Minimal Binary Abelian Codes of length $p^m q^n$

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Abstract—We consider binary abelian codes of length $p^m q^n$ where $p$ and $q$ are prime rational integers under some restrictive hypotheses. In this case, we determine the idempotents generating minimal codes and either the respective weights or bounds of these weights. We give examples showing that these bounds are attained in some cases.

Index Terms—group algebra, code weight, primitive idempotent, minimal abelian codes.

I. INTRODUCTION

GENERATORS of minimal abelian codes were determined in [1] together with their corresponding dimensions and weights under the following hypotheses:

- $G$ a finite abelian group of exponent $p^m$
  (or $2p^m$, with $p$ odd)
- $F$ a field with $q$ elements,
- $q$ with multiplicative order $\varphi(p^m)$ mod $p^m$.

Here we extend these ideas to study groups of the form $G \times G_q$, where $G_p$ and $G_q$ denote abelian groups, the first a $p$-group and the second a $q$-group, satisfying the following conditions which will allow us to use the results in [1]. We shall denote by $U(Z_n)$ the group of units of $Z_n$.

(i) $\gcd(p - 1, q - 1) = 2$,
(ii) $2$ generates the groups $U(Z_{p^2})$ and $U(Z_{q^2})$ (2)
(iii) $\gcd(p - 1, q) = \gcd(p, q - 1) = 1$.

Notice that hypothesis (i) above implies that at least one of the primes $p$ and $q$ is congruent to 3 (mod 4). In what follows, to fix notations, we shall always assume $q \equiv 3$ (mod 4).

Let $F_2$ be the field with 2 elements. In Section III we compute, for each minimal code in $F_2(G_p \times G_q)$, the generating primitive idempotent, the dimension and give explicitly a basis over $F_2$.

Primitive idempotents are usually determined using the characters of the group over a splitting field $L$ and then using Galois descent (see [3] Lemma 9.18). In what follows, we avoid the use of this technique by deriving the expression of the primitive idempotents from the subgroup structure of $G$.

In the later sections we show how to study the corresponding code weights in several cases.

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II. BASIC FACTS

Let $F_p$ be the Galois field with $p$ elements. In this section we list some results on finite fields and elementary number theory that will be needed in the sequel. Our first result is well-known.

Lemma II.1. Let $p$ be a positive prime number and $r, s \in \mathbb{N}^*$. Then

$$F_{p^r} \otimes_{F_p} F_{p^s} \cong \mathbb{F}_{p^{\gcd(r,s)}}.$$ (3)

Remark II.2. Notice that any extension $L$ of $F_2$ of even degree contains a subfield $K$ with four elements, hence there exists an element $1 \neq a \in L$ such that $a^3 = 1$.

Lemma II.3. Let $r, s \in \mathbb{N}$ be non-zero numbers such that $\gcd(r, s) = 2$. Let $u \in \mathbb{F}_{2^r}$ and $v \in \mathbb{F}_{2^s}$ be elements satisfying the equation $x^2 + x + 1 = 0$. Then

$$\mathbb{F}_{2^r} \otimes_{\mathbb{F}_2} \mathbb{F}_{2^s} \cong \mathbb{F}_{2^r} \oplus \mathbb{F}_{2^s}.$$ (3)

And $e_1 = (u \otimes v) + (u^2 \otimes v^2)$ and $e_2 = (u \otimes v^2) + (u^2 \otimes v)$ are the primitive idempotents generating the simple components of $F$.

Proof: The decomposition of $\mathbb{F}_{2^r} \otimes_{\mathbb{F}_2} \mathbb{F}_{2^s}$, as a direct sum follows from Lemma II.1.

Since $u, u^2 \in \mathbb{F}_{2^r}$ (resp. $v, v^2 \in \mathbb{F}_{2^s}$) are linearly independent over $\mathbb{F}_2$, we have that $(u \otimes v), (u^2 \otimes v^2), (u \otimes v^2)$ and $(u^2 \otimes v)$ are linearly independent in $\mathbb{F}_{2^r} \otimes_{\mathbb{F}_2} \mathbb{F}_{2^s}$. Hence $e_1 \neq 0$ and $e_2 \neq 0$. As $1 + v + v^2 = 0$, $1 + u + u^2 = 0$, and also $u^3 = v^3 = 1$, we obtain:

$$e_1 \cdot e_2 = (u^2 \otimes 1) + (1 \otimes v^2) + (1 \otimes v) + (u \otimes 1) = (u^2 + u) \otimes 1 + 1 \otimes (v + v^2) = 0$$

and also

$$e_1 + e_2 = (u \otimes v) + (u^2 \otimes v^2) + (u \otimes v^2) + (u^2 \otimes v) = u \otimes (v + v^2) + u^2 \otimes (v + v^2) = 1 \otimes 1.$$ (4)

As $\mathbb{F}_{2^r} \oplus \mathbb{F}_{2^s}$ has two simple components, $e_1$ and $e_2$ are, in fact, the corresponding primitive idempotents.

We shall also need the following result whose proof is elementary.

Lemma II.4. Let $p$ and $q$ be two distinct odd primes such that $\gcd(p - 1, q - 1) = 1$ and $2$ generates both groups of units $U(\mathbb{Z}_p)$ and $U(\mathbb{Z}_q)$. Then the least positive integer $k$ such that $2^k \equiv 1 \pmod {pq}$ is $\text{lcm}(p - 1, q - 1) = \frac{(p-1)(q-1)}{2}$.

Remark II.5. Let $p$ be an odd prime. Denote by $Q$ the set of non-zero elements in $\mathbb{Z}_p$ which are quadratic residues modulo $p$ and by $N$ the set of non-zero elements in $\mathbb{Z}_p$ which are non-quadratic residues modulo $p$. By [22] Theorem 79], we have $\mathbb{Z}_p = \{0\} \cup Q \cup N$, hence $|Q| = |N| = \frac{p-1}{2}$. 

We note that \( \mathbb{Z}_p \) is the union of the following six disjoint subsets:

\[
\begin{align*}
\{0\}, & \quad \{1\}, \\
Q_Q & = \{ x \in \mathbb{Z}_p : x \in Q \text{ and } x - 1 \in Q \}, \\
Q_N & = \{ x \in \mathbb{Z}_p : x \in Q \text{ and } x - 1 \in N \}, \\
N_Q & = \{ x \in \mathbb{Z}_p : x \in N \text{ and } x - 1 \in Q \}, \\
N_N & = \{ x \in \mathbb{Z}_p : x \in N \text{ and } x - 1 \in N \}.
\end{align*}
\]

For any odd prime \( p \), clearly \( \bar{1} \in Q \) and if \( \bar{2} \) generates \( U(\mathbb{Z}_p) \), then \( \bar{2} \in N \). As \( 1 \in Q \), we have \( Q = \{1\} \cup Q_Q \cup Q_N \) and \( N = N_Q \cup N_N \).

**Lemma II.6. (H Theorem 83)** Let \( p \) be an odd prime. The element \( \bar{1} \) is a quadratic residue in \( \mathbb{Z}_p \) if and only if \( p \equiv 1 \pmod{4} \).

The next elementary result is known and appears as Exercise 18, p. 149 in [7].

**Lemma II.7. (a) Let \( p \equiv 3 \pmod{4} \). Then \( |Q_Q| = |Q_N| = \frac{p-1}{2} \) and \( |N_Q| = \frac{p+1}{2} \).**

(b) Let \( p \equiv 1 \pmod{4} \). Then \( |N_Q| = |Q_N| = \frac{p-1}{2} \) and \( |Q_Q| = \frac{p+1}{2} \).

### III. Primitive Idempotents

Let \( G \) be a group. For a subgroup \( H \leq G \), we set \( \hat{H} = \sum_{h \in H} h \) and, for an element \( x \in G \), we set \( \hat{x} = (x) \).

It was shown in [8] that, under the hypotheses in [1], if \( A \) is an abelian \( p \)-group, then the primitive idempotents of \( A \) can be constructed as follows: for each subgroup \( H \) of \( A \) such that \( A/H \neq 1 \) is cyclic, consider the unique subgroup \( H^* \) of \( A \) containing \( H \) such that \( [H^*/H] = p \). We define \( e_H = \hat{H} - \hat{H}^* \). Then, these elements \( e_H \), together with \( e_A = A \) are precisely the primitive idempotents of \( FA \).

Moreover, if \( |A : H| = p^r \) then the dimension of the ideal generated by an idempotent of the form \( e_H \) is

\[
dim_{F}(FA)e_H = dim_{F}[A/H] - dim_{F}[A/H^*] = p^r - (p - 1)
\]

(see [8] formula(1),p. 390)).

Let \( G_p \) and \( G_q \) be finite abelian groups, the first a \( p \)-group and the second a \( q \)-group and set \( G = G_p \times G_q \). For each subgroup \( H \) in \( G \) such that \( G_p/H \neq 1 \) is cyclic, consider the idempotent \( e_H = \hat{H} - \hat{H}^* \) as above and, similarly, consider primitive idempotent of the form \( e_K = \hat{K} - \hat{K}^* \) for \( G_q \).

Clearly \( \hat{G_p} \cdot \hat{G_q} = \hat{G_p} \times \hat{G_q} \) is a primitive idempotent of \( \mathbb{F}_2(G_p \times G_q) \).

We claim that idempotents of the form \( \hat{G_p} \cdot e_K \) are primitive. In fact, we have \( (\mathbb{F}_2(G_p \times G_q) \cdot e_K = (\mathbb{F}_2G_p \cdot G_q) \cdot e_K \cong (\mathbb{F}_2G_q) \cdot e_K \) which is a field. In a similar way, it follows that idempotents of the form \( e_H \cdot \hat{G_q} \) are primitive.

Finally, we wish to show that an idempotent of the form \( e_H \cdot e_K \) decomposes as the sum of two primitive idempotents in \( \mathbb{F}_2G \).

To do so, write \( e_H = \hat{H} - \hat{H}^* \) and set \( a \in H^* \setminus H \). Notice that \( aH \) is a generator of \( H^*/H \). Set

\[
\begin{align*}
u & = \begin{cases} a^q + a^{q^2} + \cdots + a^{q^{p-3}}, & \text{if } p \equiv 1 \pmod{4} \\ a^{q^2} + a^{q^3} + \cdots + a^{q^{p-3}}, & \text{if } p \equiv 3 \pmod{4} \end{cases}
\end{align*}
\]

and

\[
\begin{align*}u & = \begin{cases} a^q + a^{q^2} + \cdots + a^{q^{p-2}}, & \text{if } p \equiv 1 \pmod{4} \\ a^q + a^{q^3} + \cdots + a^{q^{p-2}}, & \text{if } p \equiv 3 \pmod{4} \end{cases}
\end{align*}
\]

Setting \( (u\hat{H})^2 = u\hat{H} \), a direct computation shows \( (u\hat{H})^2 = u\hat{H} \) and \( (u\hat{H})^2 + u\hat{H} + e_H = 0 \) (recall that the unity of \( \mathbb{F}_2G_p \cdot e_H \) is precisely \( e_H \)).

Also we have

\[
\begin{align*}u\hat{H} - \hat{H}^* = u\hat{H} - u\hat{H}^* = u\hat{H} - \varepsilon(u)\hat{H}^*,
\end{align*}
\]

where \( \varepsilon(u) \) is the number of summands in \( u \), which is always even, hence \( \varepsilon(u) = 0 \) and so \( u\hat{H} = \hat{H} = u\hat{H}^* = u\hat{H} \). Similarly, we also have \( u\hat{H} = \hat{H} \). Consequently, we see that both \( u\hat{H} = u\hat{H} \) and \( u\hat{H} \) lie in \( \mathbb{F}_2G_p(\hat{H} - \hat{H}^*) \).

Similarly, for \( e_K = \hat{K} \), let \( b \in K^* \setminus K \) and define \( v \) as in (5) and \( v' \) as in (7) (replacing \( a \) by \( b \)). Hence \( v\hat{K} = v'\hat{K} = (v\hat{K})^2 \) in \( \mathbb{F}_2G_q(\hat{K} - \hat{K}^*) \).

Notice that, by equation (5), we have \( dim_{F}(\mathbb{F}_2G_p) e_H = p^{r-1}(p - 1) \) and \( dim_{F}(\mathbb{F}_2G_q) e_K = q^{s-1}(q - 1) \), where \( r = [G_p : H] \) and \( s = [G_q : K] \). Thus

\[
\begin{align*}(\mathbb{F}_2G_p)e_H \cong \mathbb{F}_{2^{p r-1}(p - 1)} \quad \text{and} \quad (\mathbb{F}_2G_q)e_K \cong \mathbb{F}_{2^{q s-1}(q - 1)}.
\end{align*}
\]

As \( \gcd(p^{r-1}(p - 1), q^{s-1}(q - 1)) = 2 \) we can apply Lemma II.4 to see that

\[
\begin{align*}e_1 & = e_1(H, K) = u\hat{H} \cdot v\hat{K} + u'\hat{H} \cdot v'\hat{K} \\
e_2 & = e_2(H, K) = u\hat{H} \cdot v'\hat{K} + u'\hat{H} \cdot v\hat{K}
\end{align*}
\]

are primitive orthogonal idempotents such that \( e_1 + e_2 = e_{HEK} \).

Hence, we have shown the following.

**Theorem III.1.** Let \( G_p \) and \( G_q \) be abelian \( p \) and \( q \)-groups, respectively, satisfying the conditions in (2). For a group \( G \), denote by \( S(G) \) the set of subgroups \( N \) of \( G \) such that \( G/N \neq 1 \) is cyclic. Then the set of primitive idempotents in \( \mathbb{F}_2(G_p \times G_q) \) is:

\[
\begin{align*}G_p \cdot \hat{G_q}, & \\
G_p \cdot e_K & \quad K \in S(G_q), \\
e_H \cdot \hat{G_q}, & \quad H \in S(G_p),
\end{align*}
\]

\[
\begin{align*}e_1(H, K), & \\
e_2(H, K), & \quad H \in S(G_p), K \in S(G_q).
\end{align*}
\]

In the following sections we shall use this result to study minimal codes in some special cases and, in these cases, we shall also study the corresponding weights and give explicit bases for these minimal codes.
IV. CODES IN $\mathbb{F}_2(C_p \times C_q)$

Let $p \neq q$ be odd primes. In this section we shall consider the group $G = \langle g \mid g^{pq} = 1 \rangle$, denote $a = g^q$, $b = g^p$ and write $G = C_p \times C_q$, where $C_p = \langle a \rangle$ and $C_q = \langle b \rangle$. Theorem I.1 above, in this context gives the following.

**Theorem IV.1.** Let $G = \langle a \times b \rangle$ be as above and assume that $p$ and $q$ satisfy $[2]$. Then the primitive idempotents of $\mathbb{F}_G$ are $e_0 = \bar{G}$, $e_1 = \bar{\alpha} (1 - \bar{b})$, $e_2 = (1 - \bar{a} \bar{b})$, $e_3 = uv + u^2v^2$ and $e_4 = uv^2 + u^2v$, where $u$ and $v$ are as in (6) and (7) above.

We introduce some notation. For each $0 \leq i \leq pq - 1$, we write

$$g^i = g^{(i_1,i_2)},$$

where $i_1 \equiv i \pmod{p}$ and $i_2 \equiv i \pmod{q}$, and, for given sets $X \subset \mathbb{Z}_p$ and $Y \subset \mathbb{Z}_q$, we write $g^j = (X,Y)$ to indicate $g^{(j_1,j_2)}$, with $(j_1,j_2) \in (X,Y)$.

For a prime $p > 0$, we write $Q_p$ for the set of quadratic residues modulo $p$, $N_p$ for the set of non quadratic residues modulo $p$ and specialize the notation (2) to

$$Q_p^p = \{ x \in \mathbb{Z}_p : x \in Q_p \text{ and } x \equiv 1 \pmod{4} \},$$

$$Q_p^N = \{ x \in \mathbb{Z}_p : x \in Q_p \text{ and } x \equiv 1 \pmod{4} \},$$

$$N_p^p = \{ x \in \mathbb{Z}_p : x \in N_p \text{ and } x \equiv 1 \pmod{4} \},$$

$$N_p^N = \{ x \in \mathbb{Z}_p : x \in N_p \text{ and } x \equiv 1 \pmod{4} \}.$$

**Lemma IV.2.** Let $G$ be an abelian group of order $pq$ as in Theorem IV.1. Then the $2$-cyclic classes of $G$ are:

- $C_1 = \{1\}$
- $C_g = \{g^j \mid (i,j) \in (Q_p^p, Q_p^N) \cup (N_p^p, N_p^N)\}$
- $C_a = C_{gq} = \{g^j g^{2q} g^{2q^2} g^{2q^3} \cdots g^{pq-2q} \}$
- $C_a = C_{gq} = \{g^j (i,j) \in (Q_p^N, 0) \cup (N_p^p, 0)\}$
- $C_b = C_{gq} = \{g^j g^{2q} g^{2q^2} g^{2q^3} \cdots g^{pq-2q} \}$
- $C_a = C_{gq} = \{g^j (i,j) \in (Q_p^N, 0) \cup (N_p^p, 0)\}$

and either

- (a) $C_{g^{p+q}} = \{g^{(i,j)} \mid (i,j) \in (Q_p^q, N_p^q) \cup (N_p^q, Q_p^q)\}$
- (b) $C_{g^{-1}} = \{g^{(i,j)} \mid (i,j) \in (Q_p^N, N_p^N) \cup (N_p^N, Q_p^N)\}$

**Proof:** Since 2 generates $U(\mathbb{Z}_p)$ and $U(\mathbb{Z}_q)$, the least positive integer $i$ such that $(g^i)^2 = g^i$ is $p-1$ and, similarly, the least positive integer $j$ such that $(g^j)^2 = g^j$ is $q-1$. Hence, $|C_a| = p-1$ and $|C_b| = q-1$. By Lemma I.4 we have $|C_g| = \frac{(p-1)(q-1)}{4}$.

Note that, using the notation in (5), we can write

$$C_y = \{g^{(i,j)} \mid i \equiv 2k \pmod{p}, j \equiv 2k \pmod{q}, 0 \leq k \leq |C_g| - 1\}.$$

Since $(2, 2) \in (N_p^q, N_p^q)$, all elements in $C_y$ are of the form $g^{2i} = g^{(i_1,i_2)}$, with $(i_1, i_2) \in (Q_p^q, Q_p^q) \cup (N_p^q, N_p^q)$. We claim that all pairs in $(Q_p^q, Q_p^q) \cup (N_p^q, N_p^q)$ do appear as powers of elements in $C_y$. This is so because $|\{Q_p^q, Q_p^q) \cup (N_p^q, N_p^q)\} = \frac{(p-1)(q-1)}{2}$. Hence, $C_{g^{p+q}} = \{g^{(2k,2k)} : k \in \mathbb{Z}\}$.

Note that $C_{g^v} \subset \{g^{(2k,2k)} : v \mid i \in \mathbb{Z}\}$. As 2 generates $U(\mathbb{Z}_p)$ and $q$ is invertible mod $p$, we can write

$$C_{g^v} = \{g^{(2k,2k)} : (2k,0) \in (Q_p^N, 0) \cup (N_p^N, 0)\}.$$ Similarly, $C_{g^v} = \{g^{(0,2k)} : (0,2k) \in (Q_p^N, 0) \cup (N_p^N, 0)\}$.

To determine the last cyclotomic class, we shall consider two separate cases.

**Case 1.** $p = 3 \pmod{4}$ and $q = 3 \pmod{4}$. By the Quadratic Reciprocity Law we have $p \not\mid Q_q$ (mod $q$) if and only if $q \not\mid P_q$ (mod $p$), hence the element $g^{pq-1} = g^{(p,0)} \in (N_p^q, Q_p^q) \cup (Q_p^q, N_p^q)$ so $g^{pq-1} \not\in C_y$. Obviously, $1 \not\in C_y \not\in C_{g^p} \cup C_{g^q}$.

Consequently, in this case, the fifth 2-cyclotomic class is:

$$C_{g^{-1}} = \{g^{-1}, g^{-2}, g^{-2}, \cdots, g^{-2} (2(p-1)q^{-1}) \}.$$
Proof: The validity of (i) is obvious. To prove (ii), notice that
\[(F_2G)e_1 = (F_2G)\hat{a}(1 - \hat{b}) \cong (F_2C_q)(1 - \hat{b})\]
and this isomorphism maps the element \(x \in (F_2C_q)(1 - \hat{b})\) to \(x\hat{a} \in (F_2G)e_1\). As the set \(\{b^i - 1 | 0 < j \leq q - 1\}\) is a basis of \((F_2C_q)(1 - \hat{b})\) (see Proposition 3.2.10, p.133), it follows that \(B_1\) is a basis of \((F_2G)e_1\).

To prove that \(B_1\) is also a basis of \((F_2G)e_1\), we prove first that \(\{b^i - \hat{b} | 1 \leq j \leq q - 1\}\) is a basis of \((F_2C_q)(1 - \hat{b})\). To do so, it suffices to show that it is linearly independent, as it contains precisely \(q - 1\) elements.

Assume that there exist coefficients \(x_j \in F_2, 1 \leq j \leq q - 1\), such that \(\sum_{j=1}^{q-1} x_j(b^j - \hat{b}) = 0\). If \(\sum_{j=1}^{q-1} x_j = 0\), then \(x_j = 0\) for all \(j, 1 \leq j \leq q - 1\). If \(\sum_{j=1}^{q-1} x_j = 1\), then \(\sum_{j=1}^{q-1} x_j(b^j + \hat{b}) = 0\) so we must have \(x_j = 1\), for all \(1 \leq j \leq q - 1\), which implies \(\sum_{j=1}^{q-1} x_j = 0\), a contradiction.

Because of the isomorphism above, it follows that also \(B_1\) is a basis of \((F_2G)e_1\).

The proof of (iii) is similar.

To prove (iv), notice that, by Lemma A.1,
\[\dim_{F_2}(F_2G)e_3 = \frac{(p-1)(q-1)}{2}\]. Also, \((F_2G)e_3 = F_2(g_e)\) is a finite field and \(g_e\) is a root of an irreducible polynomial of degree \(\frac{(p-1)(q-1)}{2}\). Hence, the set \(\{e_3, g_e^3, g_e^2, \ldots, g_e^{\frac{(p-1)(q-1)}{2} - 1}g\}\) is a basis of \((F_2G)e_3\).

We shall prove independently in Lemma IV.5 that the element
\[y = (1 + a^s)(1 + b^t)e_3 \in (F_2G)e_3\]
is nonzero. Then \(\{y, g_y^3, g_y^2, \ldots, g_y^{\frac{(p-1)(q-1)}{2} - 1}g\}\) is also a basis of \((F_2G)e_3\).

The proof of (v) is a consequence of the isomorphism \((F_2G)e_3 \cong (F_2G)e_4\).

Corollary IV.4. Let \(G\) be as above. The dimensions of the minimal ideals of \(F_2G\) are:

(i) \(\dim_{F_2}(F_2G)e_0 = 1\).
(ii) \(\dim_{F_2}(F_2G)e_1 = q - 1\).
(iii) \(\dim_{F_2}(F_2G)e_3 = p - 1\).
(iv) \(\dim_{F_2}(F_2G)e_4 = (p - 1)(q - 1)/2\).
(v) \(\dim_{F_2}(F_2G)e_5 = (p - 1)(q - 1)/2\).

We now compute the weight of a particular element of \(F_2G\).

Lemma IV.5. With the same hypothesis of the Theorem IV.7 and notation above, the element
\[y = (1 + g^{sq})(1 + g^{tp})e_3 = (1 + a^s)(1 + b^t)e_3 \in (F_2G)e_3,\]
with \(s, t \in \mathbb{Z}\) such that \(sq \equiv 1 \mod p\) and \(tp \equiv 1 \mod q\), has weight \(p + q\).

Proof: Since \(p \in U(\mathbb{Z}_q)\) and \(q \in U(\mathbb{Z}_p)\), by Theorem 69, we can choose \(s, t \in \mathbb{Z}^*\) such that \(sq \equiv 1 \mod p\) and \(tp \equiv 1 \mod q\). For this choice of \(s, t \in \mathbb{Z}^*\), we have \(y = (1 + g^{sq})(1 + g^{tp})e_3 = (1 + g^{(1,0)})(1 + g^{(0,1)})e_3 \in (F_2G)e_3\). We shall consider two cases.

Case 1: \(p \equiv 3 \mod 4\) and \(q \equiv 3 \mod 4\). In this case, we have
\[e_3 = \frac{(g + g^2 + g^{2^2} + \cdots + g^{2^{\frac{(p-1)(q-1)}{2}}})}{S_q} + \frac{(g^{p} + g^{2p} + \cdots + g^{(q-1)p})}{S_p} + \frac{(g^q + g^{2q} + \cdots + g^{(p-1)q})}{S_{qr}}\]
As in the proof of Lemma IV.2, \(S_q\) is the sum of all elements of \(G\) with exponents in \((Q^p, Q^q) \cup (N^p, N^q)\), \(S_{qr}\) is the sum of all elements with exponents in \((0, Q^q) \cup (0, N^q)\) and \(S_{qr}\) is the sum of all elements with exponents in \((Q^p, 0) \cup (N^p, 0)\). Thus \(e_3 = (p-1)(q-1) + (p-1) + (q-1)\).

Now we set \(e_3 = S_q + (1 + S_{qr}) + (1 + S_{qr})\) and claim that \(y(1 + S_{qr}) + 1 + S_{qr}) = 0\). Indeed,
\[(1 + g^{\frac{1}{2}})(1 + g^{\frac{p-1}{2}}) + (1 + S_{qr}) = (1 + S_{qr}) + (1 + S_{qr}) + g^{(1,0)}[(1 + S_{qr}) + 1 + S_{qr})] + g^{(1,1)}[(1 + S_{qr}) + (1 + S_{qr})] = (1 + S_{qr}) + (1 + S_{qr}) + g^{(1,0)}(1 + S_{qr}) + (1 + S_{qr}) + g^{(1,1)}[(1 + S_{qr}) + (1 + S_{qr})] = 0\]

To prove the last equality it is convenient to write all the elements in the form \(g^{(1,1)}\). In this way it is clear that each element appears twice in the sum, hence its coefficient is zero.

Thus \(\omega(y) = \omega(yS_q)\) and it is enough to compute the weight of \(yS_q\). To do so, note that \(\mathbb{Z}_p\) is the union of six disjoint subsets as in Remark 1.5 and, for \(\mathbb{Z}_p \times \mathbb{Z}_q\), we have:
\[N^p \times N^q = \{(N^p_0, N^q_0), (N^p_0, N^q_1), (N^p_1, N^q_0), (N^p_1, N^q_1)\}\]
and \(Q^p \times Q^q = \{(Q^p_0, Q^q_0), (Q^p_0, Q^q_1), (Q^p_1, Q^q_0), (Q^p_1, Q^q_1)\}\)

Now we get
\[\{(1, 0) + (Q^p \times Q^q)\} = \{(N^p_0, N^q_0), (N^p_0, N^q_1), (N^p_1, N^q_0), (N^p_1, N^q_1)\}\]
\[\{(0, 0) + (N^p \times N^q)\} = \{(N^p_0, N^q_0), (N^p_0, N^q_1), (N^p_1, N^q_0), (N^p_1, N^q_1)\}\]
\[\{(1, 0) + (Q^p \times Q^q)\} \cup (N^p \times N^q)\} = \{(Q^p_0, Q^q_0), (Q^p_0, Q^q_1), (Q^p_1, Q^q_0), (Q^p_1, Q^q_1)\}\]
\[\{(0, 0) + (Q^p \times Q^q)\} \cup (N^p \times N^q)\} = \{(Q^p_0, Q^q_0), (Q^p_0, Q^q_1), (Q^p_1, Q^q_0), (Q^p_1, Q^q_1)\}\]

Thus the elements that appear in \(yS_q\) are all those with exponents in the set \(\{(Q^p_0, 1), (Q^p_1, 1), (0, N^q_1), (N^p_0, 0), (Q^p_0, 0), (0, Q^q_0), (1, 0), (0, 0)\}\), whose cardinality, by Lemma A.7 is \(2 + 2(Q^p_0) + 2Q^q_0 + 2N^p_1 + 2N^q_1 + 2N^q_0 + 2 + 2\frac{p+1}{q} + 2\frac{q+1}{p} + 2\frac{p+3}{q} + 2\frac{q+3}{p} = p + q\).

Case 2: \(p \equiv 1 \mod 4\) and \(q \equiv 3 \mod 4\). In this case, we have
where, as in the proof of Lemma IV.5, $S_g$ is the sum of all elements of $G$ with exponents in $(Q^p, Q^p) \cup (N^p, N^p)$ and $S_{g^q}$ is the sum of all elements with exponents in $(Q^p, 0) \cup (N^p, 0)$. Thus $\omega(e_3) = \frac{(p-1)(q-1)}{2} + (p-1)$.

We claim that $\omega(y) = p + q$. Indeed, we can rewrite $S_{g^q}$ as $S_{g^q} = g^{(1,0)} + g^{(2,0)} + \cdots + g^{(p-1,0)}$. Thus $(1 + g^{(1,0)})(1 + g^{(0,0)})S_{g^q} = g^{(1,0)} + g^{(2,0)} + \cdots + g^{(p-1,0)} + g^{(1,0)} + g^{(2,1)} + \cdots + g^{(p-1,1)} + g^{(1,0)} + g^{(2,1)} + \cdots + g^{(p-1,1)} = 1 + g^{(1,0)} + g^{(0,1)} + g^{(1,1)}$.

In $(1 + g^{(1,0)})(1 + g^{(0,1)})S_g$, as shown in Case 1, all the elements have exponents in the set $\{(Q^n_1, 1), (Q^n_1, 0), (N^n_1, 1), (N^n_1, 0), (Q^n_1, 0), (0, Q^n_1), (1, (0))\}$. Therefore, the elements that appear in $a$ are all those with exponents in the set $\{g^{(1,0)}, (1, Q^n_1), (1, 0), (N^n_1, 1), (N^n_1, 0), (Q^n_1, 0), (0, Q^n_1), (1, (0))\}$, whose cardinality, by Lemma IV.7, is also $2 + 2|Q^n_1| + 2|Q^n_1| + 2|N^n_1| + 2|N^n_1| = 2 + 2\frac{|Q^n_1|}{p} + 2\frac{|Q^n_1|}{q} + 2\frac{|Q^n_1|}{p} + 2\frac{|Q^n_1|}{q} = p + q$.

Remark IV.6. Lemma IV.5 shows that the elements in the bases defined in parts (iv) and (v) of Proposition IV.3 have all the same weight $p + q$.

Theorem IV.7. Let $G = < g >$ be an abelian group of order $pq$ as in Theorem IV.7. Then:

(i) $\omega(\mathbb{F}_G(e_0)) = p + q$.

(ii) $\omega(\mathbb{F}_G(e_1)) = 2p$.

(iii) $\omega(\mathbb{F}_G(e_2)) = 2q$.

(iv) $4 \leq \omega(\mathbb{F}_G(e_3)) \leq p + q$.

(v) $4 \leq \omega(\mathbb{F}_G(e_4)) \leq p + q$.

Proof: (i) follows immediately as $\mathbb{F}_G(e_0) \cong \mathbb{F}_2$.

(ii) Recall that $e_1 = a(1 - b)$. Since $b = b^2 e_1 = (b + b^2) a \in (\mathbb{F}_2)[e_1]$ and $\text{supp}(a) \cap \text{supp}(b^2 a) = \emptyset$ then $\omega((b + b^2) a) = 2p$ hence $\omega(\mathbb{F}_G(e_1)) \leq 2p$.

An arbitrary element $a \in (\mathbb{F}_2)[e_1]$ is of the form

$$\alpha = \sum_{i,j} k_{ij} a^j b^i \omega(1 - b) = \left( \sum_{i = 0}^{p-1} \left( \sum_{j = 0}^{q-1} k_{ij} \right) b^j \right) a,$$

with $k_{ij} \in \mathbb{F}_2$, hence also an element of $(\mathbb{F}_2)[a]$. An element $\beta \in (\mathbb{F}_2)[a]$ is of the form

$$\beta = \sum_{i,j} \ell_{ij} a^j b^i \omega(1 - b) = \left( \sum_{i = 0}^{p-1} \left( \sum_{j = 0}^{q-1} \ell_{ij} \right) b^j \right) \omega(a),$$

with $\ell_{ij} \in \mathbb{F}_2$. Thus a nonzero element $\beta \in (\mathbb{F}_2)[a]$ has weight $\omega(\beta) = np$, with $n \geq 1$, as the elements $b^j \omega(1 - b)$, for different values of $j$ have disjoint supports.

Now as $b \omega(a) = 1 \neq b^2 \omega(1 - b)$, the element $b \omega(a) \notin (\mathbb{F}_2)[e_1]$. Hence, for an element $\alpha \in (\mathbb{F}_2)[e_1]$ to have weight $p$, we must have $\alpha = b \omega(a)$, for some $j$, a contradiction. Therefore, $2p$ is the minimum weight of the code $(\mathbb{F}_2)[e_1]$.

(iii) follows as (ii) interchanging $a$ with $b$ and $p$ with $q$.

For (iv) and (v) it is enough to compute the weight of one of these codes, since there exists an automorphism of $\mathbb{F}_2 G$ induced by a group automorphism of $G$ that maps one code into the other, hence they are equivalent.

As $(1 + a)(1 + b)(e_3 + e_4) = (1 + a)(1 + b)(1 + a)(1 + b) = (1 + a)(1 + b)$, then $(1 + a)(1 + b) \in (\mathbb{F}_2 G)(e_3 + e_4)$. Besides, it is easy to prove that there is no element of weight 2 in $(\mathbb{F}_2 G)(e_3 + e_4)$ and, as $e_3, e_4 \in (\mathbb{F}_2 G)(e_3 + e_4)$, then $4 \leq \omega([\mathbb{F}_2 G](e_3 + e_4)) \leq \omega([\mathbb{F}_2 G] c)$, for $j = 3, 4$. By Lemma IV.5 we have $\omega([\mathbb{F}_2 G] c) \leq p + q$.

A. Examples

Example IV.8. The upper bound for the weights of the codes in parts (iv) and (v) of Theorem IV.7 is sharp, as it is attained by the code generated by the primitive idempotent

$e = g + g^2 + g^3 + g^4 + g^6 + g^8 + g^9 + g^{12} \in \mathbb{F}_2 C_{15}$.

Indeed, the group code $I = (\mathbb{F}_2 C_{15})$ has dimension 4 over $\mathbb{F}_2$ and it is easy to see that $I = \{g^j e | j = 0, \ldots, 14\} \cup \{0\}$. Hence all non zero elements in $I$ have weight equal to $\omega(e) = 8$.

However, this is not always the case as we can see below.

Example IV.9. Let $C_{33} = \{g | g^{33} = 1\}$ be the cyclic group with 33 elements and $(\mathbb{F}_2 C_{33})$ be the group code generated by the primitive idempotent $e = g + g^2 + g^3 + g^4 + g^6 + g^8 + g^9 + g^{11} + g^{12} + g^{15} + g^{16} + g^{17} + g^{18} + g^{21} + g^{22} + g^{24} + g^{25} + g^{27} + g^{29} + g^{30} + g^{31} + g^{32}$. Then the weight distribution of $(\mathbb{F}_2 C_{33}) e_3$ is as follows:

<table>
<thead>
<tr>
<th>Vector Weight</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Vectors</td>
<td>165</td>
<td>165</td>
<td>165</td>
<td>330</td>
<td>165</td>
<td>33</td>
</tr>
</tbody>
</table>

In fact, notice first the ideal $(\mathbb{F}_2 C_{33}) e$ is a field over $(\mathbb{F}_2 C_{33}) e$. Also $((g + g^2)^{32} = (g - 1 + g)e)$, thus $x = (g + g^2)^{32}$ is an element of order equal to either 1 or 31 inside $U((\mathbb{F}_2 C_{33}) e)$. But $x \neq e$, as $\omega(x) = 18$ and $\omega(e) = 22$. Hence $U((\mathbb{F}_2 C_{33}) e) = C_{33} \cdot e \times \{x\}$. Computing the 2-cyclographic classes in $\{x\}$ we get:

$U_0^* = \{0\}$.

$U_1^* = \{x, x^2, x^4, x^8, x^{16}\}$.

$U_2^* = \{x^3, x^6, x^{12}, x^{24}, x^{17}\}$.

$U_3^* = \{x^5, x^{10}, x^{30}, x^9, x^{18}\}$.

$U_4^* = \{x^7, x^{14}, x^{28}, x^{25}, x^{19}\}$.

$U_5^* = \{x^{11}, x^{22}, x^{13}, x^{26}, x^{21}\}$.

$U_6^* = \{x^{15}, x^{30}, x^{29}, x^{27}, x^{23}\}$.

For a fixed $0 \leq k \leq 31$, we have $\omega(g^x k) = (x^k)$, for all $0 \leq j \leq 32$ and for each $0 \leq t \leq 6$, all $y \in U_1^*$ have the same weight.

Using these facts to compute the weights, we have that:

- There are 33 distinct elements of weight 22 in $(\mathbb{F}_2 C_{33}) e_3$.
- Since $\omega(e_3) = 22$. 

There are 330 distinct elements in $(F_2C_{33})e_3$ with weight 18, as $\omega(x) = 18$, $\omega(x^4) = 18$ and $U'_1 \cap U'_3 = \emptyset$.

There are 165 distinct elements in $(F_2C_{33})e_3$ with weight 16, since $\omega(x^3) = 16$.

There are 165 distinct elements in $(F_2C_{33})e_3$ with weight 20, since $\omega(x^7) = 20$.

There are 165 distinct elements in $(F_2C_{33})e_3$ with weight 12, since $\omega(x^{11}) = 12$.

There are 165 distinct elements in $(F_2C_{33})e_3$ with weight 14, since $\omega(x^{15}) = 14$.

**Theorem IV.10.** Let $p_1$, $p_2$, and $p_3$ be three distinct positive odd prime numbers such that $gcd(p_i - 1, p_j - 1) = 2$, for $1 \leq i \neq j \leq 3$, and 2 generates the groups of units $U(Z_{p_i})$. Then the primitive idempotents of the group algebra $F_2G$ for the finite abelian group $G = C_{p_1} \times C_{p_2} \times C_{p_3}$, with $C_{p_i} = \langle a \rangle$, $C_{p_2} = \langle b \rangle$ and $C_{p_3} = \langle c \rangle$, are

$$e_0 = \hat{a} \hat{b} \hat{c}, e_1 = \hat{a} \hat{b} (1 - \hat{c}), e_2 = \hat{a} (1 - \hat{b}) \hat{c}, e_3 = (1 - \hat{a}) \hat{b} \hat{c},$$

$$e_4 = (u + w) v \hat{c}, e_5 = (u^2 w + w^2) \hat{c},$$

$$e_6 = (u w + u^2 w^2) \hat{b}, e_7 = (u^2 w + w^2) \hat{b},$$

$$e_8 = (u v + u^2 v^2) \hat{a}, e_9 = (u^2 v + v^2) \hat{a},$$

$$e_{10} = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u^3 v^2 + uvw^2,$$

$$e_{11} = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u^2 v^2 + uvw,$$

$$e_{12} = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u v^2 w + uvw^2,$$

and

$$e_{13} = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u^2 v^2 + uvw^2,$$

where $u, v$ are defined as in (3) and (4), respectively, and

$$w = \begin{cases} c^3 + c^2 + \cdots + c^{p_3 - 3}, & \text{if } p_3 \equiv 1 \pmod{4} \\
1 + c^3 + c^2 + \cdots + c^{p_3 - 3}, & \text{if } p_3 \equiv 3 \pmod{4} \end{cases}$$

(11)

**Proof:** As in the proof of Theorem IV.1 by [1] Lemma 3.1, for each $i = 1, 2, 3$, $F_2C_{p_i}$ contains two primitive idempotents, namely, $C_{p_i}$ and $1 - C_{p_i}$, so

$$F_2C_{p_i} \cong (F_2C_{p_i}) \cdot C_{p_i} \oplus (F_2C_{p_i}) \cdot (1 - C_{p_i}) \cong F_2 \oplus F_{2p_i - 1}.$$ By Lemma [1.1], we have

$$F_2 = (F_2C_{p_i}) \cdot C_{p_i} \oplus (F_2C_{p_i}) \cdot (1 - C_{p_i}) \cong F_2 \oplus F_{2p_i - 1}.$$ Thus:

$$F_2G = F_2C_{p_1} \times C_{p_2} \times C_{p_3}$$

$$= (F_2C_{p_1}) \otimes F_2C_{p_2} \otimes F_2C_{p_3}$$

$$= (F_2 \otimes F_{2p_1 - 1} \oplus F_{2p_2 - 1} \oplus 2 \cdot F_{2(p_3 - 1)(p_2 - 1)})$$

$$\otimes (F_2 \otimes F_{2p_1 - 1} \oplus F_{2p_2 - 1} \oplus 2 \cdot F_{2(p_3 - 1)(p_2 - 1)})$$

(12)

Therefore, there exist 14 simple components in this decomposition. First note that

$$\hat{a} \hat{b} \hat{c} + (1 - \hat{a}) \hat{b} \hat{c} + \hat{a} (1 - \hat{b}) \hat{c} + \hat{a} \hat{b} (1 - \hat{c}) + (1 - \hat{a}) \hat{b} (1 - \hat{c}) + (1 - \hat{a}) (1 - \hat{b}) (1 - \hat{c}) = 1.$$

These components are as follows:

$$F_2G \cdot \hat{a} \hat{b} \hat{c} = F_2G \cdot \hat{a} \hat{b} \hat{c}$$

$$F_2G \cdot (1 - \hat{a}) \hat{b} \hat{c} = F_2G \cdot (1 - \hat{a}) \hat{b} \hat{c}$$

$$F_2G \cdot \hat{a} (1 - \hat{b}) \hat{c} = F_2G \cdot \hat{a} (1 - \hat{b}) \hat{c}$$

$$F_2G \cdot \hat{a} \hat{b} (1 - \hat{c}) = F_2G \cdot \hat{a} \hat{b} (1 - \hat{c})$$

Therefore

$$F_2G \cong F_2C_{p_1} \otimes F_2C_{p_2} \otimes F_2C_{p_3}$$

$$\cong (F_2C_{p_1} \cdot \hat{a} \hat{b} \hat{c} + F_2C_{p_1} \cdot (1 - \hat{a}) \hat{b} \hat{c})$$

$$\otimes (F_2C_{p_2} \cdot \hat{a} \hat{b} (1 - \hat{c}) + F_2C_{p_3} \cdot (1 - \hat{a}) \hat{b} (1 - \hat{c}))$$

$$\cong (F_2C_{p_1} \cdot \hat{a} \hat{b} \hat{c} + F_2C_{p_1} \cdot (1 - \hat{a}) \hat{b} \hat{c})$$

$$\otimes (F_2C_{p_2} \cdot \hat{a} \hat{b} (1 - \hat{c}) + F_2C_{p_3} \cdot (1 - \hat{a}) \hat{b} (1 - \hat{c}))$$

$$= (F_2C_{p_1} \cdot \hat{a} \hat{b} \hat{c} + F_2C_{p_1} \cdot (1 - \hat{a}) \hat{b} \hat{c})$$

$$\otimes (F_2C_{p_2} \cdot \hat{a} \hat{b} (1 - \hat{c}) + F_2C_{p_3} \cdot (1 - \hat{a}) \hat{b} (1 - \hat{c}))$$

Let $0 \neq u \in F_2C_{p_1} (1 - \hat{a})$ be an element such that $u^3 = (1 - \hat{a})$ and $u \neq (1 - \hat{a})$; $0 \neq v \in F_2C_{p_2} (1 - \hat{b})$ such that $v^3 = (1 - \hat{b})$ and $v \neq (1 - \hat{b})$ and $0 \neq w \in F_2C_{p_3} (1 - \hat{c})$ such that $w^3 = (1 - \hat{c})$ and $w \neq (1 - \hat{c})$.

Then:

$$F_2C_{p_1} (1 - \hat{a}) \otimes F_2C_{p_2} (1 - \hat{b}) \otimes F_2C_{p_3} \hat{c}$$

$$= (F_2G) c_{\hat{a} \hat{b} \hat{c}} \otimes (F_2G) c_{\hat{a} \hat{b} \hat{c}} \otimes (F_2G) c_{\hat{a} \hat{b} \hat{c}}$$

$$= (F_2G) c_{\hat{a} \hat{b} \hat{c}} \otimes (F_2G) c_{\hat{a} \hat{b} \hat{c}} \otimes (F_2G) c_{\hat{a} \hat{b} \hat{c}}$$

$$= (F_2G) c_{\hat{a} \hat{b} \hat{c}} \otimes (F_2G) c_{\hat{a} \hat{b} \hat{c}} \otimes (F_2G) c_{\hat{a} \hat{b} \hat{c}}$$

where $c_{\hat{a} \hat{b} \hat{c}} = uv + u^2 v^2$ and $c_{\hat{a} \hat{b} \hat{c}} = u^2 v + u^2 v^2$, $c_{\hat{a} \hat{b} \hat{c}} = uv + u^2 v^2$ and $c_{\hat{a} \hat{b} \hat{c}} = u^2 v + u^2 v^2$.
For $i, j, k \in \{1, 2, 3\}$ and pairwise different, we have
\[
\begin{align*}
2 \cdot \mathbb{F}_2 \cdot (p^{(p^j)_{(p^k)_{(p^l)}}}) & \cong \mathbb{F}_2 \cdot (1 - C_{p^k}) \cdot (1 - C_{p^j}) \cdot C_{p^l} \\
& = \mathbb{F}_2 \cdot f_{ij}^{(j)} \oplus \mathbb{F}_2 \cdot f_{ij}^{(k)}
\end{align*}
\]
where
\[
\begin{align*}
f_{ij}^{(j)} & = (1 - C_{p^k}) \cdot (1 - C_{p^j}) \cdot C_{p^k} + u_i v_j C_{p^k} + u^2 v^2 \mathbb{C}_{p^k} \\
\text{and} \\
f_{ij}^{(k)} & = (1 - C_{p^k}) \cdot (1 - C_{p^j}) \cdot C_{p^k} + u_i v_j^2 C_{p^k} + u^2 v^2 \mathbb{C}_{p^k}
\end{align*}
\]
with $u_i$ expressed as in (6) in terms of the generator of $C_{p^k}$ and $v_j$ expressed as in (7) in terms of the generator of $C_{p^j}$.

Now we calculate the four simple components of
\[
\mathbb{F}_2 \cdot C_{p^1} \cdot (1 - \hat{a}) \oplus \mathbb{F}_2 \cdot C_{p^2} \cdot (1 - \hat{b}) \oplus \mathbb{F}_2 \cdot C_{p^3} \cdot (1 - \hat{c}).
\]

In $\mathbb{F}_2 \cdot C_{p_1} \cdot (1 - \hat{a}) \oplus \mathbb{F}_2 \cdot C_{p_2} \cdot (1 - \hat{b})$, we have $(uv)^3 = (1 - \hat{a})(1 - \hat{b})$ and $uv \neq \hat{a}^3 \hat{b}^3$. Now take $e = \alpha uv \hat{a}^3 \hat{b}^3 = uv + u^2 v + uv^2$. Similarly, $(u^2 v^2)^3 = (1 - \hat{a})(1 - \hat{b})$ and $\alpha^2 = u^2 v^2 e_3 = u^2 v^2 + u^2 v + v^2$.

Hence the elements $A = \alpha w + \alpha^2 w^2$ and $B = \alpha^2 w + \alpha w^2$ are the primitive idempotents of $(\mathbb{F}_2 \cdot C_{p_1} \cdot C_{p_2}) e_3^2 \otimes \mathbb{F}_2 \cdot C_{p_3} \cdot (1 - \hat{c})$.

Similarly, in $\mathbb{F}_2 \cdot C_{p_1} \cdot (1 - \hat{a}) \otimes \mathbb{F}_2 \cdot C_{p_2} \cdot (1 - \hat{b})$, we have $(uv)^3 = (1 - \hat{a})(1 - \hat{b})$ and $uv \neq \hat{a}^3 \hat{b}^3$. Set $\beta = uv^2 e_4^b = uv^2 + u^2 v^2 + uv$.

The elements $C = \beta w + \beta^2 w^2$ and $D = \beta^2 w + \beta w^2$ are the primitive idempotents of $(\mathbb{F}_2 \cdot C_{p_1} \cdot C_{p_2}) e_3^2 \otimes \mathbb{F}_2 \cdot C_{p_3} \cdot (1 - \hat{c})$.

Finally, it is an easy computation to show that $A, B, C, D$ are orthogonal idempotents and that $A + B + C + D = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c})$. Notice that:
\[
A = \alpha w + \alpha^2 w^2 = (uv + u^2 v + uv^2)w + (uv + u^2 v + uv^2)^2 w^2 = uvw + u^2 v^2 + uv^2 w + u^2 v^2 + uv^2 w^2 + uv^2 w^2 = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u^2 v^2 + uv^2 + uvw.
\]
\[
B = \alpha^2 w + \alpha w^2 = (uv + u^2 v + uv^2)w + (uv + u^2 v + uv^2)^2 w^2 = u^2 v^2 w + uv^2 w + u^2 v^2 + u^2 v^2 w^2 + u^2 v^2 w^2 = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u^2 v^2 + uv^2 + uvw.
\]
\[
C = \beta w + \beta^2 w^2 = (uv + u^2 v + uv^2)w + (uv + u^2 v + uv^2)^2 w^2 = u^2 v^2 w + u^2 v^2 + u^2 v^2 + u^2 v^2 w^2 = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + u^2 v^2 + uv^2 + uvw.
\]
\[
D = \beta^2 w + \beta w^2 = (uv + u^2 v + uv^2)w + (uv + u^2 v + uv^2)^2 w^2 = u^2 v^2 w + uv^2 w + u^2 v^2 + u^2 v^2 + u^2 v^2 w^2 + uv^2 w^2 = (1 - \hat{a})(1 - \hat{b})(1 - \hat{c}) + uv^2 w + u^2 v w.
\]

\[
\mathbb{F}_2 \cdot C_{p^{m}} \times C_{q^{n}} \text{ are described in the following table.}
\]

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Primitive Idempotent</th>
<th>Dimension</th>
<th>Code Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{0}$</td>
<td>$\hat{a} \hat{b}$</td>
<td>1</td>
<td>$p^{m} q^{n}$</td>
</tr>
<tr>
<td>$I_{0}$</td>
<td>$\hat{a} (b^{n^2} + b^{n^2-1})$</td>
<td>$q^{1} (q - 1)$</td>
<td>$2p^{m} q^{n-1}$</td>
</tr>
<tr>
<td>$I_{1}$</td>
<td>$(a^{p^2} + a^{p^2-1}) \hat{b}$</td>
<td>$p^{1} (p - 1)$</td>
<td>$2p^{m} q^{n-1}$</td>
</tr>
<tr>
<td>$I_{ij}$</td>
<td>$uv + u^2 v + (p^{m} q^{n-1}) (q - 1)$</td>
<td>$uv^2 + u^2 v$</td>
<td>$p^{1} (p - 1)$</td>
</tr>
</tbody>
</table>

where
\[
\begin{align*}
u &= \hat{a} p^{2} (a^{2p^2-1} + a^{2p^2-1} + \cdots + a^{2p^2-1}), \\
& \quad \text{if } p \equiv 1 (\text{mod } 4) \text{ or } \hat{b} q^{2} (b^{2q^2-1} + b^{2q^2-1} + \cdots + b^{2q^2-1}), \\
& \quad \text{if } q \equiv 1 (\text{mod } 4) \text{ or } \hat{a} p^{2} (a^{2p^2-1} + a^{2p^2-1} + \cdots + a^{2p^2-1}), \\
& \quad \text{if } p \equiv 3 (\text{mod } 4)
\end{align*}
\]

and
\[
\begin{align*}
u &= \hat{a} q^{2} (b^{2q^2-1} + b^{2q^2-1} + \cdots + b^{2q^2-1}), \\
& \quad \text{if } q \equiv 1 (\text{mod } 4) \text{ or } \hat{b} q^{2} (b^{2q^2-1} + b^{2q^2-1} + \cdots + b^{2q^2-1}), \\
& \quad \text{if } q \equiv 3 (\text{mod } 4)
\end{align*}
\]

Proof: Since 2 generates $U(Z_{p^2})$ by 11 Lemma 5], we have
\[
\mathbb{F}_2 \cdot C_{p^m} = \mathbb{F}_2 \cdot C_{p^m} \cdot \hat{a} \oplus \bigoplus_{i=1}^{m} \mathbb{F}_2 \cdot C_{p^m} \cdot (a^{p^i} + a^{p^{i-1}})
\]
\[
\cong \mathbb{F}_2 \oplus \bigoplus_{i=1}^{m} \mathbb{F}_2 \cdot (p^{i-1} - 1),
\]
\[
\mathbb{F}_2 \cdot C_{q^n} = \mathbb{F}_2 \cdot C_{q^n} \cdot \hat{b} \oplus \bigoplus_{j=1}^{n} \mathbb{F}_2 \cdot C_{q^n} \cdot (b^{q^j} + b^{q^j-1})
\]
\[
\cong \mathbb{F}_2 \oplus \bigoplus_{j=1}^{n} \mathbb{F}_2 \cdot (q^j - 1),
\]

Notice that since 2 generates $U(Z_{p^2})$ also 2 generates $U(Z_{q^2})$, hence, using (2) and Lemma V.3, we have the following decomposition:
\[
\mathbb{F}_2 \cdot C_{p^m} \times C_{q^n} = \mathbb{F}_2 \cdot C_{p^m} \oplus \mathbb{F}_2 \cdot C_{q^n}
\]
\[
\cong \bigoplus_{i=1}^{m} \mathbb{F}_2 \cdot (p^{i-1} - 1) \oplus \bigoplus_{j=1}^{n} \mathbb{F}_2 \cdot (q^j - 1)
\]
\[
\oplus 2 \cdot \bigoplus_{i,j} \mathbb{F}_2 \cdot (p^{i-1} - 1) (q^j - 1).
\]

For each pair $i, j$, the idempotent $e_{ij} = (a^{p^i} + a^{p^i-1}) \cdot (b^{q^j} + b^{q^j-1})$ is not primitive and, by Lemma V.3, it decomposes as sum of two primitive idempotents, namely $e_{ij}^{(1)} = uv + u^2 v^2$ and $e_{ij}^{(2)} = uv^2 + u^2 v$, where $u$ and $v$ are as in the statement.
of the theorem. Thus the minimal ideals $I_{ij} = \langle uv + u^2v^2 \rangle$ and $I_{ij}^* = \langle u^2v + uv^2 \rangle$ are such that $I_{ij} \oplus I_{ij}^* = \langle e_{ij} \rangle$.

The dimension of each code follows from (13).

Consider the code $I_{0j} = \langle e = \bar{a}(b^q + b^{q-1}) \rangle$. The element $(b^{q-1} - 1)e = (1 + b^{q-1}) \bar{a}b^q \in I_{0j}$ has weight $2p^m q^{n-1}$. Since $b^q = b^{q-1} + (b^{q-1} - 1)$, we have

$$\langle \mathbb{F}_2^q \rangle \bar{a}b^q \in \langle \mathbb{F}_2^q \rangle \bar{a}b^{q-1} + \langle \mathbb{F}_2^q \rangle e,$$

hence $I_{0j} \subseteq \langle \mathbb{F}_2^q \rangle \bar{a}b^q$.

An element of $\langle \mathbb{F}_2^q \rangle \bar{a}b^q \bar{a}b^q$ is of the form $(\sum_{i,k} \lambda_{i,k} a^i b^k) \bar{a}b^q$ and has weight $\ell p^m q^{n-1}$, as $\text{supp}(b^q \bar{a}b^q) \cap \text{supp}(b^q \bar{a}b^q) = \emptyset$ or $\text{supp}(b^q \bar{a}b^q) = \text{supp}(b^q \bar{a}b^q)$, for $0 \leq k \leq t \leq q^n - 1$.

Since $b^q \bar{a}b^q \cdot e = b^q \bar{a}b^q \bar{a}(b^q + b^{q-1}) = 0$, we have $b^q \bar{a}b^q \not\subseteq I_{0j}$. Therefore, $\omega(I_{0j}) = 2p^m q^n$. Similarly, we have $\omega(I_{00}) = 2p^m q^n$.

Let $1 \leq i \leq m$ and $1 \leq j \leq n$. To compute the weight of the code $I_{ij}$ (and similarly for $I_{ij}^*$), we set

$$H = \langle b^{q-1} \rangle \times \langle b^q \rangle$$

so $K \leq H \leq G$ and $\frac{H}{K} \cong C_p \times C_q = \langle a^{p^{-1}} K \rangle \times \langle b^{q-1} K \rangle$.

Consider the isomorphism

$$\psi : (\mathbb{F}_2 H) K \rightarrow \mathbb{F}_2 \left( \frac{H}{K} \right)$$

such that $a^{p^{-1}} K \rightarrow a^{p^{-1}} K$ and $b^{q-1} K \rightarrow b^{q-1} K$. Since $a^{p^{-1}} K = b^{q-1} K = K$, an element $\alpha \in (\mathbb{F}_2 H) K$ is of the form

$$\alpha = \sum_{i=0}^{p-1} \sum_{t=0}^{q-1} \alpha_{it} a^{p^{-1}} b^{q-1} K$$

For $1 \leq t_1, t_2 \leq p - 1, 1 \leq \ell_1, \ell_2 \leq q - 1$ and $(t_1, \ell_1) \neq (t_2, \ell_2)$, we have

$$t_1 p^{q-1} \neq t_2 p^{q-1} \mod p^i \text{ or } t_1 q^{q-1} \neq t_2 q^{q-1} \mod q^j,$$

hence $\text{supp}(a^{t_1 p^{q-1}} b^{q^{-1}} K) \cap \text{supp}(a^{t_2 p^{q-1}} b^{q^{-1}} K) = \emptyset$. So

$$\omega(\alpha) = \omega \left( \sum_{i=0}^{p-1} \sum_{t=0}^{q-1} \alpha_{it} a^{p^{-1}} b^{q-1} K \right)$$

Notice that $e = uv + u^2v^2 = \tilde{K} f$, with $f \in \mathbb{F}_2 H$. Consider $\langle \mathbb{F}_2 H \rangle e \subseteq \langle \mathbb{F}_2 G \rangle e = I_{ij}^*$, take

$$\beta = \sum_{\mu=0}^p \sum_{\lambda=0}^q \alpha_{\mu,\lambda} a^\mu b^\lambda e \in \langle \mathbb{F}_2 G \rangle e.$$

Since $a^{k^{-1}} b^{q-1} e \in \langle \mathbb{F}_2 H \rangle e$, we may write

$$\beta = \sum_{\mu=0}^p \sum_{\lambda=0}^q \delta_{\mu,\lambda} a^\mu b^\lambda,$$

where $\delta_{\mu,\lambda} \in \langle \mathbb{F}_2 H \rangle e$.

For $0 \leq \mu_1, \mu_2 \leq p - 1$, $0 \leq \lambda_1, \lambda_2 \leq q - 1$ and $(\mu_1, \lambda_1) \neq (\mu_2, \lambda_2)$, we have $\text{supp}(\gamma a^{\mu_1} b^\lambda) \cap \text{supp}(\gamma a^{\mu_2} b^\lambda) = \emptyset$, where $\gamma \in \langle \mathbb{F}_2 H \rangle e$. Indeed, note that the exponents of $a$ and $b$ which appear in $\gamma a^{\mu_1} b^\lambda$ and $\gamma a^{\mu_2} b^\lambda$ are, respectively, $\mu_1 + t_1 p^{q-1}$, $\lambda_1 + s_1 q^{q-1}$ and $\mu_2 + t_2 p^{q-1}$, $\lambda_2 + s_2 q^{q-1}$. If $\text{supp}(\gamma a^{\mu_1} b^\lambda) \cap \text{supp}(\gamma a^{\mu_2} b^\lambda) \neq \emptyset$, we should have $\mu_1 + t_1 p^{q-1} \equiv \mu_2 + t_2 p^{q-1} \mod p^m$ and $\lambda_1 + s_1 q^{q-1} \equiv \lambda_2 + s_2 q^{q-1} \mod q^n$, but this does not occur.

Hence,

$$\omega(\beta) = \omega \left( \sum_{\mu=0}^p \sum_{\lambda=0}^q \delta_{\mu,\lambda} a^\mu b^\lambda \right) = \sum_{\mu=0}^p \sum_{\lambda=0}^q \omega(\delta_{\mu,\lambda}).$$

Thus, given a non-zero element $\beta \in \langle \mathbb{F}_2 G \rangle e$, there exists a non-zero element in $\beta' \in \langle \mathbb{F}_2 H \rangle e$ such that $\omega(\beta') \geq \omega(\beta')$.

Thus, $\omega(\langle \mathbb{F}_2 H \rangle e) = \omega(\langle \mathbb{F}_2 G \rangle e)$.

\section*{Example V.2.}

For $p = 3$ and $q = 5$, let $G = C_3^m \times C_5^n = \langle a \rangle \times \langle b \rangle$, with $o(a) = 3^m$ and $o(b) = 5^n$. According to Theorem V.1 in $\langle \mathbb{F}_2 (C_3^m \times C_5^n) \rangle$ with $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$, the code $I_{ij}^*$ is $\langle uv + u^2v^2 \rangle$ is generated by the element

$$e_{ij}^{(1)} = \langle uv + u^2v^2 \rangle$$

$$= a^{3^{i+1}} b^{5^{j+1}} \left( b^{5^3} + b^{2 \cdot 5^3} + b^{2^2 \cdot 5^3} + b^{2^3 \cdot 5^3} \right)$$

Using Example V.2 and the computations above, we get

$$\omega(e_{ij}^{(1)}) = \omega(I_{ij}^*) = 3^{m-(i+1)} \cdot 5^{n-(j+1)} \cdot 8.$$