Computing Compromise Sets in Polyhedral Framework

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Abstract—In this note, we describe the compromise set for a special polyhedral convex feasible set. This procedure gives the monotonicity of the compromise set. This scenario appears in some engineering and economic applications like the determination of the consumer’s equilibrium. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In [1], the authors stated that, under certain conditions, the compromise set in multicriteria problems enjoys monotonicity properties similar to those in the bicriteria case (see [2,3] for basic definitions and results of Compromise Programming).

The proofs in [1] were not constructive, that is, we established different properties without an explicit description of the compromise set. It seems to be a hard problem to get a suitable parameterization of that set in the general case.

The aim of this note is to give an explicit description of the compromise set provided the linearity of the production-transformation function. This particular case arises in some engineering and economics problems. Thus, the feasible set in consumer theory is bounded by a budgetary constraint defined by a hyperplane with coefficients equal to the prices of the different goods.

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It is well known that fixing a p-norm \((1 < p < \infty)\) in \(\mathbb{R}^n\) and a nonempty convex subset, the best approximation problem (for every point in \(\mathbb{R}^n\)) has a unique solution. Therefore, when we consider the utopia point \(X^*\) of the feasible set \(U\) (see [2]), we obtain a function \(L(p), p \in (1, \infty)\), where \(L(p)\) denotes the point where the minimum of the \(p\)-distance from \(X^*\) to \(U\) is reached.

An example given by Freimer and Yu [4] shows that, in general, the set \(L((1, \infty))\) can have some pathologies related to the possibility of ordering that set in such a way that the function \(L\) is monotone (maybe not strictly). This problem is connected to some topological properties of \(L((1, \infty))\) as topological subspace of \(\mathbb{R}^n\).

A different type of difficulty can be found if we consider \(p = 1\) and \(p = \infty\). In this case, the function

\[
L : [1, \infty] \longrightarrow U,
\]

\[p \mapsto L(p)\]

is, in general, multivalued. In this paper, we show, by constructing the compromise set, that for a linear production-transformation function, the set \(L((1, \infty))\) can be naturally ordered in such a way that \(L\) is monotone. On the other hand, it is possible to obtain a natural continuous selection of the multivalued map, simply taking limits \(p \to \infty\) and \(p \to 1\) \(L : [1, \infty] \longrightarrow U\) such that it can be used to redefine the compromise set in the absence of unicity of \(L(1)\) and \(L(\infty)\).

### 2. FINDING THE MINIMUM FOR \(p\)-NORMS

**STEP 1.** Let \(a_1, \ldots, a_n > 0\), take for each \(i \in \{1, \ldots, n\}\) the point \(A_i = (0, \ldots, \frac{1}{a_i}, \ldots, 0)\) and assume that the feasible set in the problem we are studying is the convex hull of the points \(A_1, \ldots, A_n\) and the origin \(O\); i.e.,

\[
U = \text{co}(A_1, A_2, \ldots, A_n, O)
\]

\[= \left\{ \lambda_1 A_1 + \cdots + \lambda_n A_n + \lambda_{n+1} O : 0 \leq \lambda_i \leq 1 \text{ for each } i \in \{1, \ldots, n+1\}, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.
\]

Thus, the efficient set is the part of the hyperplane which contains the points \(A_1, A_2, \ldots, A_n\) that lies in the positive cone (this corresponds to a transformation hypersurface with constant marginal rates of substitution). The ideal point is \(X^* = (a_1, \ldots, a_n)\). We can suppose that \(a_1 \geq a_2 \geq \cdots \geq a_n > 0\) (in other words, it is possible to change the variables in order to get it).

**STEP 2.** Fix \(k \in \{1, \ldots, n\}\). The affine subvariety and the convex set spanned by \(A_1, \ldots, A_k\) will be denoted by \(\Pi_k\) and \(C_k\), respectively.

Let \(R_k(p) \in \Pi_k\) the \(p\)-projection \((1 < p \leq \infty)\) of \(X^*\) over \(\Pi_k\), i.e., the only (see [4, Lemma 2.1]) point in \(\Pi_k\) such that

\[
d_p(X^*, R_k(p)) = \min_{X \in C_k} d_p(X^*, X).
\]

These points \(R_k(p)\) will help us to find the compromise solution corresponding to the \(p\)-metric. For every \(p \in (1, \infty)\), the affine coordinates of \(R_k(p)\) can be computed as they minimize the function

\[
(x_1, \ldots, x_k) \mapsto \|X^* - (x_1 A_1 + \cdots + x_k A_k)\|_p = \|(a_1 - x_1 a_1, \ldots, a_k - x_k a_k, a_{k+1}, \ldots, a_n)\|_p
\]

\[= a_1^p |1 - x_1|^p + \cdots + a_k^p |1 - x_k|^p + a_{k+1}^p + \cdots + a_n^p,
\]

subject to \(x_1 + \cdots + x_k = 1\).
By using Lagrange multipliers method, we get that

\[
R_k(p) = \left( \begin{array}{c}
1 - \frac{(k - 1) a_1^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}} \\
\vdots \\
1 - \frac{(k - 1) a_k^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}}
\end{array} \right) a_1, \ldots, \left( \begin{array}{c}
1 - \frac{(k - 1) a_1^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}} \\
\vdots \\
1 - \frac{(k - 1) a_k^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}}
\end{array} \right) a_k, 0, \ldots, 0.
\]

Clearly, \( R_k \) is a continuous map.

We also define \( L_k(p) \in C_k \) as the \( p \)-projection of \( X^* \) over \( C_k \). When \( 1 < p \leq \infty \), it is given by the expression

\[
d_p(X^*, L_k(p)) = \min_{X \in C_k} d_p(X^*, X)
\]

and it is continuous. Note that \( L_n(p) \) is just the compromise solution, \( L(p) \), corresponding to the \( p \)-metric defined in the first section.

The affine coordinates of \( L_k(p) \) minimize the mapping

\[
(x_1, \ldots, x_k) \mapsto ||X^* - (x_1 A_1 + \cdots + x_k A_k)||_p^p = \|(a_1 - x_1 a_1, \ldots, a_k - x_k a_k, a_{k+1}, \ldots, a_n)\|_p^p
\]

subject to \( x_1 + \cdots + x_k = 1; x_1, \ldots, x_k \geq 0 \).

**STEP 3.** Note that \( R_k(p) \in C_k \) if and only if \( R_k(p) = L_k(p) \), because of \( \Pi_k \supset C_k \). In the other case, when \( R_k(p) = x_1 A_1 + \cdots + x_k A_k \notin C_k \), we have that at least one of its affine coordinates \( x_1, \ldots, x_k \) is negative. Since \( a_1 \geq a_2 \geq \cdots \geq a_k \), when there is \( i_0, 1 \leq i_0 \leq k \), such that \( x_{i_0} < 0 \), we have that \( x_j < 0 \) for each \( j \in \{i_0, \ldots, k\} \).

Furthermore, when \( R_k(p) \notin C_k \), then \( L_k(p) \in C_{k-1} \) (and therefore, \( L_k(p) = L_{k-1}(p) \)). In fact, each time that \( R_k(p) \notin C_k \), \( L_k(p) \) lies in \( \partial C_k \), the boundary of \( C_k = \text{co}(A_1, \ldots, A_k) \).

If \((y_1, \ldots, y_k)\) are the affine coordinates of a point in \( \partial C_k \), at least one of them is null. Take \( y_1 A_1 + \cdots + 0 A_i + \cdots + y_k A_k \in \partial C_k \). Clearly,

\[
||X^* - (y_1 A_1 + \cdots + 0 A_i + \cdots + y_k A_k)||_p^p \geq ||X^* - (y_1 A_1 + \cdots + y_k A_k + \cdots + 0 A_k)||_p^p.
\]

Therefore, \( L_k(p) \in \text{co}(A_1, \ldots, A_{k-1}) = C_{k-1} \).

**STEP 4.** The above results allow us to compute \( L_k(p) \) using the known expression for \( R_k(p) \) and the procedure shown in Figure 1. Since \( R_1(p) = L_1(p) \), the process finishes at most in \( k \) iterations.

**STEP 5.** As \( L(p) = L_n(p) \), the described procedure leads to the \( p \)-compromise solution. So, for each \( p \in (1, \infty] \), there exists some \( k \in \{1, \ldots, n\} \) such that \( L(p) = L_k(p) = R_k(p) \).

In this sense, when \( p \in (1, \infty] \) verifies \( L(p) = L_k(p) \neq L_{k-1}(p) \), equivalently \( L(p) \in C_k - C_{k-1} \), by virtue of Step 3, necessarily \( R_k(p) \in C_k \), thus \( L(p) = L_k(p) = R_k(p) \).

On the other hand, take the \( k \)-component function of \( R_k \), \( f_k : (1, \infty) \rightarrow \mathbb{R} \), defined by

\[
f_k(p) = 1 - \frac{(k - 1) a_k^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}}.
\]
Figure 1. Computation of $L_k(p)$.

Since

$$f'_k(p) = \frac{(k-1)a_k^{-p/(p-1)} \sum_{j=1}^{k-1} a_j^{-p/(p-1)}(\log a_j - \log a_k)}{(p-1)^2 (\sum_{j=1}^{k} a_j^{-p/(p-1)})^2} \geq 0$$

(and $f'_k(p) = 0 \iff a_k = \cdots = a_1$), whenever $R_k(q_0) = L_k(q_0)$ it is $R_k(q) = L_k(q)$ for all $q \geq q_0$.

STEP 6. With the purpose of studying the monotonicity of the compromise set, define $p_1 = 1$, $p_{n+1} = \infty$, and for each $k \in \{2, \ldots, n\}$, $p_k \in [1, \infty]$ such that

$$\inf\{p \in (1, \infty]: L(p) \not\in C_{k-1}\},$$

if $\emptyset \neq \{p \in (1, \infty]: L(p) \not\in C_{k-1}\}$,

$$p_k = \begin{cases} \infty, & \text{if } \emptyset = \{p \in (1, \infty]: L(p) \not\in C_{k-1}\}. \end{cases}$$

In this manner, $1 = p_1 \leq p_2 \leq \cdots \leq p_n \leq p_{n+1} = \infty$ and the family $\{(p_k, p_{k+1}]: k = 1, \ldots, n\}$ covers $(1, \infty]$.

We are going to analyze the geometric meaning of that partition of the set $(1, \infty]$. As $C_k$ is closed and $L$ is continuous, it follows that for every $p \leq p_{k+1}$, we have $L(p) \in C_k$.

Moreover, when $p \in (p_k, p_{k+1}]$, it is $L(p) \in C_k - C_{k-1}$; roughly speaking, if $L(p)$ leaves $C_j$, does not return. In fact, suppose the existence of $p_0 \in (p_k, p_{k+1}]$ with $L(p_0) \in C_{k-1}$. Through the definition of $p_k$, we can take a sequence $\{z_m\}_{m \in \mathbb{N}}$ such that $z_m \in (p_k, p_{k+1}]$, $L(z_m) \in C_k - C_{k-1}$ for every $m \in \mathbb{N}$ and $\lim_{m \to \infty} z_m = p_k$. Hence, by Step 5,

$$L(z_m) = L_k(z_m) = R_k(z_m),$$

for all $m \in \mathbb{N}$. So, as $\lim_{m \to \infty} z_m = p_k$, from the final remark in Step 5, $R_k(q) = L_k(q)$ for every $q > p_k$, in particular, $R_k(p_0) = L_k(p_0) = L(p_0)$.

Our assumption, $L(p_0) \in C_{k-1}$, implies that the $k^{th}$ coordinate of $R_k(p_0)$, $f_k(p_0)$, is nonpositive. Recall that $R_k(q) = L_k(q)$ for all $q > p_k$, thus, the $k^{th}$ coordinate of $R_k(q)$, $f_k(q)$, is nonnegative. In this way, since the function $f_k$ is not decreasing and $p_k < p_0$, necessarily
\( f_k(q) = f_k(p_0) = 0 \) \((q > p_k)\) and \( a_k = \cdots = a_1 \). Consequently, \( L_1, \ldots, L_k \) will be constant functions and
\[
L_j(r) = \frac{1}{j} A_1 + \cdots + \frac{1}{j} A_j,
\]
for each \( r \in (1, \infty] \) and \( j \in \{1, \ldots, k\} \), which is a contradiction with \( L_k(p_0) = L(p_0) \in C_{k-1} = \text{co}(A_1, \ldots, A_{k-1}) \).

Finally, we conclude that if \( p \in (p_k, p_{k+1}] \), then
\[
L(p) = R_k(p) = \left( \begin{array}{c}
1 - \frac{(k-1)a_1^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}} a_1, \\
1 - \frac{(k-1)a_2^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}} a_2, \\
\vdots \\
1 - \frac{(k-1)a_k^{-p/(p-1)}}{\sum_{j=1}^{k} a_j^{-p/(p-1)}} a_k, 0, \ldots, 0
\end{array} \right).
\]

STEP 7. Thus, for \( p \in (p_{k-1}, p_k] \), the last \( n-k+1 \) coordinates of \( L(p) \) are not decreasing (in fact, the last \( n-k \) coordinates are null). Hence, if \( L(p) = L(q) \) for \( p, q \in (1, \infty] \), \( L(r) = L(p) = L(q) \) for every \( r \in [p, q] \), i.e., the mapping \( p \mapsto L(p) \) is monotone. If \( a_1 \neq a_2 \), the mapping \( p \mapsto L(p) \) is even injective.

We can now enunciate the following generalization of Freimer-Yu's Theorem.

**Theorem 2.1.** Let \( a_1, \ldots, a_n \in \mathbb{R} \) such that \( a_1 \geq a_2 \geq \cdots \geq a_n > 0 \). Consider for each \( i \in \{1, \ldots, n\} \) the point \( A_i = (0, \ldots, \check{a_i}, \ldots, 0) \in \mathbb{R}^n \) and suppose that the feasible set in the utility space is the convex hull of the points \( A_1, \ldots, A_n \) and the origin \( O \); i.e.,
\[
U = \text{co}(A_1, A_2, \ldots, A_n, O).
\]

Then the compromise set is a piecewise \( C^\infty \) monotone curve, i.e.; if \( L(p) = L(q) \) for some \( p, q \in (1, \infty] \), then \( L(r) = L(p) = L(q) \) for every \( r \in [p, q] \).

Theorem 2.1. allows us to give an order to the points in the compromise set.

STEP 8. For the sake of clearness, we will study in depth the three-criteria case. So, consider \( U = \text{co}(A, B, C, O) \). The problems with three criteria involved lead to six different classes of compromise sets, depending on the relations between the coordinates of the feasible set vertices. They are described in Figure 2.

(a) If \( U = \text{co}((a, 0, 0), (0, a, 0), (0, 0, a), O) \), then \( X^* = (a, a, a) \) and \( L(p) = (a/3, a/3, a/3) \) for every \( p \in [1, \infty] \), and thus, \( L = \{(a/3, a/3, a/3)\} \). It can be derived as a consequence of our method, but it can be also thought of as a consequence of Jensen's inequality. In this case, each point of \( \text{co}(A, B, C) \) is a one-compromise solution.

(b) If \( U = \text{co}((a, 0, 0), (0, b, 0), (0, 0, b), O) \), \( a > b \), we have that \( L(\infty) \) lies in the interior of the triangle, \( L(1) = (a, 0, 0) \) and the compromise set \( L \) is the segment joining \( L(1) \) and \( L(\infty) \).

(c1) If \( U = \text{co}((a, 0, 0), (0, a, 0), (0, 0, b), O) \), \( a \geq 2b \), every \( p \)-compromise solution, \( p \in (1, \infty] \), coincides with the barycentre of the segment delimited by \( (a, 0, 0) \) and \( (0, a, 0) \). Note that in this case, every point of the segment \( AB \) is a one-compromise solution.

(c2) If \( U = \text{co}((a, 0, 0), (0, a, 0), (0, 0, b), O) \), \( a > b > a/2 \), the compromise set is a segment in the triangle. Anec, each point of the segment \( AB \) is a one-compromise solution.

(d1) In the general case, when \( U = \text{co}((a, 0, 0), (0, b, 0), (0, 0, c), O) \), \( a > b > a/2 \) or \( c < 1/2 \), the compromise set is a segment along the side \( AB \), which is delimited by \( L(1) = A \) and \( L(\infty) \).
In the general case, when \( U = \text{co}((a, 0, 0), (0, b, 0), (0, 0, c), O) \), \( a > b > c \) and \( \frac{1}{a} + \frac{1}{b} > \frac{1}{c} \), we have that the compromise set is formed by a segment along the side \( AB \) and a curve in the interior of the triangle, but it remains delimited by \( L(1) = A \) and \( L(\infty) \).

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