Optimizing algebraic connectivity by edge rewiring

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Abstract

Robustness in complex networks is an ongoing research effort that seeks to improve the connectivity of networks against attacks and failures. Among other measures, algebraic connectivity has been used to characterize processes such as damped oscillation of liquids in connected pipes. Similar characterizations include the number of edges necessary to disconnect a network: the larger the algebraic connectivity, the larger the number of edges required to disconnect a network and hence, the more robust a network. In this paper, we answer the question, "Which edge can we rewire to have the largest increase in algebraic connectivity?". Furthermore, we extend the rewiring of a single edge to rewiring multiple edges to realize the maximal increase in algebraic connectivity. The answer to this question above can provide insights to decision makers within various domains such as communication and transportation networks, who seek an efficient solution to optimize the connectivity and thus increase the robustness of their networks. Most importantly, our analytical and numerical results not only provide insights to the number of edges to rewire, but also the location in the network where these edges would effectuate the maximal increase in algebraic connectivity and therefore, enable a maximal increase in robustness.

1. Introduction and motivation

Complex networks including biological, power grid, Internet, and transportation networks can range from hundreds to millions of vertices, and their characterization provides mechanisms to enhance their performance and realize their impact to our standard of living [1–3]. For example, the US freight network (a classic example of a transportation network), transports over $200 billion worth of products every year. The ailing transportation infrastructure is unable to sustain the economic needs of the US and a predicted increase in population (an estimated 70 million additional individuals by 2035) and trade only exacerbates the situation. To further substantiate this, in 2009 Americans spent 4.2 billion hours sitting in traffic: a cost of $115 billion dollars [4]. From a post analysis of the transportation infrastructure report for 2011, the deployment of new roadways contributes to an overall solution to alleviate traffic congestion. For this example, our results provide insights to select endpoints for roadways to increase connectivity and robustness of the transportation network. These results can apply similarly to power grid utility companies such as AEP that propose an increase of 19,000 miles to the national transmission line network, in an effort to upgrade the aging power grid infrastructure [5].

Algebraic connectivity is a spectral measure to determine the robustness of a graph. As a topological measure, we recognize the limitations of algebraic connectivity when used as the determining factor to increase the robustness of a real-world network [6–9]. For such networking domains, other measures particular to the behavior of the network under consideration can be used in addition to algebraic connectivity to provide a comprehensive solution to increase the robustness of a network.

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In this paper, we endeavor to answer the question of where an edge should be rewired to increase algebraic connectivity the most. Our approach is based on studies conducted to determine where an edge should be added to increase algebraic connectivity the most \cite{10,11}. Given a network \( G(V, E) \) such that \( |V| \) is the number of vertices, and \( |E| \) is the number of edges, the number of possibilities to rewire an edge is given by \( \binom{|E|}{2} - |E| \). For complex networks, comparing each edge to find the optimal one that maximizes algebraic connectivity is infeasible. Furthermore, as a complimentary problem, it has been proven that maximum algebraic connectivity augmentation is NP-Hard \cite{12}. For this reason, we propose a strategy that rewire edges to maximally increase the algebraic connectivity of a network.

In our approach, we consider the rewiring of an edge as a two step process where we either insert an edge then remove an edge or we remove an edge then insert an edge. Hence, our original question of “Where should an edge be rewired to increase algebraic connectivity the most?” is subdivided into two parts:

1. “Where should an edge be removed to decrease algebraic connectivity the least?”
2. “Where should an edge be added to increase algebraic connectivity the most?”

The latter question is addressed \cite{10,11}. Therefore, this paper focuses on the first question and contributes the following:

- Two corollaries to develop the framework for constructing the upper and lower bounds for algebraic connectivity when an edge is removed.
- A method to select the edge that when removed, decreases algebraic connectivity the least.
- An algorithm that removes edges to numerically validate our analytical results for the upper and lower bounds. Additionally, we present a second algorithm to rewire edges, and a third algorithm to add edges to maximally increase algebraic connectivity. All algorithms have a running time \( O(|V|^2) \).
- The comparison of three network models to determine which network model realizes the highest increase in algebraic connectivity when a small percent of the edges are rewired while keeping the number of nodes and edges constant.

The structure of this paper is outlined as follows: Section 2 builds on the introduction by providing the necessary background and state of the art for algebraic connectivity. Section 3 reviews theorems, definitions, and introduces two corollaries to two of the theorems presented and Section 4 presents the lower and upper bounds for algebraic connectivity when an edge is removed. In Section 5, we review the three network models used in our analysis: Watts–Strogatz model, Gilbert’s stochastic model, and Barabási–Albert Scale Free model. Section 6 describes an algorithm for edge removal. Furthermore, we provide the numerical analysis for edge removal for the three classes of networks. In Section 7, we compare graphs from the three different models to determine which model realizes the greatest increase in algebraic connectivity through rewiring. Section 8 presents a second algorithm and the corresponding implementation to rewire edges to maximally increase algebraic connectivity. In Section 9, we present a third algorithm to add edges to increase algebraic connectivity. A comparison is then drawn based on the results of adding edges to that of rewiring edges to maximally increase algebraic connectivity. Section 10 presents a discussion on the applicability of this work in the real-world and finally, Section 11 discusses the benefits and shortcomings of the rewiring approach and highlights the future direction of this work.

2. Background and related work

The classical approach for determining robustness of networks entails the use of basic concepts from graph theory. For instance, the connectivity of a graph is an important, and probably the earliest, measure of robustness of a network \cite{13}. Vertex (edge) connectivity, defined as the size of the smallest vertex (edge) cut, determines in a certain sense the robustness of a graph to the deletion of vertices (edges). However, the vertex or edge connectivity only partly reflects the ability of graphs to retain certain degrees of connectedness after deletion. Other improved measures were introduced and studied, including super connectivity \cite{14}, conditional connectivity \cite{15}, restricted connectivity \cite{16}, fault diameter \cite{17}, toughness \cite{18}, scattering number \cite{19}, tenacity \cite{20}, expansion parameter \cite{21}, and isoperimetric number \cite{22}. In contrast to vertex (edge) connectivity, these new measures consider both the cost to damage a network and how badly the network is damaged.

Subsequent measures consider the size of the largest connected component as vertices are attacked \cite{23}. Furthermore, percolation models were used to assess the damage incurred by random graphs \cite{24}. More recent efforts present a topological analysis of robustness in networks such as the power grid \cite{25}. Other metrics in networking literature include the average node degree \cite{26}, betweenness \cite{27}, heterogeneity \cite{28}, and characteristic path length \cite{29}.

The measures reviewed thus far, consider the network structure to assess robustness. However, recent efforts have incorporated the behavior of the network to assess robustness \cite{6,8,9}. The authors maximized flows in the network while imposing constraints on routers and links.

From spectral analysis, experimentalists have generally utilized the second smallest Laplacian eigenvalue to guarantee connectivity of a graph: if this value is 0, a graph is disconnected \cite{30}. Furthermore, several relationships have been established between algebraic connectivity and graph theoretical measures such as network diameter and are relevant to domains such as the Internet to understanding the implications of protocols such as spanning tree \cite{31,32}. With regards to robustness, the second smallest eigenvalue has also been considered as a measure of how difficult it is to break the network into...
3. Principles of algebraic connectivity

Throughout this paper $G = (V, E)$ is an undirected, connected graph with vertex set $V = 1, \ldots, N$ and edge set $E$, such that $N = |V|$ is the number of vertices. $u, w, z$ are vectors, $\lambda$ is an eigenvalue, and $\text{deg}(v)$ is the vertex degree of vertex $v \in V$.

**Definition 1.** Given a graph $G$, the Laplacian $L(G)$ of $G$ is an $N \times N$ matrix $L$ defined by

$$
L = \begin{cases} 
\text{deg}(i) & \text{if } i = j, \\
-1 & \text{if } i \neq j \text{ and } (i, j) \in E, \\
0 & \text{if } i \neq j \text{ and } (i, j) \notin E.
\end{cases}
$$

$L(G)$ is a symmetric positive semidefinite matrix with all real and non-negative eigenvalues. The set of eigenvalues denoted by $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_N(G)$, is the Laplacian spectrum of graph $G$.

**Definition 2.** The algebraic connectivity of a graph $G$ is the second-smallest eigenvalue of $L(G): \lambda_2(G)$

**Theorem 1.** Let $G$ be a graph with $N$ vertices. Let $G + e$ be the augmented graph obtained by adding edge $e$ between two vertices in $G$. Then the eigenvalues of $G$ and $G + e$ are intertwined as follows [35]:

$$0 = \lambda_1(G) \leq \lambda_1(G + e) \leq \lambda_2(G) \leq \lambda_2(G + e) \leq \cdots \leq \lambda_N(G) \leq \lambda_N(G + e).$$

If $\lambda_2(G)$ is a multiple eigenvalue such that $\lambda_2(G) = \lambda_2(G + e)$, the result of adding an edge does not improve the algebraic connectivity. Given that the trace($L$) = $\sum_{i=1}^N \lambda_i(G) = 2|E|$, it follows that

$$\sum_{i=1}^N (\lambda_i(G + e) - \lambda_i(G)) = 2,$$

which implies that $0 \leq \lambda_2(G + e) - \lambda_2(G) \leq 2$. Additionally, we deduce that given a graph with $N$ vertices, the magnitude of $\lambda_i$ for $i \in N$ tends to increase as $|E|$ increase.

**Corollary 1.** Let $G$ be a graph with $N$ vertices. Let $G - e$ be the augmented graph obtained by removing an edge $e$ between two vertices in $G$ such that the removal of an edge does not disconnect the graph. Then the eigenvalues of $G$ and $G - e$ are intertwined as follows:

$$0 = \lambda_1(G - e) \leq \lambda_1(G) \leq \lambda_2(G - e) \leq \lambda_2(G) \leq \cdots \leq \lambda_N(G - e) \leq \lambda_N(G).$$

We can also deduce that:

$$\sum_{i=1}^N (\lambda_i(G) - \lambda_i(G - e)) = 2.$$ (2)

This implies that $0 \leq \lambda_2(G) - \lambda_2(G - e) \leq 2$ and that given a graph with $N$ vertices, the magnitude of $\lambda_i$ for $i \in N$ tends to increase as $|E|$ increase.

**Theorem 2.** Let $G$ be a connected graph with $N$ vertices and let $i$ and $j$ be two non-adjacent vertices in $G$. The largest possible increase in algebraic connectivity occurs if and only if $G = K_N \setminus \{i, j\}$: the complete graph with one edge removed [36].

**Corollary 2.** Let $G$ be a connected graph with $N$ vertices and let $i$ and $j$ be two non-adjacent vertices in $G$. The largest possible decrease in algebraic connectivity occurs if and only if $G = K_N$: the complete graph.

**Theorem 3.** Let $G$ be a simple connected graph with $N > 2$. If $G$ has a pendant vertex (i.e. a vertex with degree 1), $\lambda_2 \leq 1$. Moreover, $\lambda_2 < 1$ if the pendant vertex is not adjacent to the highest degree vertex [37].

Complex networks typically contain pendant vertices and for this reason $\lambda_2(G) < 1$. This implies that $\lambda_2(G - e) < 1$. 
The removal of edge \( v_i v_j \) from \( G \) for \( i, j \in V \) can be achieved using a positive semidefinite matrix \( B \). An example of \( B \) such that \( i = 1 \) and \( j = 2 \) is shown below:

\[
\begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

Thus, for the spectrum \( \lambda_1(G-e), \ldots, \lambda_n(G-e) \) of \( L-B \), we have

\[
0 = \lambda_1(G-e) = \lambda_1(G) \leq \lambda_2(G-e) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G-e) \leq \lambda_n(G).
\]

### 4.1. Upper bound for \( \lambda_2(G-e) \)

Given that \( v_i v_j \) are the vertices from which an edge is removed, let \( z \) be a vector with \(+1\) for the \( i \)th component, \( -1 \) for the \( j \)th component, and 0 otherwise. Additionally, let \( u^{(2)}_i \) represent the \( i \)th element of the eigenvector that corresponds to \( \lambda_2 \) the second smallest eigenvalue. It follows that our matrix \( B = zz^T \). Also, let \( z := |(z, u^{(2)}_i)| = |u^{(2)}_i| - u^{(2)}_i(G) \), such that \((i, j) \in E: \) the set of edges of \( G \). For a vector \( w \perp u^{(1)}(G-e) \), and assuming \( u^{(2)}(G) = w \) the Rayleigh quotient has the following property:

\[
R(u^{(2)}) = u^{(2)T}L u^{(2)} - u^{(2)T}zz^T u^{(2)} = \lambda_2 - u^{(2)T}zz^T u^{(2)} = \lambda_2 - \alpha^2.
\]

Therefore,

\[
\lambda_2(G-e) \leq \lambda_2(G) - \alpha^2.
\]

From the upper bound for \( \lambda_2(G-e) \), we deduce that the lower \( \alpha \) is that is, the smaller the difference between elements of the eigenvector corresponding to the second smallest eigenvalue, the higher the upper bound.

### 4.2. Lower bound for \( \lambda_2(G-e) \)

To obtain the lower bound, we use the technique of intermediate value problems [38]. Our new Laplacian \( L' = L - zz^T \). To make \( zz^T \) positive definite, we replace it by \( C := -zz^T - \epsilon I \). If we let \( k = 2 \), \( p^{(r)} := C^{-1}u^{(r)} \), such that \( r = 1, \ldots, k \), we get the matrix \( (\gamma_{rs})_{r,s=1,2} := \left( \langle p^{(r)}, Cp^{(s)} \rangle \right)^{-1} \)

\[
\begin{pmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{21} & \gamma_{22}
\end{pmatrix}
\]

In particular, each element of the matrix can be obtained by first expanding \( \gamma_{rs} \) as follows:

\[
\gamma_{rs} = \left[ (-p^{(r)})^T zz^T p^{(s)} - \epsilon (p^{(r)})^T p^{(s)} \right]^{-1}.
\]

Secondly, given the nonsingular \( N \times N \) matrix \( C \) and vector \( z \), we obtain the inverse of \( C \) using Sherman–Morrison’s formula as follows [39]:

\[
C^{-1} = \frac{1}{\epsilon} I - \frac{zz^T}{\epsilon^2 + |z|^2}.
\]

Therefore, if \( r = s = 1 \), \((\gamma_{11})^{-1}\) can be computed as follows:

\[
(\gamma_{11})^{-1} = -\left[ \frac{1}{C^{-1}u^{(1)}} \right]^T zz^T \left[ \frac{1}{C^{-1}u^{(1)}} \right] - \epsilon \left[ \frac{1}{C^{-1}u^{(1)}} \right]^T \frac{1}{C^{-1}u^{(1)}} ,
\]

\[
= -\left[ \frac{1}{(u^{(1)})^T zz^T u^{(1)}} \right] - \epsilon \left[ \frac{u^{(1)}}{\epsilon} \right]^2 \frac{1}{\epsilon^2 + |z|^2} ,
\]

\[
= -\epsilon (z^T u^{(1)})^2 - \left[ |u^{(1)}(\epsilon + |z|^2) - z(z^T u^{(1)})|^2 \right] \frac{1}{\epsilon (\epsilon + |z|^2)^2} ,
\]

\[
= -\epsilon (z^T u^{(1)})^2 - \frac{|\epsilon + |z|^2|^2}{\epsilon (\epsilon + |z|^2)^2} - \frac{|z^T u^{(1)}|^2 |z|^2 - 2(\epsilon + |z|^2)(z^T u^{(1)})^2}{\epsilon (\epsilon + |z|^2)^2} .
\]
Since $u^{(1)}$ is constant, $z^j u^{(1)} = 0$. Therefore,

$$- \frac{\epsilon + |z|^2}{\epsilon(\epsilon + |z|^2)^2} = - \frac{1}{\epsilon}. \quad (10)$$

From this we obtain $\gamma_{11} = -\epsilon$. Using our previous formulations for $\gamma_{11}$, if $r = s = 2$, we compute $\gamma_{22}$ as follows:

$$\gamma_{22} = \frac{(z^j u^{(2)})^2 - |z|^2}{\epsilon(\epsilon + |z|^2)^2}. \quad (11)$$

$$= \frac{(z^j u^{(2)})^2 - |z|^2}{\epsilon(\epsilon + |z|^2)^2}. \quad (12)$$

Let $x = |z^j u^{(2)}| = |u^{(2)} - u^{(2)}|$; the difference between the $ith$ and $jth$ elements of $u^{(2)}$, the vector corresponding to the second smallest eigenvalue. Since $|z|^2 = 2$, it follows that:

$$\gamma_{22} = \frac{x}{\frac{\epsilon}{22} - 1}. \quad (13)$$

For $r \neq s$, $\gamma_{rs} = 0$. Therefore, the matrix $\gamma_{rs}$ is constructed as follows:

$$\begin{pmatrix}
-\epsilon & 0 \\
0 & \frac{x}{\epsilon}
\end{pmatrix}.$$ 

The intermediate eigenvalue problem corresponding to the second Rayleigh quotient becomes:

$$Lu + \langle u, u^{(1)} \rangle \gamma_{11} u^{(1)} + \langle u, u^{(2)} \rangle \gamma_{22} u^{(2)} = \tau u. \quad (14)$$

We then use a matrix $S$ to extract the spectrum of $L'$ as follows:

$$\begin{pmatrix}
\tau_1 & -1 & 0 & \cdots & 0 \\
0 & \tau_2 & 0 & \cdots & 0 \\
0 & 0 & \tau_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \tau_N
\end{pmatrix} = S^{-1}L'S.$$

The spectrum of $L'$ becomes:

$$\tau_1 = -\epsilon, \quad \tau_2 = \lambda_2 + \frac{\epsilon}{\frac{\epsilon}{22} - 1}, \quad \tau_3 = \lambda_3, \ldots, \tau_N = \lambda_N. \quad (15)$$

Since our objective value is the second smallest in the sequence, the lower bound for $\lambda_2(G - e)$ is as follows:

$$\lambda_2(G - e) \geq \min \{ \tau_2 + \epsilon, \tau_3 + \epsilon \}. \quad (16)$$

Substituting the values for $\tau_2$ and $\tau_3$, we get:

$$\lambda_2(G - e) \geq \min \left\{ \lambda_2 + \frac{\epsilon}{\frac{\epsilon}{22} - 1} + \epsilon, \lambda_3 + \epsilon \right\}. \quad (17)$$

The best lower bound is therefore achieved by the choice of $\epsilon$ that makes both term equal. For $\xi = \lambda_3(G) - \lambda_2(G)$, we let

$$\epsilon = - \frac{\xi + 2}{2} + \frac{\left(\frac{\xi + 2}{2} + \xi(\xi^2 - 2) \right)}{2}. \quad (18)$$

Hence, a decrease in $\alpha$ decreases $\epsilon$ and increases the lower bound. Finally, combining the upper and lower bounds, we obtain the following bounds for algebraic connectivity after removing an edge:

$$\min \left\{ \lambda_2(G) + \frac{\epsilon x^2}{x^2 + (-2 - \epsilon)}, \lambda_3(G) + \epsilon \right\} \leq \lambda_2(G - e) \leq \lambda_2(G) - \alpha^2.$$
As shown, a smaller \(a\) leads to a higher upper bound and also tends to increase the lower bound. This means that a smaller \(a\) leads to the minimal decrease in algebraic connectivity. In other words, we should remove an edge with the smallest 
\[
\left| u^{(2)}(G) - u^{(2)}_i(G) \right|
\]
that is an edge that connects two strongly connected vertices in \(G\). Combining the removal and addition of edges, we obtain the following approach to rewiring edges such that algebraic connectivity increases the most:

1. Remove an edge such that \(\left| u^{(2)}(G) - u^{(2)}_i(G) \right|\) is the lowest
2. Insert an edge such that \(\left| u^{(2)}(G) - u^{(2)}_i(G) \right|\) is the highest

or

1. Insert an edge such that \(\left| u^{(2)}(G) - u^{(2)}_i(G) \right|\) is the highest
2. Remove an edge such that \(\left| u^{(2)}(G) - u^{(2)}_i(G) \right|\) is the lowest

5. Network models

This section reviews the three network models studied in this paper:

1. Watts–Strogatz model
2. Gilbert stochastic model
3. Barabási–Albert Scale Free model

5.1. Watts–Strogatz model (WS)

The Watts–Strogatz model is constructed by interpolating between a regular ring lattice and a random network. The construction begins with a ring of \(N\) vertices. Each vertex is then connected to its \(k\) nearest neighbors. Then in a clockwise manner, vertex \(i\) is selected. The edge that connects to \(i\)'s nearest neighbor is randomly rewired with a probability of \(p\) (or left untouched with a probability of \(1 - p\)), considering the constraint that no self-loops or duplicate loops can exist. This procedure is repeated cyclically for each successive vertex until vertex \(i\) is once again selected. At this point, the edge that connects to \(i\)'s second nearest neighbor undergoes similar rewiring procedures. This cycle of vertex selection and rewiring recurs until the edge that connects all vertices \(i\) to their furthest neighbor is considered [40].

In the Watts–Strogatz model, the parameter \(p\) determines the level of randomness in the graph while maintaining the initial number of vertices and edges [24]. For intermediate values of \(p\), Watts–Strogatz model produces a Small-world network, which captures the high clustering properties of regular graphs and the small characteristic path length of random graph models. Fig. 1 shows three snapshots of graphs obtained for different values of \(p\).

For the Watts–Strogatz networks used in this paper, we generated three networks with the respective sizes of \(N = 100, 400, 800\) and a rewiring probability of 0.6 [41].

5.2. Gilbert stochastic model (Gi)

A random graph is obtained by random addition of edges between \(N\) vertices. Erdős–Rényi (ER) stochastic model is one of the most studied of these models. In the construction of an ER graph \(G(V, E)\), \(|E|\) edges are connected at random to \(N = |V|\) vertices [24]. For this model, each of the \(\frac{N(N-1)}{2}\) edges have an equal probability of being selected. However, this paper

![Fig. 1. The construction of Watts–Strogatz model. For the regular graph \(p = 0\). The random graph is obtained at \(p = 1\) and for intermediate values of \(p\), a Small-world network is realized [40].](image-url)
considers the Gilbert stochastic model $G(V, p)$, a modified version of the ER model, in which edges are connected to vertices with a probability of $p$. As opposed to the ER model, the number of edges in a graph produced by the Gi model is not known in advance. Below are a few key properties of random graphs:

- The average node degree $k$, such that $k = \text{deg}(v)$, determines the connectivity of the graph. Therefore, if $k < 1$, a disconnected component exists. At $k = 1$, a phase transition occurs, and a giant component exists when $k > 1$ [24].
- The node degree $k$ exhibits a binomial distribution and thus, given $N$ vertices and a probability of $p$,

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}.$$  \hfill (18)

However, the model in this paper was based on the poisson distribution, an approximation of the binomial distribution when the limit of $N$ is large and $pN = k$ [24].

$$P(k) = e^{-k} \frac{k^k}{k!}$$ \hfill (19)

- As $k$ becomes large, the degree distribution decays exponentially.

For this paper we generated three networks of size $N = 100$, 400, and 800 with $p = 0.6$, 0.05, and 0.02 respectively [41]. Fig. 2 shows the node degree distribution for $N = 400$.

5.3. Barabási–Albert Scale Free model (BA)

Barabási–Albert Scale Free models (also referred to as preferential attachment (PA) models) highlight a class of topologies associated with a heavy tailed node degree distribution [42]. This distribution is also known as a power-law distribution. In particular, given a graph $G$ with $N$ vertices, the degree distribution is power-law if $P(k) \sim k^{-\sigma}$, where $\sigma > 1$ [26]. Furthermore, the power law distribution cuts-off at the maximum degree, $k_{\text{cut-off}} = n^{\sigma-1}$. The node degree distribution is defined as,

$$P(k) = \frac{n(k)}{N}, \quad k = 0, 1, \ldots, k_{\text{max}}$$ \hfill (20)

These networks pervade numerous real world domains. For example, within the sphere of social networks, an individual with few friends is more likely to form a new friendship with a more popular person. Likewise, new Internet websites will more likely establish ties with the most popular websites.

From their origin, BA models have been considered vulnerable to targeted attacks while robust to random failures [42]. This model constitutes popular vertices called “hubs”, which have a large number of neighbors compared to other vertices with few neighbors. The rules for construction are governed by the two key principles of growth and preferential attachment. The initial number of vertices at construction must be greater than two and each vertex must have at least one neighbor. At each time step, a new vertex is added to the graph. The probability of attracting this new vertex is determined by the node degree of preexisting vertices. Thus, the higher the node degree of preexisting vertices, the higher their probability of attracting new vertices. The attachment probability is given in [41] by

![Fig. 2. The node degree distribution for $N = 400$ and $p = 0.05$.](image-url)
\[ P(k_i) = \frac{k_i}{\sum_{j=0}^{N} k_j} \tag{21} \]

where \( P(k_i) \) is the probability that a new vertex will connect to an existing vertex \( i \) with degree \( k_i \).

For this paper we generated three networks of size \( N = 100, 400, \) and 800 [41]. Fig. 3 shows the node degree distribution for \( N = 400 \).

6. Numerical analysis for edge removal

In this section we generate three graphs which are representative of the three models presented in Section 5. We then use Algorithm 1 to realize the decrease in \( \lambda_2(G) \) for all instances when an edge is removed.

**Algorithm 1.** Edge removal

\[
A := \text{Adjacency matrix of graph } G; \quad N := |A| \\
L := \text{The Laplacian matrix of } G \\
L' := \text{The Laplacian matrix of } (G - e) \\
R := \text{Matrix to store Lower bound, } \lambda_2(G - e), \text{ and Upper bound, such that } e \text{ is edge } (i,j) \\
\text{for } i = 1 \text{ to } N \text{ do} \\
\quad \text{for } j = 1 \text{ to } N \text{ do} \\
\quad \quad \text{if } (i \neq j) \text{ and } A(i,j) = 1 \text{ and } \lambda_2(G - e) > 0 \text{) } \\
\quad \quad \text{Remove } e \\
\quad \quad \text{Compute } L' \\
\quad \quad \text{Store } \lambda_2(G - e) \text{ in } R \\
\quad \quad \text{Insert } e \\
\quad \quad \text{Compute } L \\
\quad \quad \text{Compute } \epsilon \\
\quad \quad \text{Store Lower and Upper bounds for } \lambda_2(G - e) \text{ in } R \\
\quad \text{end if} \\
\text{end for} \\
\text{end for} \\
\text{Output } R
\]

Fig. 4 shows the decrease in algebraic connectivity for all realizations of an edge removal. These numerical results complement the analytical conclusions that removing an edge with the smallest absolute difference in the elements of the eigenvector (that is \( |u_i^2| - |u_j^2| \) for vertices \( i, j \in V \)) corresponding to the second smallest eigenvalue (\( \lambda_2 \)), tends to have the smallest decrease in algebraic connectivity. Furthermore, for these examples the coefficient of determination (\( R^2 \)) shows that 99.4%, 99.5%, and 93.7% of the variation of \( \lambda_2(G) - \lambda_2(G - e) \) for the Gi, WS, and BA networks respectively are accounted for by the polynomial relationship with \( |u_i - u_j| \).

![Degree vs. Number of Nodes](image-url)  

**Fig. 3.** The node degree distribution for \( N = 400 \).
7. Comparative analysis of the increase in algebraic connectivity via edge rewiring

In this section, we compare the increase in algebraic connectivity through rewiring, for the three graph models presented in Section 5. In particular, for each network model, we first generate 10,000 networks, each with 100 nodes and 300 edges. For each network from the same model, we compute the initial value of algebraic connectivity ($k_i$). We then rewire 7% of the edges and compute the final value of algebraic connectivity after rewiring ($k_f$) and the difference between the final and initial values ($k_f/k_i$). This procedure is conducted for all 10,000 networks of a particular model and we averaged the results. Finally, we repeat this procedure for each network model. Fig. 5 illustrates that for the Gi graphs, the average of $k_f/k_i$ is much lower than that of the BA and WS graphs and the average of $k_f$ is also higher for Gi than for the other two. With respect to the level of connectivity, this implies that networks from the BA and WS models tend to be more robust than that of the Gi model. Furthermore, if we compare the results of Figs. 5 and 6, we can deduce that graphs from the Gi model tend to have the highest gain in algebraic connectivity for the proposed rewiring procedure.

8. Edge rewiring to maximally increase algebraic connectivity

With the knowledge of which edge to remove to decrease algebraic connectivity the least, and also which edge to insert to increase algebraic connectivity the most, we combine these two strategies to obtain Algorithm 2. In particular, Algorithm 2 rewire an edge by:
1. Removing an edge with the smallest \( \alpha \)
2. Inserting an edge with the largest \( \alpha \)

Similarly from Algorithm 2, if we reverse the removal/insertion order in the “while” statement such that first, \( A(e_{\text{max}}) = 1 \) and second \( A(e_{\text{min}}) = 0 \), we would rewire an edge by:

1. Inserting an edge with the largest \( \alpha \)
2. Removing an edge with the smallest \( \alpha \)
**Algorithm 2.** Edge rewiring to maximally increase $\lambda_2(G)$

- $A$ := Adjacency matrix of graph $G$
- $L$ := Laplacian matrix of $G$
- $\psi$ := % of edges to rewire
- $e_{\text{max}}$ := Edge $(i,j) \in E$ corresponding to $x_{\text{max}}$
- $e_{\text{min}}$ := Edge $(i,j) \in E$ corresponding to $x_{\text{min}}$
- $\text{flag}$ := Variable to ensure validity of while statement

\begin{algorithm}
\For{$i = 1$ to $\psi$}
    \begin{algorithmic}
      \State $\text{flag} = 0$
      \State Compute $L$
      \State Extract $u^{(2)}$, the eigenvector corresponding to $\lambda_2(G)$
      \State Compute $x_{\text{max}}$ and $x_{\text{min}}$
      \While{$\text{flag} = 0$}
        \If{$(e_{\text{min}} \in G \text{ and } e_{\text{max}} \notin G \text{ and } \lambda_2(G \setminus e_{\text{min}}) > 0)$}
          \State $A(e_{\text{min}}) = 0$
          \State $A(e_{\text{max}}) = 1$
          \State $\text{flag} = 1$
        \Else
          \State Find alternates for $e_{\text{min}}$, $e_{\text{max}}$, and $\lambda_2(G \setminus e_{\text{min}})$
        \EndIf
      \EndWhile
    \EndFor
\end{algorithmic}
\end{algorithm}

In the following simulations, Table 1 highlights the number of nodes and edges in the original nine graphs that were generated.

<table>
<thead>
<tr>
<th>Networks</th>
<th>$N = 100$</th>
<th>$N = 400$</th>
<th>$N = 800$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Watts–Strogatz</td>
<td>1000</td>
<td>2000</td>
<td>4000</td>
</tr>
<tr>
<td>Random</td>
<td>2940</td>
<td>3925</td>
<td>6392</td>
</tr>
<tr>
<td>Barabási–Albert 1</td>
<td>451</td>
<td>1923</td>
<td>3913</td>
</tr>
</tbody>
</table>

**Fig. 7.** The increase in algebraic connectivity for Watts–Strogatz (WS), the Barabási–Albert Scale Free (BA), and the Gilbert stochastic (Gi) networks as edges are rewired by first inserting an edge then removing another. The “*” variation captures the results when rewiring is conducted by first removing an edge and then rewiring another. In this figure $N = 100$ and the values of $\lambda_2$ for 0% rewiring are 9.117, 2.757, and 42.834 for the WS, BA, and Gi networks respectively.
From Theorem 1, since \( \text{trace}(L) = \sum_{i=1}^{N} \lambda_i(G) = 2|E| \), given a graph \( G \) with \( N \) vertices and \( |E| \) edges, the magnitude of the eigenvalues increase with the \( |E| \). This explains the huge variance in the magnitude of the eigenvalues in Figs. 7 and 8 for the different classes of networks. As a result, in Fig. 7 we expect the Gi network’s eigenvalues to be the highest (since it has the most edges), followed by that of the WS network, and the BA network. Similarly, in Fig. 8 we expect the Gi network to have the highest eigenvalues and the eigenvalues for the WS and BA to be comparable.

Fig. 7 illustrates the propensity for algebraic connectivity to increase as 30% of the edges are rewired. The “\( \psi \)” denotes the variation in the rewiring procedure where first, an edge with the smallest \( \alpha \) was removed and second, an edge with the largest \( \alpha \) was inserted (as opposed to the default rewiring procedure where first, an edge with the highest \( \alpha \) is inserted and second, an edge with the smallest \( \alpha \) is removed). As shown, both variations result in identical increases in algebraic connectivity. Finally, as shown in Fig. 7 and more apparent in Fig. 8, there exists a rewiring threshold such that the algebraic connectivity is constant when this threshold is exceeded. From Fig. 8 in particular, for the ER graph there is no increase in algebraic connectivity beyond 8% rewiring. For the WS and BA networks, this phenomenon occurs at 20% rewiring.

9. Rewiring vs adding edges to maximally increase algebraic connectivity

In this section, we compare the results of rewiring to that of adding edges to maximally increase algebraic connectivity. For the addition of edges, we introduce Algorithm 3.

Algorithm 3. Edge addition to maximally increase \( \lambda_2(G) \)

\[ A := \text{Adjacency matrix of graph } G \]
\[ L := \text{Laplacian matrix of } G \]
\[ \psi := \% \text{ of edges to rewire} \]
\[ e_{\text{max}} := \text{Edge } (i,j) \in E \text{ corresponding to } \alpha_{\text{max}} \]
\[ \text{flag} := \text{Variable to ensure validity of while statement} \]

for \( i = 1 \) to \( \psi \)
   flag = 0
   Compute \( L \)
   Extract \( u^{(2)} \), the eigenvector corresponding to \( \lambda_2(G) \)
   Compute \( \alpha_{\text{max}} \)
   while flag = 0 do
      if \( e_{\text{max}} \neq G \) then
         \[ A(e_{\text{max}}) = 1 \]
         flag = 1
      else
         Find alternates for \( e_{\text{max}} \)
      end if
   end while
end for

Fig. 8. The increase in algebraic connectivity for Watts–Strogatz (WS), the Barabási–Albert Scale Free (BA), and the Gilbert stochastic (Gi) networks such that \( N = 400 \).
Fig. 9 compares the increase in algebraic connectivity for rewiring and adding edges. It is immediately apparent that there is the large difference between rewiring and adding edges when the percentage of edges augmented (rewired/added) exceeds 5%. However, in a real-world scenario, the percentage of edges augmented can reasonably revolve around 1% depending on perhaps the size and financial constraints of an organization.

10. Discussion

These results are important not only in the domain of graph theory but also in numerous complex networking domains such as the Smart Grid communication network, and even the transportation network. In the communication network domain, network engineers are constantly faced with the challenge of upgrading or, under certain circumstances, partially redesigning the network topology to increase connectivity. To accomplish such upgrades in most real-world cases, the number of edges to rewire or add is relatively small compared to the total number of edges in the network.

For $N = 100$, a 1% augmentation to the WS network is equivalent to augmenting 10 edges. For the Gi network, this equates to 29 edges, and for the BA network, this results in 5 edges. For the networks in Figs. 9(a)–(c), the increase in algebraic connectivity is comparable for both rewiring and adding edges if we are to consider a 1% augmentation. On the same note, Fig. 10 compares the increase in algebraic connectivity for rewiring and adding 30 edges for $N = 800$. For such a small resolution in the number of edges augmented, the results for adding edges are comparable to that of rewiring for all classes of networks. From a real-world perspective, this implies that for both rewiring and addition of edges, the number of edges
required to disconnect a network is the same. Therefore, a solution that considers rewiring of edges is as robust as a solution that considers addition of edges. Thus, an organization can opt for either solution depending on its economical and financial constraints.

11. Conclusion

To date, robustness in complex networks is an ongoing research effort. Among other topological measures, we use algebraic connectivity from spectral graph theory as our measure of robustness: the larger the algebraic connectivity, the more robust the network. In this paper, we answer the question of, “Where should an edge be rewired to increase algebraic connectivity the most?” by dividing this question into two parts: first, “Where should an edge be removed to decrease algebraic connectivity the least?” and second, “Where should an edge be inserted to increase algebraic connectivity the most?” From our analytical results, we conclude that to decrease algebraic connectivity the least, we should remove an edge that connects two strongly connected vertices. Conversely, to increase algebraic connectivity the most, we should insert an edge between two weakly connected vertices. From our numerical results, we implement a rewiring strategy on three classes of networks that provides the maximal increase in algebraic connectivity and hence, the maximal increase in robustness of a graph.

From our simulations, we initially compare graphs from the three classes of networks to determine the class that realizes the highest increase in algebraic connectivity. For an unbiased comparison, we set the number of nodes and edges constant for all networks and rewire a small percent of the edges. Our results reveal that graphs from Gilbert’s model (Gi) tend to have the lowest initial value for algebraic connectivity in addition to the highest gain in algebraic connectivity after rewiring. Subsequently, we compare the addition of edges to that of rewiring edges to maximally increase algebraic connectivity. We show that for edge augmentations (rewirings/additions) that exceed 5% of the network’s edges, the algebraic connectivity obtained when adding edges exceeds that obtained when rewiring edges. However, in real-world scenarios, such augmentations tend to be relatively small due to the non-negligible economical impact. In this case, the increase in algebraic connectivity is similar for both rewiring and addition of edges. From a real-world perspective, this implies that the number of edges required to disconnect the network is the same for both cases of rewiring or adding edges. Therefore, a solution that rewires edges is as robust as a solution where edges are added. Finally, our results illustrate that beyond a certain rewiring threshold, which can range from 8% to 20% for the graphs presented, algebraic connectivity is constant.

For the future direction of this work, it will be interesting to consider the impact to a network’s characteristics when algebraic connectivity is maximally increased. Such networks can include complex networks such as communication, power grid, and transportation networks. It would also be interesting to consider rewiring edges to maximally increase other spectral measures such as the spectral radius of a network. Finally, the “greedy” algorithm employed in this paper will not necessarily result in the optimal increase in algebraic connectivity. As a result, it would be interesting to explore various strategies to optimize algebraic connectivity when multiple links are rewired.

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References


