Fuzzy analytic hierarchy process with interval type-2 fuzzy sets

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1. Introduction

In the traditional formulation of multicriteria decision making (MCDM) problems human’s judgments are represented as exact numbers. However, in many practical cases, the data may be imprecise, or the decision makers might be unable to assign exact numerical values to the evaluation. Since some of the evaluation criteria are subjective and qualitative in nature, it is very difficult for the decision maker to express the preferences using exact numerical values [20]. The conventional MCDM approaches tend to be less effective in dealing with the imprecise or vague nature of the linguistic assessments [13]. Analytic Hierarchy Process (AHP) which is one of the most used MCDM approaches is a structured multicriteria technique for organizing and analyzing complex decisions including many conflicting criteria. In the literature fuzzy AHP methods based on type-1 fuzzy sets exist. The fuzzy AHP technique can be viewed as an advanced analytical method developed from the traditional AHP. Despite the convenience of AHP in handling both quantitative and qualitative criteria of multicriteria decision making problems based on decision makers’ judgments, fuzziness and vagueness existing in many decision-making problems may cause to the imprecise judgments of decision makers in conventional AHP approaches [7].

In type-1 fuzzy sets, each element has a degree of membership which is described with a membership function valued in the interval \([0,1]\) [21]. The concept of a type-2 fuzzy set was introduced by Zadeh [22] as an extension of the concept of an ordinary fuzzy set called a type-1 fuzzy set. Such sets are fuzzy sets whose membership grades themselves are type-1 fuzzy sets; they are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set; hence, they are useful for incorporating linguistic uncertainties, e.g., the words that are used in linguistic knowledge can mean different things to different people [14]. While the membership functions of type-1 fuzzy sets are two-dimensional, the membership functions of type-2 fuzzy sets are three-dimensional. It is the new third-dimension that provides additional degrees of freedom that make it possible to directly model uncertainties.

An interval type-2 fuzzy set is a special case of a generalized type-2 fuzzy set. Since generalized type-2 fuzzy sets require complex and immense computational burdensome operations, the wide spread application of generalized type-2 fuzzy systems has not occurred. Interval type-2 fuzzy sets are the most commonly used type-2 fuzzy sets because of their simplicity and reduced computational effort with respect to general type-2 fuzzy sets. For this reason, we used interval type-2 fuzzy sets.

In this paper, an interval type-2 fuzzy AHP method is developed and presented into the literature for the first time. The linguistic scale of fuzzy AHP is expressed in a more detailed and flexible way by interval type-2 fuzzy sets. New defuzzification methods for both triangular and trapezoidal type-2 fuzzy sets are also incorporated into the developed method.
The rest of the paper is organized as follows. Section 2 presents the basics of interval type-2 fuzzy sets. Section 3 gives defuzzification methods including two new methods called DTRiTr and DTRaT. Section 4 includes our proposed interval type-2 fuzzy AHP method. Section 5 gives an illustrative application of the proposed method. Finally Section 6 gives the conclusions.

2. Interval type-2 fuzzy sets

In this section, some definitions of type-2 fuzzy sets and interval type-2 fuzzy sets are briefly explained [17].

A type-2 fuzzy set \( \tilde{A} \) in the universe of discourse \( X \) can be represented by a type-2 membership function \( \mu_{\tilde{A}} \), shown as follows [22]:

\[
\tilde{A} = \left\{ (x, u) \in X \times [0, 1] \mid \forall u \in \tilde{\mu}_{\tilde{A}}(x, u) \right\}
\]

(1)

where \( \tilde{\mu}_{\tilde{A}} \) denotes an interval \( [0, 1] \). The type-2 fuzzy set \( \tilde{A} \) can also be represented as follows [17]:

\[
\tilde{A} = \int_{x \in X} \int_{u \in \tilde{\mu}_{\tilde{A}}(x, u)} (x, u),
\]

(2)

where \( \tilde{\mu}_{\tilde{A}} \subseteq [0, 1] \) and \( \tilde{\mu} \) denote union over all admissible \( x \) and \( u \).

Let \( \tilde{A} \) be a type-2 fuzzy set in the universe of discourse \( X \) represented by the type-2 membership function \( \mu_{\tilde{A}} \). If all \( \mu_{\tilde{A}}(x, u) = 1 \), then \( \tilde{A} \) is called an interval type-2 fuzzy set [3]. An interval type-2 fuzzy set \( \tilde{A} \) can be regarded as a special case of a type-2 fuzzy set, represented as follows [17]:

\[
\tilde{A} = \int_{x \in X} \int_{u \in \tilde{\mu}_{\tilde{A}}(x, u)} 1/(x, u),
\]

(3)

where \( \tilde{\mu}_{\tilde{A}} \subseteq [0, 1] \).

Arithmetic operations with trapezoidal interval type-2 fuzzy sets are given in the following.

Definition 2.1. The upper and lower membership functions of an interval type-2 fuzzy set are type-1 membership functions.

A trapezoidal interval type-2 fuzzy set is illustrated as \( \tilde{A}_{1} = (\tilde{A}_{1}, \tilde{A}_{1}) = (((a_{11}, a_{12}, a_{13}, a_{14}), H_{1}(\tilde{A}_{1})), (a_{21}, a_{22}, a_{23}, a_{24}), H_{2}(\tilde{A}_{1})) \) where \( \tilde{A}_{1} \) and \( \tilde{A}_{1} \) are type-1 fuzzy sets, \( a_{11}, a_{12}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24} \) and \( a_{13} \) are the reference points of the interval type-2 fuzzy set \( \tilde{A}_{1}, H_{1}(\tilde{A}_{1}) \); denotes the membership value of the element \( a_{1j} \) in the upper trapezoidal membership function \( \tilde{A}_{1} \), \( 1 \leq j \leq 2, H_{1}(\tilde{A}_{1}) \) denotes the membership value of the element \( a_{ij} \) in the lower trapezoidal membership function \( \tilde{A}_{1} \), \( 1 \leq j \leq 2, H_{2}(\tilde{A}_{1}) \) \( \in [0, 1] \), \( H_{1}(\tilde{A}_{1}) \) \( \in [0, 1] \), \( H_{2}(\tilde{A}_{1}) \) \( \in [0, 1] \), \( 1 \leq i \leq n \) [8].

Definition 2.2. The addition operation between the trapezoidal interval type-2 fuzzy sets \( \tilde{A}_{1} = (\tilde{A}_{1}, \tilde{A}_{1}) \), \( \tilde{A}_{2} = (\tilde{A}_{2}, \tilde{A}_{2}) \), \( \tilde{A}_{1}, H_{1}(\tilde{A}_{1})), H_{2}(\tilde{A}_{1})) \) and \( \tilde{A}_{2} = (\tilde{A}_{2}, \tilde{A}_{2}, H_{2}(\tilde{A}_{2})) \) is defined as follows [8]:

\[
\tilde{A}_{1} + \tilde{A}_{2} = ((a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}), \min (H_{1}(\tilde{A}_{1}), H_{1}(\tilde{A}_{2})), \min (H_{2}(\tilde{A}_{1}), H_{2}(\tilde{A}_{2})))
\]

(4)

\[
\tilde{A}_{1} \lor \tilde{A}_{2} = ((a_{11} - a_{21}, a_{12} - a_{22}, a_{13} - a_{23}, a_{14} - a_{24}), \min (H_{1}(\tilde{A}_{1}), H_{1}(\tilde{A}_{2})), \min (H_{2}(\tilde{A}_{1}), H_{2}(\tilde{A}_{2})))
\]

(5)

\[
\tilde{A}_{1} \lor \tilde{A}_{2} = ((a_{11}, a_{12}, a_{13}, a_{14}, H_{1}(\tilde{A}_{1})), H_{2}(\tilde{A}_{1})) \) and \( \tilde{A}_{2} = ((a_{21}, a_{22}, a_{23}, a_{24}, H_{2}(\tilde{A}_{1})), H_{2}(\tilde{A}_{1})) \) is defined as follows [8]:

\[
\tilde{A}_{1} \lor \tilde{A}_{2} = ((a_{11} - a_{21}, a_{12} - a_{22}, a_{13} - a_{23}, a_{14} - a_{24}), \min (H_{1}(\tilde{A}_{1}), H_{1}(\tilde{A}_{2})), \min (H_{2}(\tilde{A}_{1}), H_{2}(\tilde{A}_{2})))
\]

(6)

\[
\tilde{A}_{1} \lor \tilde{A}_{2} = ((a_{11}, a_{12}, a_{13}, a_{14}, H_{1}(\tilde{A}_{1})), H_{2}(\tilde{A}_{1})) \) and \( \tilde{A}_{2} = ((a_{21}, a_{22}, a_{23}, a_{24}, H_{2}(\tilde{A}_{1})), H_{2}(\tilde{A}_{1})) \) is defined as follows [8]:

\[
\tilde{A}_{1} \lor \tilde{A}_{2} = ((a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}), \min (H_{1}(\tilde{A}_{1}), H_{1}(\tilde{A}_{2})), \min (H_{2}(\tilde{A}_{1}), H_{2}(\tilde{A}_{2})))
\]

(7)

\[
\tilde{A}_{1} \lor \tilde{A}_{2} = ((k \times a_{11}, k \times a_{12}, k \times a_{13}, k \times a_{14}), H_{1}(\tilde{A}_{1}), H_{2}(\tilde{A}_{1}))
\]

(8)

3. Defuzzification methods for type-2 fuzzy sets

Defuzzification of a type-2 fuzzy set consists of two steps. In the first step, a type-2 fuzzy set is determined as a type-1 fuzzy set by using the type reduction process. Then one of the defuzzification methods for ordinary (type-1) fuzzy sets is used to find the equivalence of the type-2 fuzzy set [14]. There are a lot of type reduction methods proposed in the literature. In the following the most used
type reduction methods are given and then the proposed ranking method is presented.

3.1. Centroid of a type-2 fuzzy set

The centroid \( \mu_C(x) \) of an interval type-2 fuzzy set is the union of the centroids of all its \( n_t \) embedded type-1 fuzzy sets \( \mu_C(A_k) \) \([14]\).

\[
\mu_C(x) = \bigcup_{k=1}^{n_t} \mu_C(A_k) = \{ \mu_C(\tilde{A}), \ldots, \mu_C(\tilde{A}) \} = [c_1(\tilde{A}), c_t(\tilde{A})],
\]

where
\[
c_1(\tilde{A}) = \min_{\forall x_k} \mu_C(A_k) = \min_{\forall x_k} \mu_C(A_k) \left( \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i} \right),
\]
\[
c_t(\tilde{A}) = \max_{\forall x_k} \mu_C(A_k) = \max_{\forall x_k} \mu_C(A_k) \left( \frac{\sum_{i=1}^{N} x_i \theta_i}{\sum_{i=1}^{N} \theta_i} \right)
\]

Karnik and Mendel \([14]\) proposed iterative algorithms to calculate \( c_1(\tilde{A}) \) and \( c_t(\tilde{A}) \). The iterative algorithm for \( c_t(\tilde{A}) \) is given as follows:

Step 1. Set \( \theta_i = (\mu(C_k(x_i)) + \mu(C_{-k}(x_i))) / 2 \) for \( i = 1, \ldots, N \) and compute \( c_t(\tilde{A}) = \sum_{i=1}^{N} x_i \theta_i / \sum_{i=1}^{N} \theta_i \) using Eq. (10).

Step 2. Find \( k(1 < k < N - 1) \) such that \( x_k \in c_t \times x_{k+1} \).

Step 3. Set \( \theta_i = \mu(C_k(x_i)) \) for \( i < k \) and \( \theta_i = \mu(C_{-k}(x_i)) \) for \( i > k \) and compute \( c_t(k) \) using Eq. (12).

Step 4. Check if \( c_t(k) = c_t \). If yes, stop. \( c_t(k) \) is the minimum value of \( c_t(\tilde{A}) \). If no, go to step 5.

Step 5. Set \( c_t = c_t(k) \) and go to Step 2.

The minimum of \( c_t(\tilde{A}) \) can be obtained by using a procedure similar to the one described above. In step 3, we set \( \theta_i = \mu(C_k(x_i)) \) for \( i < k \) and \( \theta_i = \mu(C_{-k}(x_i)) \) for \( i > k \) and compute \( c_t(k) \) using Eq. (13) \([14]\).

\[
c_t = \frac{\sum_{i=1}^{k} x_i \mu(C_k(x_i)) + \sum_{i=k+1}^{N} x_i \mu(C_{-k}(x_i))}{\sum_{i=1}^{k} \mu(C_k(x_i)) + \sum_{i=k+1}^{N} \mu(C_{-k}(x_i))}
\]

The type reduction indices method

Niewiadowski et al. \([18]\) proposed optimistic, pessimistic, realistic and weighted average indices which determine different points of view for the type reduction of interval type-2 fuzzy sets. If \( \tilde{A} \) is an interval-valued fuzzy set in the universe \( X \) and \( \mu_{\tilde{A}}(x) \) and \( \mu_{\tilde{A}}(x) \) are their lower and upper membership functions, Eqs. (14)–(17) transform \( \tilde{A} \) into an ordinary fuzzy set. \( \text{TR}_{opt}(\tilde{A}) \) determines optimistic type reduction of \( \tilde{A} \). \( \text{TR}_{pes}(\tilde{A}) \) determines pessimistic type reduction of \( \tilde{A} \). \( \text{TR}_{re}(\tilde{A}) \) determines realistic type reduction of \( \tilde{A} \), and \( \text{TR}_{wa}(\tilde{A}) \) determines weighted average type reduction of \( \tilde{A} \).

\[
\text{TR}_{opt}(\tilde{A}) = \mu_{\tilde{A}}(x), \ x \in X
\]
\[
\text{TR}_{pes}(\tilde{A}) = \mu_{\tilde{A}}(x), \ x \in X
\]
\[
\text{TR}_{re}(\tilde{A}) = \frac{\mu_{\tilde{A}}(x) + \tilde{\mu}_{\tilde{A}}(x)}{2}, \ x \in X
\]
\[
\text{TR}_{wa}(\tilde{A}) = w_1 \mu_{\tilde{A}}(x) + w_2 \tilde{\mu}_{\tilde{A}}(x), \ x \in X
\]

where \( w_1 \) and \( w_2 \) are the coefficients which satisfy the equation \( "w_1 + w_2 = 1" \).

3.3. Ranking values of interval type-2 fuzzy sets

Lee and Chen \([16]\) presented the concept of ranking values of trapezoidal interval type-2 fuzzy sets. If \( \tilde{A} = (\tilde{A}_1, \tilde{A}_2) = ((a_0, a_1, a_2, a_3), (a_0, a_1, a_2, a_3), H_1(\tilde{A}_1), H_2(\tilde{A}_2)) \) is a trapezoidal interval type-2 fuzzy set, the ranking value \( \text{Rank}(\tilde{A}) \) of \( \tilde{A} \) is defined as follows \([16]\):

\[
\text{Rank}(\tilde{A}) = M_1(\tilde{A}_1) + M_2(\tilde{A}_2) + M_3(\tilde{A}_1) + M_2(\tilde{A}_2) + M_3(\tilde{A}_1)
\]
\[
+ M_3(\tilde{A}_1) - \frac{1}{4} (S_1(\tilde{A}_1) + S_1(\tilde{A}_2) + S_2(\tilde{A}_1) + S_2(\tilde{A}_1))
\]
\[
+ S_3(\tilde{A}_1) + S_3(\tilde{A}_2) + S_4(\tilde{A}_1) + S_4(\tilde{A}_1) + H_1(\tilde{A}_1)
\]
\[
+ H_1(\tilde{A}_2) + H_2(\tilde{A}_1) + H_2(\tilde{A}_2)
\]

where \( M_p(\tilde{A}) \) denotes the average of the elements \( a_p \) and \( a_{p+1} \), \( M_p(\tilde{A}) = \left( (a_p + a_{p+1}) / 2 \right) \), \( 1 \leq p \leq 3 \).

\[ S_k(\tilde{A}) = \frac{1}{4} \sum_{i=k}^{4} \left( a_i - \frac{1}{2} \sum_{i=k}^{4} a_i \right)^2 \], \( 1 \leq k \leq 3 \), \( S_k(\tilde{A}) \) denotes the standard deviation of the elements \( a_1, a_2, a_3, a_4 \), \( S_k(\tilde{A}) = \frac{1}{4} \sum_{i=k}^{4} \left( a_i - \frac{1}{2} \sum_{i=k}^{4} a_i \right)^2 \).

\( H_p(\tilde{A}) \) denotes the membership value of the element \( a_{p+1} \) in the trapezoidal membership function \( \tilde{A} \). \( 1 \leq p \leq 2, j \in \{ U, L \}, \) and \( 1 \leq i \leq n \).

In Eq. (18) the summation of \( M_1(\tilde{A}_1), M_2(\tilde{A}_1), M_3(\tilde{A}_1), M_2(\tilde{A}_1), H_1(\tilde{A}_1), H_2(\tilde{A}_1) \) and \( H_2(\tilde{A}_1) \) is called the basic ranking score, where we deduct the average of \( S_1(\tilde{A}_1), S_1(\tilde{A}_2), S_2(\tilde{A}_1), S_2(\tilde{A}_1), S_3(\tilde{A}_1), S_3(\tilde{A}_1), S_4(\tilde{A}_1) \) and \( S_4(\tilde{A}_1) \) from the basic ranking score to give the dispersive interval type-2 fuzzy set a penalty, where \( 1 \leq i \leq n \).

3.4. Chen and Lee’s likelihood based approach

Chen and Lee \([9]\) proposed the following ranking method for type-2 fuzzy sets. They first calculate the likelihood of \( \tilde{A}_1 \geq \tilde{A}_2 \) by Eq. (19):

\[
p(\tilde{A}_1 \geq \tilde{A}_2) = \max \left( 1 - \frac{\sum_{i=1}^{n} \max(\mu_1(x_i), \mu_2(x_i)) + \sum_{i=1}^{n} \max(h_1(\tilde{A}_1), h_2(\tilde{A}_2))}{\sum_{i=1}^{n} \max(\mu_1(x_i), \mu_2(x_i)) + \sum_{i=1}^{n} \max(h_1(\tilde{A}_1), h_2(\tilde{A}_2))} \right)
\]

Then the ranking values for upper and lower membership functions are given by Eqs. (20) and (21), respectively:

\[
\text{Rank}(\tilde{A}_1) = \frac{1}{n(n-1)} \left( \sum_{k=1}^{n-1} p(\tilde{A}_1 \geq \tilde{A}_2) + \frac{n}{2} - 1 \right)
\]
\[
\text{Rank}(\tilde{A}_1) = \frac{1}{n(n-1)} \left( \sum_{k=1}^{n-1} p(\tilde{A}_1 \geq \tilde{A}_2) + \frac{n}{2} - 1 \right)
\]
Rank(\(\hat{A}_i\)) = \frac{Rank(\(\hat{A}^U_i\)) + Rank(\(\hat{A}^L_i\))}{2} \tag{22}

3.5. The proposed ranking methods

In the following we propose ranking methods for triangular and trapezoidal type-2 fuzzy sets.

We adjusted the center of area (COA) method’s Best Nonfuzzy Performance (BNP) value for defuzzifying and ranking interval type-2 fuzzy sets. In our approach, we first obtain defuzzified values and then rank fuzzy sets with respect to these values. We propose the Defuzzified Triangular Type-2 Fuzzy Set (DTriT) approach as follows:

\[
DTriT = \frac{(u_L - u_U) + (\hat{m}_U - \hat{m}_L)}{3} + \frac{u_L + 2(\hat{m}_L - \hat{m}_L) + l_U}{2} \tag{23}
\]

where \(x\) is the maximum membership degree of the lower membership function of the considered type-2 fuzzy set; \(u_U\) is the largest possible value of the upper membership function; \(u_L\) is the largest possible value of the lower membership function; \(m_U\) is the most possible value of the upper membership function; \(m_L\) is the least possible value of the lower membership function; \(l_U\) is the least possible value of the lower membership function.

In the following four different cases are examined:

For Fig. 1a–d the defuzzified values are calculated as follows:

\[
\begin{align*}
DTriT_{1a} & = \frac{(48-10) + (22-10)}{3} + 10 + 0.7 \left( \frac{42-15 + 22-15}{3} + 15 \right) = 22.55 \\
DTriT_{1b} & = \frac{(48-10) + (22-10)}{3} + 10 + 0.7 \left( \frac{48-22 + 22-22}{3} + 22 \right) = 24.07 \\
DTriT_{1c} & = \frac{(48-10) + (22-10)}{3} + 10 + 0.7 \left( \frac{48-10 + 22-10}{3} + 10 \right) = 22.67 \\
DTriT_{1d} & = \frac{(48-10) + (22-10)}{3} + 10 + 0.7 \left( \frac{22-15 + 22-15}{3} + 15 \right) = 20.22
\end{align*}
\]

The ranking of the obtained results through defuzzification is consistent with the expected order of these triangular type-2 fuzzy sets.

Now we extend Eq. (23) for trapezoidal type-2 fuzzy sets (DTraT) as in Eq. (24):

\[
DTraT = \frac{(u_L - l_U) + (\hat{m}_U - \hat{m}_L) + (\hat{m}_U - \hat{m}_L) + l_U}{4} + l_U + \frac{2(\hat{m}_L - \hat{m}_L) + l_U}{4} \tag{24}
\]

where \(x\) and \(\beta\) are the maximum membership degrees of the lower membership function of the considered type-2 fuzzy set; \(u_U\) is the largest possible value of the upper membership function; \(m_U\) is the least possible value of the upper membership function; \(u_L\) and \(m_L\) are the second and third parameters of the upper membership function; \(l_U\) is the least possible value of the lower membership function; \(m_L\) is the least possible value of the lower membership function; \(l_U\) and \(m_L\) are the second and third parameters of the lower membership function, respectively.

Using Eq. (24), for Fig. 2a–d the defuzzified values are calculated as follows:

\[
\begin{align*}
DTraT_{2a} & = \frac{(52-10) + (0.85-10) + (0.85-10) + (32-10)}{4} + 10 + \frac{2(0.85-10) + (32-10)}{4} + 25.34 \\
DTraT_{2b} & = \frac{(52-10) + (0.4-10) + (0.4-10) + (32-10)}{4} + 10 + \frac{2(0.85-10) + (32-10)}{4} + 26.21 \\
DTraT_{2c} & = \frac{(52-10) + (0.85-10) + (0.85-10) + (32-10)}{4} + 10 + \frac{2(0.85-10) + (32-10)}{4} + 25.34 \\
DTraT_{2d} & = \frac{(52-10) + (0.75-10) + (0.75-10) + (32-10)}{4} + 10 + \frac{2(0.85-10) + (32-10)}{4} + 23.46
\end{align*}
\]

The proposed method gives a ranking of \(1b > 1c > 1a > 1d\) for triangular type-2 fuzzy sets whereas Chen and Lee [9] and Lee and Chen [16] gives \(1b > 1a > 1c > 1d\). For the trapezoidal type-2 fuzzy sets, the proposed method gives \(1b > 1a = 1c > 1d\) whereas the others do \(1b > 1a > 1c > 1d\). The ranking of fuzzy sets in Fig. 1a and c are somewhat controversial (see Table 1). This is an expected result since various ranking methods can give controversial results and this problem is still open to discuss.

![Illustrative examples for triangular type 2 fuzzy sets.](image)
4. The proposed type-2 fuzzy AHP

In this section, for the proposed method to be better understood, type-1 fuzzy AHP will first be given with its steps and literature review. Then, the proposed type-2 fuzzy AHP will be presented.

4.1. Type-1 fuzzy AHP

There are several fuzzy AHP applications which use the fuzzy set theory to incorporate the linguistic variables into these methods. Laarhoven and Pedrycz [15] proposed the first algorithm in fuzzy AHP by describing compared fuzzy ratios with triangular fuzzy membership functions. The computation steps in fuzzy AHP are the same as the steps in crisp AHP which is proposed by Saaty [19]. Logarithmic least square method is used to derive fuzzy weights and fuzzy performance scores. Buckley [3] presented fuzzy priorities of comparison ratios whose membership functions are trapezoidal. He also extended Saaty's AHP method to incorporate fuzzy comparison ratios. Buckley [3] claims that Laarhoven and Pedrycz's [15] linear equations do not always have a unique solution and the obtained solutions are not always triangular fuzzy numbers. To overcome these difficulties, Buckley used the geometric mean method to derive fuzzy weights and performance scores [7]. Buckley's fuzzy AHP method has received no criticism up to now. For this reason, Buckley's method has been chosen to apply interval type-2 fuzzy sets.

Chang [5] developed extent analysis method for the synthetic extent values of the pairwise comparisons. The extent analysis method is used to consider the extent of an object to be satisfied for the goal. On the basis of the fuzzy values for the extent analysis of each object, a fuzzy synthetic degree value is obtained.

Table 2 indicates the differences among the fuzzy AHP methods in the literature. It gives advantages and disadvantages, and main characteristic of each method.

In the following we briefly give the steps of Buckley's type-1 fuzzy AHP.

AHP is a structured approach to decision making developed by Saaty [19]. It is a weighted factor-scoring model and has the ability to detect and incorporate inconsistencies inherent in the decision making process. Therefore, it has been applied to a wide variety of decision making problems, including the evaluation of alternatives. Sometimes a decision maker's judgments are not crisp, and it is relatively difficult for the decision maker to provide exact numerical values. Therefore most of the evaluation parameters cannot be given as precisely and the evaluation data of the alternative project's suitability for various subjective criteria and the weights of the criteria are usually expressed in linguistic terms by the decision makers (DMs). In this case, fuzzy logic that provides a mathematical strength to capture the uncertainties associated with human cognitive process can be used [12,1]. The steps of Buckley's method are given in the following:

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</tbody>
</table>

Fig. 2. Illustrative examples for trapezoidal type 2 fuzzy sets.

Table 1
The ranking results of the methods.
Table 2
The comparison of different fuzzy AHP methods [4].

<table>
<thead>
<tr>
<th>Sources</th>
<th>The main characteristics of the method</th>
<th>Advantages (A) and disadvantages (D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Van Laarhoven and Pedrycz [15]</td>
<td>Direct extension of Saaty’s AHP method with triangular fuzzy numbers Loetsch’s logarithmic least square method is used to derive fuzzy weight and fuzzy performance scores.</td>
<td>(A) The options of multiple experts can be modeled in the reciprocal matrix. (D) There is not always a solution to linear equations. (D) The computational requirement is tremendous, even for a small problem. (D) It allows only triangular fuzzy numbers to be used.</td>
</tr>
<tr>
<td>Buckley [3]</td>
<td>Extension of Saaty’s AHP method with trapezoidal fuzzy numbers Uses the geometric mean method to derive fuzzy weights and performance scores</td>
<td>(A) It is easy to extend to the fuzzy case. (A) It guarantees a unique solution to the reciprocal comparison matrix. (D) The computational requirement is tremendous.</td>
</tr>
<tr>
<td>Boender et al. [2]</td>
<td>Modifies van Laarhoven and Pedrycz's method Present a more robust approach to the normalization of the local priorities</td>
<td>(A) The options of multiple experts can be modeled. (D) The computational requirement is tremendous.</td>
</tr>
<tr>
<td>Chang [5]</td>
<td>Synthetical degree values Layer simple sequencing Composite total sequencing</td>
<td>(A) The computational requirement is relatively low. (A) It follows the steps of crisp AHP. It does not involve additional operations. (D) It allows only triangular fuzzy numbers to be used.</td>
</tr>
<tr>
<td>Cheng [10]</td>
<td>Builds fuzzy standards Represent performance scores by membership functions</td>
<td>(A) The computational requirement is tremendous. (D) Entropy is used when probability distribution is known. (D) The method is based on both probability and possibility measures.</td>
</tr>
<tr>
<td>Zeng et al. [23]</td>
<td>Uses arithmetic averaging method to get performance scores. Extension of Saaty’s AHP method with different scales contains triangular, trapezoidal, and crisp numbers.</td>
<td>(A) It follows the steps of crisp AHP. (A) The options of multiple experts can be modeled. (A) There is a flexibility of using different scales. (D) The computational requirement is tremendous when there are much expert’s judgements.</td>
</tr>
</tbody>
</table>

**Step 1.** Pairwise comparison matrices for criteria, subcriterias and alternatives are constructed using linguistic terms. Each element \((\tilde{a}_{ij})\) of the pairwise comparison matrix \(\tilde{A}\) is a fuzzy number corresponding to its linguistic term. The pairwise comparison matrix is given by:

\[
\tilde{A} = \begin{bmatrix}
1 & \tilde{a}_{12} & \ldots & \tilde{a}_{1n} \\
\tilde{a}_{21} & 1 & \ldots & \tilde{a}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{a}_{n1} & \tilde{a}_{n2} & \ldots & 1 \\
\end{bmatrix}
\]  

(25)

Assuming these fuzzy numbers are triangular, Eq. (25) can be rewritten as in Eq. (26):

\[
\tilde{A} = \begin{bmatrix}
1 & (a_{12l}, a_{12m}, a_{12u}) & \ldots & (a_{1nl}, a_{1nm}, a_{1nu}) \\
(a_{21l}, a_{21m}, a_{21u}) & 1 & \ldots & (a_{2nl}, a_{2nm}, a_{2nu}) \\
\vdots & \vdots & \ddots & \vdots \\
(a_{nl1}, a_{nlm}, a_{nlu}) & (a_{n2l}, a_{n2m}, a_{n2u}) & \ldots & 1 \\
\end{bmatrix}
\]  

(26)

For the evaluation procedure, the linguistic terms given in Table 3 are used.

**Step 2.** The consistency of each fuzzy pairwise comparison matrix is examined. In order to check the consistency of the fuzzy pairwise comparison matrices, pairwise comparison values are defuzzified by the graded mean integration approach [6]. Assume \(\tilde{A} = [\tilde{a}_{ij}]\) is a fuzzy positive reciprocal matrix and \(\tilde{A} = [\tilde{a}_{ij}]\) is its defuzzified positive reciprocal matrix. If the result of the comparisons of is consistent, then it can imply that the result of the comparisons of is also consistent [3]. According to the graded mean integration approach, a triangular fuzzy number can be transformed into a crisp number by employing the below equation:

\[
A = \frac{l + 4m + u}{6} 
\]  

(27)

If the pairwise comparisons are not consistent, experts must reevaluate the pairwise comparisons.

**Step 3.** To weigh the criteria and alternatives, the fuzzy geometric mean for each row of matrices is computed. First the geometric means of the first parameters of triangular fuzzy numbers in each row are calculated:

\[
a_{1l} = [1 \times a_{12l} \times \ldots \times a_{1nm}]^{1/n} \\
a_{12} = [a_{21l} \times 1 \times \ldots \times a_{2nu}]^{1/n} \\
\ldots \\
\tilde{a}_{g} = [a_{nl1} \times a_{n2l} \times \ldots \times 1]^{1/n} \\
\]  

Then the geometric means of the second and third parameters of triangular fuzzy numbers in each row are calculated, respectively:

\[
b_{1m} = [1 \times b_{12m} \times \ldots \times b_{1nm}]^{1/n} \\
b_{2m} = [b_{21m} \times 1 \times \ldots \times b_{2nm}]^{1/n} \\
\ldots \\
b_{nm} = [b_{n1m} \times b_{n2m} \times \ldots \times 1]^{1/n} \\
\]  

and
\( c_{tu} = [1 \times c_{2u} \times \ldots \times c_{1m}]^{1/n} \)
\( c_{nu} = [c_{2u} \times 1 \times \ldots \times c_{2m}]^{1/n} \)
\( \ldots \)
\( c_{mu} = [c_{n1u} \times c_{n2u} \times \ldots \times 1]^{1/n} \)

Assume that the sums of the geometric mean values in the row are \( a_{ij} \) for lower parameters; and \( a_{ij} \) for upper parameters. Finally \( \tilde{r}_j \) matrix is obtained by using \( a_{ij} \) values obtained above:

\[
\tilde{r}_j = \begin{bmatrix}
\text{Criterion 1} & \text{Criterion 2} & \text{Criterion } j
\end{bmatrix}
\begin{bmatrix}
\left( \frac{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}, \frac{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}} \right), & \ldots & \left( \frac{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}, \frac{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}} \right)
\end{bmatrix}
\vdots
\begin{bmatrix}
\left( \frac{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}, \frac{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}} \right), & \ldots & \left( \frac{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}, \frac{v_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}}{u_{11} \cdot b_{12} \cdot \ldots \cdot b_{1m}} \right)
\end{bmatrix}
\end{equation}

Step 4. The fuzzy weights and fuzzy performance scores are aggregated as follows:

\[
\tilde{U}_i = \sum_{j=1}^{n} \tilde{w}_j \tilde{r}_j , \quad \forall i
\]

where \( \tilde{U}_i \) is the fuzzy utility of alternative \( i \); \( \tilde{w}_j \) is the weight of the criterion \( j \), and \( \tilde{r}_j \) is the performance score of alternative \( i \) with respect to criterion \( j \).

Step 5. Fuzzy numbers are defuzzified in order to determine the importance ranking of the criteria and alternatives. The Center of Area (COA or Center Index, CI) method can be used for defuzzification in this step [11]. The COA method for a triangular fuzzy number can be calculated as follows:

\[
\text{BNP}_i = \left( \frac{(u_i - l_i) + (m_i - l_i)}{3} + l_i, \quad \forall i \right)
\]

where BNP means best nonfuzzy performance.

Step 6. The best alternative is determined as in the classical AHP.

4.2. Interval type-2 fuzzy AHP

In this section, Buckley’s type-1 fuzzy AHP method will be modified by using interval type-2 fuzzy sets. The procedure of this fuzzy AHP method is explained as follows:

**Table 4**

<table>
<thead>
<tr>
<th>Linguistic scales</th>
<th>Scale of fuzzy number</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1,3)</td>
<td>Equally important</td>
</tr>
<tr>
<td>(1,3,5)</td>
<td>Weakly important</td>
</tr>
<tr>
<td>(3,5,7)</td>
<td>Essentially important</td>
</tr>
<tr>
<td>(5,7,9)</td>
<td>Very strongly important</td>
</tr>
<tr>
<td>(7,9,9)</td>
<td>Absolutely important</td>
</tr>
</tbody>
</table>

**Step 1.** Fuzzy pairwise comparison matrices among all the criteria in the dimensions of the hierarchy system are constructed. The result of the comparisons is constructed as fuzzy pairwise comparison matrices as follows:

\[
\tilde{A} = \begin{bmatrix}
\tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\
\tilde{a}_{21} & 1 & \cdots & \tilde{a}_{2n} \\
\vdots & \ddots & \ddots & \vdots \\
\tilde{a}_{n1} & \cdots & 1 & \tilde{a}_{nn}
\end{bmatrix}
\]

where

\[
1/\tilde{a}_{ij} = \left( \frac{1}{\tilde{a}_{ij}^{\text{type-1}}} \right)
\]

The linguistic variables and their triangular and trapezoidal interval type-2 fuzzy scales which can be used in interval type-2 fuzzy AHP are given in **Table 4**.

**Step 2.** The consistency of each fuzzy pairwise comparison matrix is examined. Assume \( A = [a_{ij}] \) is a positive reciprocal matrix and \( \tilde{A} = [\tilde{a}_{ij}] \) is a fuzzy positive reciprocal matrix. If the result of the comparisons of \( A = [a_{ij}] \) is consistent, then it can imply that the result of the comparisons of \( \tilde{A} = [\tilde{a}_{ij}] \) is also consistent. In order to check the consistency of the fuzzy pairwise comparison matrices, the proposed DTriT or DTriT approach is used.

**Step 3.** The geometric mean of each row is calculated and then the fuzzy weights are computed by normalization. The geometric mean of each row \( \tilde{r}_i \) is calculated as follows:

\[
\tilde{r}_i = \left( \frac{\tilde{a}_{i1} \otimes \cdots \otimes \tilde{a}_{in} }{1} \right)^{1/n}
\]

where

\[
\sqrt[n]{\tilde{a}_{ij}} = \left( \sqrt[n]{a_{ij}^{\text{type-1}}} \right)
\]

**Table 4**

<table>
<thead>
<tr>
<th>Linguistic scales</th>
<th>Triangular interval type-2 fuzzy scales</th>
<th>Trapezoidal interval type-2 fuzzy scales</th>
</tr>
</thead>
<tbody>
<tr>
<td>Absolutely Strong (AS)</td>
<td>(7.5,9.10,5.1) ( \otimes ) (8.5,9.6,5.9)</td>
<td>(7.8,9.9,1.1) ( \otimes ) (7.2,8.2,8.9)</td>
</tr>
<tr>
<td>Very Strong (VS)</td>
<td>(5.5,7.8,5.1) ( \otimes ) (6.5,7.7,5.9)</td>
<td>(5.6,8.9,1.1) ( \otimes ) (5.2,6.2,7.8)</td>
</tr>
<tr>
<td>Fairly Strong (FS)</td>
<td>(3.5,5.6,5.1) ( \otimes ) (4.5,5.5,5.9)</td>
<td>(3.4,6.7,1.1) ( \otimes ) (3.2,4.2,5.6,8)</td>
</tr>
<tr>
<td>Slightly Strong (SS)</td>
<td>(1.5,3.4,5.1) ( \otimes ) (2.5,3.3,5.9)</td>
<td>(1.2,4.5,1.1) ( \otimes ) (1.2,2.3,4.8)</td>
</tr>
<tr>
<td>Exactly Equal (E)</td>
<td>(1.1,1.1) ( \otimes ) (1.1,1.1)</td>
<td>(1.1,1.1,1.1) ( \otimes ) (1.1,1.1,1.1)</td>
</tr>
</tbody>
</table>

Reciprocals of above

Reciprocals of above
The fuzzy weight of the ith criterion is calculated as follows:

$$\tilde{w}_i = \tilde{r}_i \circ \left[ \tilde{r}_1 \otimes \cdots \otimes \tilde{r}_n \right]^{-1}$$  \hspace{1cm} (33)

where

$$\tilde{r}_i = \left( \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right) = \left( \begin{array}{cccc} \min \{ H_{i1}(a), H_{i2}(b) \} & \min \{ H_{i1}(a), H_{i3}(b) \} & \cdots & \min \{ H_{i1}(a), H_{in}(b) \} \\ \min \{ H_{i2}(a), H_{i1}(b) \} & \min \{ H_{i2}(a), H_{i3}(b) \} & \cdots & \min \{ H_{i2}(a), H_{in}(b) \} \\ \vdots & \vdots & \ddots & \vdots \\ \min \{ H_{i(n-1)}(a), H_{i1}(b) \} & \min \{ H_{i(n-1)}(a), H_{i2}(b) \} & \cdots & \min \{ H_{i(n-1)}(a), H_{in}(b) \} \end{array} \right)$$

Table 5
Pairwise comparison matrix for criteria.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>E</td>
<td>1/FS</td>
<td>FS</td>
<td>VS</td>
</tr>
<tr>
<td>Q</td>
<td>FS</td>
<td>E</td>
<td>VS</td>
<td>VS</td>
</tr>
<tr>
<td>D</td>
<td>1/FS</td>
<td>1/VS</td>
<td>E</td>
<td>SS</td>
</tr>
<tr>
<td>C</td>
<td>1/VS</td>
<td>1/VS</td>
<td>1/SS</td>
<td>E</td>
</tr>
</tbody>
</table>

Table 6
Pairwise comparison matrix for alternatives.

<table>
<thead>
<tr>
<th>w.r.t. Q</th>
<th>S1</th>
<th>S2</th>
<th>w.r.t. P</th>
<th>S1</th>
<th>S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>E</td>
<td>1/FS</td>
<td>S1</td>
<td>E</td>
<td>SS</td>
</tr>
<tr>
<td>S2</td>
<td>FS</td>
<td>E</td>
<td>S2</td>
<td>1/SS</td>
<td>E</td>
</tr>
</tbody>
</table>

Table 7
Defuzzified pairwise comparison matrix for criteria.

<table>
<thead>
<tr>
<th>w.r.t. the Goal</th>
<th>P</th>
<th>Q</th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>1</td>
<td>0.2</td>
<td>4.75</td>
<td>6.65</td>
</tr>
<tr>
<td>Q</td>
<td>4.75</td>
<td>1</td>
<td>6.65</td>
<td>6.65</td>
</tr>
<tr>
<td>D</td>
<td>0.21</td>
<td>0.1</td>
<td>1</td>
<td>2.85</td>
</tr>
<tr>
<td>C</td>
<td>0.14</td>
<td>0.1</td>
<td>0.44</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 8
Defuzzified pairwise comparison matrices for alternatives.

<table>
<thead>
<tr>
<th>w.r.t. Q</th>
<th>S1</th>
<th>S2</th>
<th>w.r.t. P</th>
<th>S1</th>
<th>S2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>1</td>
<td>0.21</td>
<td>S1</td>
<td>1</td>
<td>2.85</td>
</tr>
<tr>
<td>S2</td>
<td>4.75</td>
<td>1</td>
<td>S2</td>
<td>0.44</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 9
Pairwise comparisons for the criteria.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(1,1,1,1,1)</td>
<td>(1,1,1,1,1)</td>
<td>(0.14,0.16,0.25,0.33,1,1)</td>
<td>(3.4,6,7,1,1)</td>
</tr>
<tr>
<td>Q</td>
<td>(3.4,6,7,1,1)</td>
<td>(3.2,4,2,5,8,6,8,0.8,8,0.8)</td>
<td>(0.14,0.17,0.23,0.31,0.8,0.8,0.8)</td>
<td>(5.6,8,9,1,1)</td>
</tr>
<tr>
<td>D</td>
<td>(0.14,0.16,0.25,0.33,1,1)</td>
<td>(0.11,0.12,0.16,0.2,1,1)</td>
<td>(1,1,1,1,1)</td>
<td>(1,1,1,1,1)</td>
</tr>
<tr>
<td>C</td>
<td>(0.11,0.12,0.16,0.2,1,1)</td>
<td>(0.11,0.12,0.16,0.2,1,1)</td>
<td>(0.2,0.25,0.5,1,1)</td>
<td>(1,1,1,1,1)</td>
</tr>
</tbody>
</table>

Step 4. The fuzzy weights and fuzzy performance scores are aggregated as follows:

$$U_i = \sum_{j=1}^{n} \tilde{w}_j \tilde{r}_{ij}, \forall i.$$  \hspace{1cm} (34)

where $U_i$ is the fuzzy utility of alternative $i$; $\tilde{w}_j$ is the weight of the criterion $j$, and $\tilde{r}_{ij}$ is the performance score of alternative $i$ with respect to criterion $j$.

Step 5. The classical AHP method’s procedure is applied to determine the best alternative.

5. An application

Consider a supplier selection problem with four criteria and two alternatives (S1 and S2). The hierarchy of the problem is given in Fig. 3. The considered criteria are price (P), quality (Q), delivery (D), and capacity (C). The abbreviations of the linguistic variables are represented in Table 3.

We apply the steps of the proposed type-2 fuzzy AHP method in the following.

Step 1. Construction of the pairwise comparison matrices. Tables 5 and 6 present pairwise comparison matrices for criteria and alternatives using linguistic terms.

Step 2. Consistency check of defuzzified pairwise comparison matrices. Using the proposed DTrAT method, the defuzzified values in Tables 7 and 8 are obtained.

The defuzzified pairwise comparison matrix for criteria has been checked for its consistency and it has been found that its consistency ratio is under 0.1.

Step 3. Geometric mean of each row is calculated as shown below;

$$\tilde{r}_1 = \left[ \tilde{a}_{11} \otimes \tilde{a}_{12} \otimes \tilde{a}_{13} \otimes \tilde{a}_{14} \right]^{1/4}$$

$$= [(1,1,1,1,1),(1,1,1,1,1),(0.14,0.16,0.25,0.33,1,1)]$$

$$\times (3.2,4,2,5,8,6,8,0.8,0.8) \otimes (5.6,8,9,1,1)$$

$$= (2.14,4.00,12.21,1.1)$$

$$= (1.20,1.41,1.86,2.14,1.1)$$

$$= (1.22,2.23,6.48,0.8,0.8)$$

Step 4. The priority weights of criteria and alternatives are determined by using Eq. (33). For example the priority weights of the criteria (Table 5) can be calculated as follows. The calculated geometric means are found in Step 3 as follows;

Using the data represented in Table 10, the priority weight of $P$ can be calculated as shown below:
Geometric means of pairwise comparison matrix for the criteria.

\[
\omega_1 = \sqrt[n]{\prod_{i=1}^{n} a_{ii}}^{-1} = (1.20, 1.41, 1.86, 2.14, 1.1) \times (1.25, 1.45, 1.81, 2.07, 0.8) \times \ldots \times (1.20, 1.41, 1.86, 2.14, 1.1) \times (1.25, 1.45, 1.81, 2.07, 0.8) \times (2.94, 3.46, 4.42, 4.87, 1.1) \times (3.04, 3.56, 4.33, 4.79, 0.8) \times (0.35, 0.45, 0.63, 0.75, 1.1) \times (0.37, 0.46, 0.61, 0.73, 0.8) \times (0.22, 0.25, 0.34, 0.44, 1.1) \times (0.22, 0.25, 0.32, 0.41, 0.8) \times (4.73, 5.58, 7.27, 8.22, 1.1) \times (4.90, 5.74, 7.09, 8.02, 0.8) \times (0.12, 0.13, 0.17, 0.21, 1.1) \times (0.12, 0.14, 0.17, 0.20, 0.8) \times (0.14, 0.19, 0.33, 0.45, 1.1) \times (0.15, 0.20, 0.31, 0.42, 0.8) \times (0.004, 0.006, 0.16, 0.28, 1.1) \times (0.001, 0.002, 0.01, 0.02, 1.1) \times (0.0001, 0.0001, 0.0003, 0.001, 0.01) \times (0.0002, 0.0002, 0.0006, 0.001, 0.08) \times (0.0002, 0.0005, 0.0009, 0.1) \times (0.0002, 0.0005, 0.0008, 0.08) \times (0.0001, 0.0001, 0.0003, 0.001, 0.08) \times (0.0001, 0.0001, 0.0003, 0.001, 0.08) \times (0.0001, 0.0001, 0.0003, 0.001, 0.08) \times (0.0001, 0.0001, 0.0003, 0.001, 0.08) \times (0.0001, 0.0001, 0.0003, 0.001, 0.08) \times (0.0001, 0.0001, 0.0003, 0.001, 0.08)
\]

Table 10
Geometric means of pairwise comparison matrix for the criteria.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>P</th>
<th>Q</th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>(1.20, 1.41, 1.86, 2.14, 1.1)</td>
<td>(1.25, 1.45, 1.81, 2.07, 0.8)</td>
<td>(2.94, 3.46, 4.42, 4.87, 1.1)</td>
<td>(3.04, 3.56, 4.33, 4.79, 0.8)</td>
</tr>
</tbody>
</table>

Table 11
Type 2 fuzzy weights of the criteria and alternatives.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>With respect to the Goal</th>
<th>Priority weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(0.14, 0.19, 0.33, 0.45, 1.1)</td>
<td>(0.15, 0.20, 0.31, 0.42, 0.8)</td>
</tr>
<tr>
<td>Q</td>
<td>(0.35, 0.47, 0.79, 1.03, 1.1)</td>
<td>(0.38, 0.50, 0.75, 0.97, 0.8)</td>
</tr>
<tr>
<td>D</td>
<td>(4.31, 2.10, 1.10, 1.61, 1.1)</td>
<td>(4.69, 6.61, 1.01, 1.44, 0.8)</td>
</tr>
<tr>
<td>C</td>
<td>(2.70, 3.43, 6.15, 9.45, 1.1)</td>
<td>(2.83, 3.61, 7.38, 5.84, 0.8)</td>
</tr>
</tbody>
</table>

Table 12
Type 2 fuzzy global weights of the alternatives.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Q</th>
<th>P</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weights</td>
<td>(0.35, 0.47, 0.79, 1.03, 1.1)</td>
<td>(0.14, 0.19, 0.33, 0.45, 1.1)</td>
<td>(0.15, 0.20, 0.31, 0.42, 0.8)</td>
</tr>
<tr>
<td>Alternatives</td>
<td>Local weights</td>
<td>Local weights</td>
<td>Local weights</td>
</tr>
<tr>
<td>S1</td>
<td>(0.11, 0.13, 0.20, 0.27, 1.1)</td>
<td>(0.30, 0.52, 1.04, 1.54, 1.1)</td>
<td>(0.12, 0.14, 0.19, 0.25, 0.8)</td>
</tr>
<tr>
<td>S2</td>
<td>(0.53, 0.67, 1.01, 1.25, 1.1)</td>
<td>(0.30, 0.52, 1.04, 1.54, 1.1)</td>
<td>(0.10, 0.19, 0.33, 0.58, 0.8)</td>
</tr>
<tr>
<td>S1</td>
<td>(0.0009, 0.01, 0.11, 0.12, 1.1)</td>
<td>(0.10, 0.19, 0.33, 0.45, 1.1)</td>
<td>(0.78, 0.84, 0.94, 1.00, 1.1)</td>
</tr>
<tr>
<td>S2</td>
<td>(0.0009, 0.01, 0.11, 0.12, 1.1)</td>
<td>(0.10, 0.19, 0.33, 0.45, 1.1)</td>
<td>(0.78, 0.84, 0.92, 0.98, 0.8)</td>
</tr>
</tbody>
</table>

Table 13
Type 2 fuzzy and defuzzified overall weights of the alternatives.

<table>
<thead>
<tr>
<th>Type 2 fuzzy global weights</th>
<th>Defuzzified weights</th>
<th>Normalized crisp weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>(0.10, 0.20, 0.63, 1.24, 1.1)</td>
<td>0.502</td>
</tr>
<tr>
<td>S2</td>
<td>(0.23, 0.39, 1.03, 1.81, 1.1)</td>
<td>0.805</td>
</tr>
</tbody>
</table>
Comparison of type-1 and type-2 fuzzy AHP methods.

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Type-1 AHP</th>
<th>Type-2 AHP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Weights with respect to the goal</td>
<td>Weights with respect to the goal</td>
</tr>
<tr>
<td></td>
<td>Rank</td>
<td>Rank</td>
</tr>
<tr>
<td>P</td>
<td>0.282</td>
<td>2</td>
</tr>
<tr>
<td>Q</td>
<td>0.418</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>0.168</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>0.132</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Weights of the alternatives</td>
<td>Weights of the alternatives</td>
</tr>
<tr>
<td>S1</td>
<td>0.590</td>
<td>0.410</td>
</tr>
<tr>
<td>S2</td>
<td>0.338</td>
<td>0.662</td>
</tr>
<tr>
<td></td>
<td>Overall priority</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.440</td>
<td>0.560</td>
</tr>
</tbody>
</table>

Fig. 5. Defuzzified value of S2.

There are limited number of defuzzification and ranking methods for type-2 fuzzy sets in the literature. New defuzzification and ranking methods can be proposed and compared with the existing ones.

References


6. Conclusion

Type-1 fuzzy sets are somewhat problematic in defining membership functions since it is not possible to model uncertainty and imprecision sufficiently. Type-2 fuzzy sets capture this problem by incorporating footprint of uncertainty into type-1 fuzzy sets. Fuzzy AHP based on interval type-2 fuzzy sets has been developed for the first time in this paper. Linguistic scales have been also developed to be used in the proposed fuzzy AHP method. Thus a flexible definition opportunity to decision makers has been provided.

In this paper we extended Buckley’s fuzzy AHP approach using interval type-2 fuzzy sets. Nonsymmetrical interval type-2 membership functions may create significant differences in the results with respect to the ones of type-1 AHP methods.

For further research, other fuzzy AHP methods like Chang’s [5] or Laarhoven and Pedrycz’s [15] may be extended by type-2 fuzzy sets and the obtained results can be compared with ours. Other multi attribute decision making techniques such as; ELECTRE, VIKOR, and MACBETH can be also extended by using interval type-2 fuzzy sets.