfectly conducting surfaces. Here, we also consider the case where a crack runs from the periphery to the center as shown in Fig. 1. Analytical solutions for the uncracked and cracked domains are presented in [25] and [26]. The crack is modeled just as one would model a sector of a circle, i.e., using ‘double noding’ along the crack. For both problems we use uniform $8 \times 32$ meshes along the $r$ and $\theta$ directions as shown in Fig. 2, with 6-node triangular elements in the layer around the origin and 9-node elements elsewhere. Table 6 presents the results for both cases. For the full circular domain both B6/B9 and T6/A9 show good agreement with the analytical solution. In the case of the cracked circular domain, the singular eigenvalue is not predicted by the B6/B9 elements, and there is a spurious nonzero eigenvalue of 6.431327. On the other hand, there is excellent agreement between the results obtained using the proposed T6/A9 elements and the analytical solution, including the multiplicity of the eigenvalues.

We now present some three-dimensional examples.

3.1.5. Cube with perfectly conducting surfaces

Consider a cube of dimension $\pi$. We discretize this domain with a uniform $7 \times 7 \times 7$ mesh of 27-node hexahedral elements and an equivalent (i.e., having the same number of nodes) mesh of 8-node hexahedral elements. Table 7
Figure 1: Cracked circular domain

Figure 2: Mesh for the circular domain problem
Table 6: Values of $k_0^2$ for the circular domain problem (bracketed values show the multiplicity).

3.1.6. Three-dimensional L-shaped domain with perfectly conducting surfaces

The three-dimensional L-shaped domain as described in Reference [24] is considered here. The domain is a rectangular parallelepiped $[-1,1] \times [0,1] \times [-1,1]$ with the cube $[0,1] \times [0,1] \times [-1,0]$ removed. A uniform mesh of 192 27-node elements, each element having dimension 0.25, is used to discretize the domain, and an equivalent mesh comprising of 1536 elements is used for the S8 element. Table 8 presents the comparison of the proposed elements with the benchmark results presented in [24]. Analogous to the two-dimensional L-shape domain case, not only does the B27 element fail to predict the singular eigenvalue, but also predicts a spurious eigenvalue of 13.888613. Both the proposed elements S8 and S27 yield good results.

3.1.7. Fichera corner with perfectly conducting surfaces

The domain is obtained by excluding the cube $[-1,0] \times [-1,0] \times [-1,0]$ from the cube $[-1,1] \times [-1,1] \times [-1,1]$ (see Figure 5 in [20]). We use uni-
### Table 7: Values of $k_0^2$ for the cubical domain problem.

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<th>Analytical $k_0^2$</th>
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<th>Mixed FEM $k_0^2$</th>
<th>S27</th>
<th>S8</th>
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<td>2.000112 (3)</td>
<td>2.008406 (3)</td>
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<td>5.003518 (6)</td>
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<td></td>
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<td>6.024536 (3)</td>
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<td>6.003557 (3)</td>
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</tr>
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<td>8.006925 (3)</td>
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<td>9.007090 (3)</td>
<td>9.006809 (3)</td>
<td>9.087945 (3)</td>
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</tr>
<tr>
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<td>10.037677 (6)</td>
<td>10.348978 (6)</td>
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<td>11.038031 (3)</td>
<td>11.037700 (3)</td>
<td>11.305340 (3)</td>
<td></td>
</tr>
<tr>
<td>$12 \ (2)$</td>
<td>12.010531 (2)</td>
<td>12.008580 (2)</td>
<td>11.994460 (2)</td>
<td></td>
</tr>
<tr>
<td>$13 \ (6)$</td>
<td>13.041083 (6)</td>
<td>13.041083 (6)</td>
<td>13.412356 (6)</td>
<td></td>
</tr>
<tr>
<td>$14 \ (12)$</td>
<td>14.041171 (6)</td>
<td>14.033585 (6)</td>
<td>14.067889 (6)</td>
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</tr>
<tr>
<td>-</td>
<td>14.041952 (6)</td>
<td>14.040744 (6)</td>
<td>14.333187 (6)</td>
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</tbody>
</table>

Number of Computed Zeros

<table>
<thead>
<tr>
<th></th>
<th>1123</th>
<th>2079</th>
<th>1099</th>
</tr>
</thead>
</table>

Table 7: Values of $k_0^2$ for the cubical domain problem.
## 3.2. Driven problems

The stiffness matrix for the mixed formulation includes the null space of the curl operator, and hence is singular. So, we use a modal approach to solve (frequency) driven problems. Since each mode satisfies the divergence-free condition $\nabla \cdot (\varepsilon E) = 0$, within the context of the modal method and similar to edge elements, one can only solve driven problems where $\rho$ is zero. In contrast, the conventional formulation based on potentials excludes the null-space by adding a penalty term as already discussed, and hence one can directly solve the system of equations in Eqn. (24). Since the problems that we consider here do not have singularities (we are unable to find benchmark problems involving singularities for driven problems in the literature), the solutions obtained using conventional elements are marginally better than the proposed elements.

<table>
<thead>
<tr>
<th>Existing Strategies</th>
<th>Mixed FEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark [24] B27</td>
<td>S27</td>
</tr>
<tr>
<td>9.639724</td>
<td>9.673084</td>
</tr>
<tr>
<td>11.345226</td>
<td>11.007909</td>
</tr>
<tr>
<td>13.403636</td>
<td>13.403901</td>
</tr>
<tr>
<td>-</td>
<td>13.888613</td>
</tr>
<tr>
<td>15.197252</td>
<td>15.208324</td>
</tr>
<tr>
<td>19.509328</td>
<td>19.749318</td>
</tr>
<tr>
<td>19.739209</td>
<td>19.749318</td>
</tr>
<tr>
<td>19.739209</td>
<td>19.749318</td>
</tr>
<tr>
<td>19.739209</td>
<td>20.733765</td>
</tr>
<tr>
<td>21.259084</td>
<td>21.331241</td>
</tr>
<tr>
<td>Number of Computed Zeros</td>
<td></td>
</tr>
<tr>
<td>-</td>
<td>1127</td>
</tr>
<tr>
<td></td>
<td>1484</td>
</tr>
<tr>
<td></td>
<td>1484</td>
</tr>
</tbody>
</table>

Table 8: Values of $k_0^2$ for the three-dimensional L-shaped domain problem.
### Existing Strategies

<table>
<thead>
<tr>
<th>Benchmark [20]</th>
<th>B27</th>
<th>S27</th>
<th>S8</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.234320</td>
<td>-</td>
<td>3.032632</td>
<td>2.982422</td>
</tr>
<tr>
<td>5.882670</td>
<td>6.330217</td>
<td>5.856172</td>
<td>5.865666</td>
</tr>
<tr>
<td>5.883710</td>
<td>6.330217</td>
<td>5.856172</td>
<td>5.865666</td>
</tr>
<tr>
<td>-</td>
<td>7.044993</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10.678900</td>
<td>10.736292</td>
<td>10.576651</td>
<td>7.951000</td>
</tr>
<tr>
<td>10.683200</td>
<td>10.736292</td>
<td>10.719786</td>
<td>10.652077</td>
</tr>
<tr>
<td>10.694500</td>
<td>12.424268</td>
<td>10.719786</td>
<td>10.819827</td>
</tr>
<tr>
<td>12.365300</td>
<td>13.419777</td>
<td>12.098698</td>
<td>10.819827</td>
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<tr>
<td>12.372300</td>
<td>13.482654</td>
<td>12.098698</td>
<td>12.139104</td>
</tr>
<tr>
<td>Number of Computed Zeros</td>
<td>-</td>
<td>1513</td>
<td>733</td>
</tr>
</tbody>
</table>

Table 9: Values of $k_0^2$ for the Fichera corner problem.

#### 3.2.1. Two dimensional problem

A square of dimension $\pi$ with $\epsilon_r = \mu_r = 1$, and perfectly conducting surfaces is subjected to a forcing $\mathbf{j}$ given by

$$
\mu_0 \omega j_x = \frac{i}{\pi^2} \left[ 2 - k_0^2 (\pi y - y^2) \right],
$$

$$
\mu_0 \omega j_y = \frac{i}{\pi^2} \left[ 2 - k_0^2 (\pi x - x^2) \right],
$$

$$
\mu_0 \omega^2 \rho = 0.
$$

The analytical solution is given by

$$
E_x = \frac{y}{\pi} - \left( \frac{y}{\pi} \right)^2,
$$

$$
E_y = \frac{x}{\pi} - \left( \frac{x}{\pi} \right)^2,
$$

$$
E_z = H_x = H_y = 0,
$$

$$
\mu_0 \omega H_z = \frac{2i(y-x)}{\pi^2}.
$$

Table 10 presents the comparison for different elements at four different points in the domain for $\omega = 3 \times 10^8$ rad/sec. Since there is no singularity in this problem, the B9 element yields marginally better results compared to the mixed formulation.
3.2.2. Three dimensional problem

A cube of dimension \( \pi \) with perfectly conducting surfaces is subjected to

\[
\begin{align*}
\mu_0 \omega j_x &= (2 - 2i)(k_0^2 - 3) \cos x \sin y \sin z, \\
\mu_0 \omega j_y &= (-1 + i)(k_0^2 - 3) \sin x \cos y \sin z, \\
\mu_0 \omega j_z &= (-1 + i)(k_0^2 - 3) \sin x \sin y \cos z, \\
\mu_0 \omega^2 \rho &= 0.
\end{align*}
\]

The analytical solution is

\[
\begin{align*}
E_x &= (2 + 2i) \cos x \sin y \sin z, \\
E_y &= (-1 - i) \sin x \cos y \sin z, \\
E_z &= (-1 - i) \sin x \sin y \cos z, \\
\mu_0 \omega H_x &= 0, \\
\mu_0 \omega H_y &= (-3 + 3i) \cos x \sin y \cos z, \\
\mu_0 \omega H_z &= (3 - 3i) \cos x \cos y \sin z.
\end{align*}
\]

Table 11 presents the results at four different points in the domain with \( \omega = 3 \times 10^8 \text{ rad/sec} \). Similar to the two-dimensional case, the conventional elements perform marginally better compared to the mixed ones due to the absence of a singularity.
<table>
<thead>
<tr>
<th>Co-ordinate</th>
<th>Analytical</th>
<th>B27</th>
<th>S27</th>
<th>S8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[E_x, E_y, E_z]^T$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\frac{\pi}{3}, 0, \frac{\pi}{6})$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$-0.4330 -0.4330i$</td>
<td>$-0.4330 -0.4330i$</td>
<td>$-0.4424 -0.4424i$</td>
<td>$-0.4405 -0.4405i$</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$-0.8660 -0.8660i$</td>
<td>$-0.8664 -0.8664i$</td>
<td>$-0.8847 -0.8847i$</td>
<td>$-0.8811 -0.8811i$</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$0.7500 -0.7500i$</td>
<td>$0.7504 -0.7504i$</td>
<td>$0.7662 -0.7662i$</td>
<td>$0.7630 -0.7630i$</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$-[H_x, H_y, H_z]^T$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(\frac{\pi}{3}, 0, \frac{\pi}{6})$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0020 $-0.0020i$</td>
<td>$0.0020 -0.0020i$</td>
<td>$0.0021 -0.0021i$</td>
<td>$0.0020 -0.0020i$</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$-0.0060 -0.0060i$</td>
<td>$-0.0061 -0.0061i$</td>
<td>$-0.0062 -0.0062i$</td>
<td>$-0.0059 -0.0059i$</td>
<td></td>
</tr>
<tr>
<td>0.0020 $-0.0020i$</td>
<td>$0.0020 -0.0020i$</td>
<td>$0.0021 -0.0021i$</td>
<td>$0.0020 -0.0020i$</td>
<td></td>
</tr>
<tr>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$-0.0010 -0.0010i$</td>
<td>$-0.0010 -0.0010i$</td>
<td>$-0.0010 -0.0010i$</td>
<td>$-0.0010 -0.0010i$</td>
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<tr>
<td>0.0030 $-0.0030i$</td>
<td>$0.0031 -0.0031i$</td>
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</tr>
<tr>
<td>0.0</td>
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<td>0.0</td>
<td>0.0</td>
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</tr>
<tr>
<td>$0.0052 -0.0052i$</td>
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<tr>
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<td>$0.0018 -0.0018i$</td>
<td>$0.0017 -0.0017i$</td>
<td></td>
</tr>
</tbody>
</table>

Table 11: Values of $E$ and $H$ at different points in the cube.
4. Conclusion and Future Work

Existing nodal finite element methods for electromagnetic problems either use some kind of a penalty or non-standard interpolation functions. A formulation based on potentials works well for convex domains, even in the presence of inhomogeneities, but fails to predict singular eigenvalues in the case of nonconvex domains besides predicting some spurious eigenvalues.

In this work, we show that mixed finite element formulations are capable of yielding accurate solutions for any domain (convex or non-convex), even in the presence of inhomogeneities. The proposed formulation not only yields accurate values for the eigenvalues but even predicts their multiplicity accurately. In contrast to existing formulations, no penalty terms are added to the original variational formulation, and no factors need to be adjusted in order to obtain the results. Thus, there is very strong numerical evidence that the proposed elements (except the A4 element) satisfy the LBB (inf-sup) conditions [27].

Currently, although the two-dimensional formulation works for arbitrary geometries, the three-dimensional elements work only for regular geometries, but otherwise yield accurate results even in the presence of singularities.

Acknowledgment

The first author gratefully acknowledges many helpful discussions with Prof. Daniele Boffi.

References


