Incentives-Based Mechanism for Efficient Demand Response Programs

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Abstract

In this work we evaluate the inefficiency in the sense of Pareto of the electricity system and its similarity to the tragedy of the commons. Also, we propose an incentives scheme that promotes optimal outcomes in the inefficient electricity market. The economic incentive might be seen as an indirect revelation mechanism that uses a one dimensional message space, and it is valid for any concave electricity valuation function. These incentives are calculated for each user only based on the consumption of the population. We propose a distributed implementation of the mechanism using population games and evaluate the performance of four popular dynamics methods in terms of the cost to implement the mechanism. We find that the achievement of efficiency in strategic environments might be achieved at a cost, which is dependent on both the users’ preferences and the dynamic evolution of the system. Some simulation results illustrate the ideas presented throughout the paper.

Index Terms

Smart grid, demand response, electricity market, dynamic pricing, game theory, mechanism design, indirect revelation, budget balanced, resource allocation.

I. INTRODUCTION

The smart grid (SG) concept has entailed profound changes in the conception of electricity systems. Specifically, efficiency in both electricity generation and consumption is one of the

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main goals of the SG [1]. One of the main characteristics of the electricity system is that, since consumers become active agents in the SG, the technological development might be insufficient by itself to achieve the desired goals [2]. Instead, efficiency is expected to be achieved by means of an active cooperation of consumers in the electricity systems. Thus, demand response (DR) programs arise as a tool intended to promote cooperation of consumers with the electricity system. DR programs deal with the problem of providing economic incentives to consumers in order to modify their electricity consumption behavior. Incentives proposed in the literature use pricing mechanisms, by which either higher or lower consumption is encouraged through changes in the electricity prices along the day. However, the design of incentives is not a trivial task, because the pricing scheme might impact the volatility or robustness of the system [3]. Also, incentives must be designed having into account the characteristics of the consumers. Consumers might be either price takers or price anticipators. On the one hand, a price taker makes decisions without considering future implications of its decisions. On the other hand, a price anticipator (or strategic agent in the context of game theory) makes decisions taking into account the consequences of its actions on the system. Thus, the behavior of consumers might lead to different outcomes. Particularly, the degradation of the system’s efficiency due to selfish agents is known as the price of anarchy [4].

An efficient outcome in a population of strategic agents might be achieved by means of mechanism design. Mechanism design addresses the problem of designing the rules of a game in order to achieve the desired outcome [5], [6]. An outstanding result in mechanism design is the Vickrey-Clarke-Groves mechanism (VCG) [7]–[9]. The VCG mechanism is a direct revelation mechanism that guarantees efficient outcomes in dominant strategies. The VCG mechanism requests private information from users (such as preferences or utility functions) and assigns a central agent, which gathers information and computes the optimal resource allocation. In particular, the VCG mechanism provides incentives to elicit private information, even if users are unwilling to report their true preferences due to strategic or privacy issues. Some disadvantages of this class of mechanism are its high requirements of both communication and computation resources [6].

A similar mechanism designed for decentralized resource allocation without private information of users is the Kelly mechanism [10], [11]. In this mechanism users send a bid to a central planner based on their own convenience. Then, the central planner assign resources following a
proportional fairness criteria. However, there is an efficiency loss of the Kelly mechanism [12], which inspired the design of efficient mechanisms in strategic environments. In [13], an efficient one-dimensional mechanism based on the Kelly and VCG mechanisms is proposed. In [14], a more general mechanism for convex environments is presented, in which each user reports the utility function. Note that these mechanisms use a one-dimensional message space because utility functions are parametrized by a single parameter. Some works have used the VCG mechanism to coordinate the demand in electricity systems [15]–[17]. The work in [18] presents a distributed algorithm based on a pricing mechanism to minimize the energy cost in smart grids. In [18] users solve an optimization problem in function of the system’s aggregated consumption and tariffs, where preferences of users are considered as restrictions of the problem, rather than part of the objective function. Consequently, users do not reveal preferences and details of their appliances, but they have to reveal the aggregated consumption. In particular, each user has to broadcast the scheduled daily energy consumption to all other users, because that information is used by other users to find their optimal schedule.

In this work, we show a relationship between the electricity system and the tragedy of the commons. The tragedy of the commons describes the inefficiency of a system due to overuse of resources [19], which implies that the optimal outcome has lower demand peaks, even though the objective function is not designed with that purpose. Therefore, we propose a scheme of economic incentives, which can be seen as an indirect revelation mechanism based on the Clarke pivot mechanism [20]. The main feature of this mechanism is that it does not require private information from users and employs a one-dimensional message space to coordinate the demand of users to achieve the optimal outcome. These properties facilitate the decentralized implementation of the mechanism. Specifically, the mechanism entrusts the computational tasks among users, who should maximize their own utility function based on the aggregated demand, which is calculated and broadcasted by a central agent. Thus, users avoid revelation of private information (e.g., preferences), but are required to report the aggregated consumption of their appliances during some time periods. It is noteworthy that the mechanisms implement efficient outcomes regardless of the form of the valuation function (as long as it is concave).

We extend the work presented previously in [21] by implementing the mechanism using evolutionary dynamics to distribute efficiently the power consumption of each user. Note that the classical implementation of mechanisms involves a negotiation phase, which is necessary to find
the social equilibrium before doing any transaction of goods. However, in this implementation we relax the negotiation process by introducing some dynamics that model smooth changes in the demand of each user. Thus, we can observe the evolution of the system as a consequence of the continuous interaction between agents. The optimization problem of each user is seen as a local resource allocation problem, in which the electric energy is the resource that should be allocated in different time periods to achieve maximum benefit. Under concave environments, the individual allocations made by each user lead to the global optimum. However, this incentives scheme might require external subsidies to be implemented. That is, the benefit obtained in the optimal outcome is not enough to fund the incentives scheme. In the literature this property is known as *weak budget balance* [22]. We investigate four evolutionary dynamics to evaluate the mechanism cost in terms of incentives in the long run, i.e., the cost incurred when moving the system state to the optimal equilibrium. To the best of our knowledge, the budget balance property has not been considered in previous work of applications of demand response. Particularly, the VCG mechanism (the most used in the literature) might not be budget balanced. For example, [23] and [24] demonstrate the impossibility of implementing efficient and budget balanced mechanisms in dominant strategies.

The main contributions of this paper can be summarized as follows: 1) we formulate the demand response problem as a tragedy of the commons dilemma, highlighting the efficiency loss due to overuse of resources in the electricity system when price signals are not controlled; 2) we propose a novel scheme of economic incentives for achieving optimal demand profiles in a population of strategic agents that uses a one dimensional message space, which is valid for any concave electricity valuation function, and we prove properties of weak budget balance and individual rationality; 3) we present a distributed implementation of the mechanism using population games; 4) we evaluate the performance of four popular evolutionary dynamics, namely logit dynamics, replicator dynamics, Brown-von Neumann-Nash dynamics, and Smith dynamics in terms of the cost to implement the mechanism; and 5) we note that the achievement of efficiency in strategic environments might be achieved at a cost, which is dependent on both the users’ preferences and the dynamic evolution of the system.

This work is inspired in previous applications of mechanism design in efficient resource allocation problems, notably [25]. In particular, we consider the DR problem as a resource allocation problem of multiple goods. We depart from other resource allocation approaches,
such as [26], in which we do not specify the utility functions that lead to a desired outcome. Instead, we design the rules of the game to achieve the desired result without modifying the valuation function of each agent.

The rest of the paper is outlined as follows. In Section II the model of the electricity system is presented and its proved that its inefficiency is caused by selfish agents that have incentives to over-exploit resources. In Section III, a design of an indirect revelation mechanism that uses an incentive scheme to achieve the optimal outcomes in strategic frameworks is presented. In Section IV, we introduce the distributed implementation of the mechanism. In this case, we use evolutionary dynamics to model the dynamics that, when implemented locally by each user, lead to the global efficient equilibrium. Section V presents some examples to illustrate the ideas developed throughout the paper, and finally in Section VI some conclusions and future directions are drawn.

II. Problem Statement

In this section we introduce the market model of an electricity system and present two solution concepts arising in both ideal and strategic environments. Then, we prove the inefficiency of the electricity system in strategic environments and discuss its consequences in electricity markets.

A. Electricity System Model

We consider an electricity market composed by three parties, namely the generator, the customers, and the independent system operator (ISO). In this case, a single generator provides energy to several customers, while the ISO maintains a balance between supply and demand (market clearing). We define the society as a set with $N$ customers denoted by $\mathcal{P} = \{1, \ldots, N\}$.

We take into account users that might have time varying consumption preferences. These preferences are associated with their changing necessities of energy along the day. This feature is modeled by partitioning a day in $T$ time intervals in which users have roughly the same consumption preferences. In other words, we divide a period of 24 hours into $T$ disjoint time intervals, denoted by $\pi_1, \ldots, \pi_T$, that satisfies $\bigcup_{k \in \{1, \ldots, T\}} \pi_k = 24h$ and $\bigcap_{k \in \{1, \ldots, T\}} \pi_k = \emptyset$. Each user $i \in \mathcal{P}$ can choose the amount of energy $q_i^k$ that is consumed in the $k^{th}$ time interval. Thus, the daily consumption profile of a user is represented with the vector $\mathbf{q}_i = [q_i^1, \ldots, q_i^T]^\top \in \mathbb{R}_{\geq 0}^T$.

Without loss of generality, we restrict the electricity consumption to null or positive values,
i.e., $q_i^k \geq 0$, for all agents $i \in \mathcal{P}$, and any time interval $\pi_k, k \in \{1, \ldots, T\}$. Likewise, the electricity consumption of the population is denoted by the vector $q = [q_1^T, \ldots, q_N^T] \in \mathbb{R}_{\geq 0}^{T \cdot N}$ and the vector $q_{-i} = [q_1^T, \ldots, q_{i-1}^T, q_{i+1}^T, \ldots, q_N^T] \in \mathbb{R}_{\geq 0}^{T \cdot (N-1)}$ represents the consumption of the population, except for the $i^{th}$ agent.

Ideally, each user regulates its electricity consumption in function of its own preferences and the electricity price. In particular, rational agents might try to maximize its profit, which is defined as the benefit earned minus the cost associated with the energy consumed. The benefit might be represented by means of a valuation function $v_i^k : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, where $v_i^k(q_i^k)$ represents the economic value that the $i^{th}$ user assigns to $q_i^k$ electricity units in the $k^{th}$ time interval. Furthermore, the daily valuation $v_i : \mathbb{R}_{\geq 0}^T \rightarrow \mathbb{R}$ is the aggregated economic value that the $i^{th}$ user gives to the daily electricity consumption and it is defined as $v_i(q_i) = \sum_{k=1}^T v_i^k(q_i^k)$.

In order to ease the analysis, let us assume that the valuation functions belong to a family of functions $v : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$. Here, the valuation function of the $i^{th}$ agent in the $k^{th}$ time interval is denoted by $v_i^k(q) = v(q, \alpha_i^k)$, where $\alpha_i^k \geq 0$ is a parameter that characterizes the valuation of $q \geq 0$ power units. We assume that the valuation of $q$ is increasing with respect to $\alpha_i^k$. Moreover, if $\alpha_i^k > \alpha_j^k$, then the $i^{th}$ agent gives higher valuation than the $j^{th}$ agent to any consumption $q \geq 0$ at the $k^{th}$ time interval. Also, we consider that a null consumption implies a null valuation. We summarize the previous properties in the following assumption.

**Assumption 1.** The family of valuation functions $v : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ satisfies the following conditions:

- If $\alpha_i^{k_1} > \alpha_j^{k_2}$, then $v(q, \alpha_i^{k_1}) > v(q, \alpha_j^{k_2})$, for all $i, j \in \mathcal{P}$ and $\alpha_i^{k_1}, \alpha_j^{k_2}, q \in \mathbb{R}_{\geq 0}$.
- $\lim_{q \rightarrow 0} v(q, \alpha) = 0$, for $\alpha \in \mathbb{R}_{\geq 0}$.

In a market economy, the resource consumption is also limited by the cost associated to it. Hence, besides their valuation of energy, users also take into account the energy price to make consumption decisions. In particular, the cost of the energy in the $k^{th}$ time interval is denoted by $\lambda^k$, and consequently, the profit of the $i^{th}$ consumer can be represented by

$$U_i(q) = v_i(q_i) - \sum_{k=1}^T q_i^k \lambda^k. \quad (1)$$

We assume that the generation $g^k$ at each time interval $k$ is coordinated by the ISO in order to guarantee the balance between generation and the total demand, i.e., $g^k = \sum_{i=1}^N q_i^k$. Also,
we assume that the energy price is enough to cover the production costs, i.e., \( \sum_{k=1}^{T} g^k \lambda^k = \sum_{k=1}^{T} C(g^k) \), where \( C(g^k) \) is the production cost associated with a generation of \( g^k \) energy units at the \( k^{th} \) time interval (note that generation cost function \( C(\cdot) \) is the same for every time interval).

Let us consider the electricity price as an average price function \( p(z) = C(z)/z \), where \( z \) represents the total energy demand in a time interval. With this average price scheme we can express the profit of the \( i^{th} \) agent in Eq. (1) as

\[
U_i(q) = U_i(q_i, q_{-i}) = v_i(q_i) - \sum_{k=1}^{T} q^k_i p \left( \|q^k\|_1 \right),
\]

where \( q^k = [q^k_1, q^k_2, \ldots, q^k_N]^\top \) is a vector with the electricity consumption of each agent at the \( k^{th} \) time interval, and \( \| \cdot \|_1 \) is the 1-norm.

### B. Market Equilibrium

A market equilibrium is achieved when the constant interaction between buyers and sellers achieves a balance. Here, we introduce the equilibrium conditions of both ideal economies and economies with strategic agents. These conditions are used to show the inefficiency of the electricity system in strategic environments.

1) Optimal Equilibrium: In an ideal market customers behave as price takers and consequently, an optimal outcome in the sense of Pareto (i.e., an outcome in which no individual can made better without making someone else worse off) can be achieved by means of an average price scheme. Accordingly, the demand profile \( \mu \in \mathbb{R}^{T \times N}_{\geq 0} \) that maximizes the social welfare is a solution to the following optimization problem:

\[
\begin{align*}
\text{maximize}_{q} & \quad \sum_{i=1}^{N} U_i(q) = \sum_{i=1}^{N} \left( v_i(q_i) - \sum_{k=1}^{T} q^k_i p \left( \|q^k\|_1 \right) \right) \\
\text{subject to} & \quad q^k_i \geq 0, \quad i = \{1, \ldots, N\}, \quad k = \{1, \ldots, T\}.
\end{align*}
\]

The existence of a unique market equilibrium \( \mu \) inside the feasible area (i.e., \( \mu \) belongs to \( \mathbb{R}^{T \times N}_{\geq 0} \)) is ensured if the following assumptions are satisfied [27]:

**Assumption 2.**

1. The valuation function \( v^k_i(\cdot) \) is differentiable, concave, and non-decreasing.
ii. The generation cost function can be expressed as $C(z) = z p(z)$, where $p(\cdot)$ is differentiable, convex, and non-decreasing.

**Assumption 3.**

$$\lim_{q_i^n \to 0} \frac{\partial}{\partial q_i^n} U_i(q_i^n, q_{-i}^n) = \lim_{q_i^n \to 0} \frac{\partial}{\partial q_i^n} v_i^n(q_i^n) - p\left(\|q_i^n\|_1\right) > 0,$$

for all agent $i \in P$.

**Remark 1.** These assumptions are reasonable in the context of economic theory [22]. Notice that Assumptions 2 and 3 imply that the problem in Eq. (3) has an optimal $\mu$ that satisfies the following first order conditions (FOC):

$$\frac{\partial}{\partial q_i^n} \sum_{i=1}^{N} U_i(q) \bigg|_{q=\mu} = \frac{\partial}{\partial q_i^n} v_i^n(q_i^n) - p\left(\|q_i^n\|_1\right) - \left(\|q_i^n\|_1\right) \frac{\partial}{\partial q_i^n} p\left(\|q_i^n\|_1\right) \bigg|_{q=\mu} = 0. \quad (4)$$

2) **Nash Equilibrium:** Let us consider a strategical environment in which agents might have conflicting interests. In the electricity system, this conflict arises because each agent might impose externalities on the society through the price signals. Note that if the society is finite, then the consumption of each agent might have a significant impact on the electricity prices, and thus might affect the profit of other agents. This conflict can be seen as a game between agents, in which the actions of each agent determine the outcome of the game, e.g., their payoff. Consequently, the electricity system might be seen as a game in which each agent is selfish and endeavors to maximize independently its own welfare. The game can be defined as the 3-tuple $G = (P, (S_i)_{i \in P}, (U_i)_{i \in P})$, where $P$ is the set of players, $S_i$ is the set of available strategies of each player, and $U_i : S_1 \times \cdots \times S_N \to \mathbb{R}$ is the utility function of the $i^{th}$ player as a function of its own actions as well as the strategies of other players. The equilibrium concept used in game theory is the Nash equilibrium [28]. In particular, the Nash equilibrium of the game $G$, denoted by $\xi \in \mathbb{R}_{\geq 0}^{N \times T}$, is given by

$$U_i(\xi_i, \xi_{-i}) \geq U_i(q_i, \xi_{-i}), \text{ for all } q_i \in \mathbb{R}_{\geq 0}^{T}.$$
for all agent $i \in \mathcal{P}$. Note that the Nash equilibrium $\xi$ is a solution to the following maximization problem that every agent $i \in \mathcal{P}$ attempts to solve independently

$$
\max_{q_i} U_i(q_i, q_{-i}) = v_i(q_i) - \sum_{k=1}^{T} q_i^k p\left(\|q^k\|_1\right)
$$

subject to $q_i^k \geq 0, i = \{1, \ldots, N\}, k = \{1, \ldots, T\}$.

Assumptions 2 and 3 imply that the Nash equilibrium $\xi$ of the game $G$ satisfies the following FOC:

$$
\left. \frac{\partial}{\partial q_i^k} U_i(q_i, q_{-i}) \right|_{q=\xi} = \left. \frac{\partial}{\partial q_i^k} v_i(q_i^k) \right|_{q=\xi}
$$

$$
- p\left(\|q^k\|_1\right) - q_i^k \left. \frac{\partial}{\partial q_i^k} p\left(\|q^k\|_1\right) \right|_{q=\xi} = 0,
$$

C. Electricity System Inefficiency

The degradation of the system’s efficiency due to selfish agents is known as the price of anarchy [4]. The research in this field has been focused on quantifying the efficiency loss in specific game environments [12], [29], [30]. In this work, we are interested in the relationship between the electricity system and the tragedy of the commons [19]. The tragedy of the commons describes the inefficiency of a system due to overuse of resources. Now, let us prove this relationship and then analyze the implications of this fact in the design of demand response programs.

**Theorem 1.** Suppose that Assumptions 1, 2, and 3 are satisfied. If $\mu$ and $\xi$ are the solutions of the optimization problems in Eq. (3) and Eq. (5), respectively. Then, the following conditions are satisfied:

i. $\xi_i^k > \mu_i^k$, for all individuals $i, j \in \mathcal{P}$ such that $i \neq j$.

ii. The Nash equilibrium $\xi$ is an inefficient outcome in the sense of Pareto.

Hence, the game defined by $(U_1, \ldots, U_N)$ is similar to the tragedy of the commons.

The proof for this and all other Theorems and Propositions are given in the Appendix.

An implication of the tragedy of the commons is that, although there are plenty of resources and consumption capacity, it is not convenient to over-exploit resources. In particular, the optimal outcome is characterized by a low consumption with respect to the inefficient outcome.
Specifically, the optimal outcome reduces not only the maximum consumption (or peak), but also the overall consumption. In consequence, using this efficiency criteria in DR schemes might seem counter intuitive, because the objective of DR is to balance the demand profile along the day (to avoid peaks and to flatten the demand), rather than lowering the consumption at each time period.

Hence, the DR objectives might not be fully captured by maximizing the aggregate surplus (see Eq. (3)). Thus, depending on the conditions of the system it might be necessary to implement another efficiency criteria that guarantees reduction of the peak to average ratio (PAR).

However, we can use (or implement) this optimality criteria to alleviate the burden on stressed systems, without incurring in costs associated with additional generation and transmission capacity. Note that by reducing peaks the supplier can eliminate the most expensive generators that are used to supply peak demand [31].

III. Incentives Based Mechanism

Given the unsatisfactory results of the Nash equilibrium in the previous section, we are motivated to design a mechanism to improve the efficiency of the equilibrium in a strategic environment. In this case, we use mechanism design to model an incentives scheme that guarantees the efficiency of the equilibrium [6], [22]. First, we introduce mechanism design and present the general structure of the incentives. Then, we analyze the mechanism as well as its properties.

Note that the optimization problems described in Eq. (3) and (5) can be separated in $T$ independent optimization problems, since valuation and generation functions are time independent. Hence, without loss of generality, we can analyze the case of $T = 1$ and the results obtained can be extended to cases with $T > 1$. Accordingly, the notation used hereafter does not have into account any particular time period.

A. Incentives-Based Mechanism

In a strategic environment, we assume that individuals are selfish and take actions that maximize their profits, based on both local (or private) and global information. From the perspective of mechanism design, the rules that govern the payoff structure (and consequently the outcome of the game) can be designed by an agent, hereinafter called the principal. The principal
might be interested in the achievement of some social goals, such as efficiency according to a particular criteria. Hence, the principal’s problem consists in designing the game rules that promote the desired outcome in function of the system characteristics (or economic environment). The economic environment is defined in terms of the private information held by each agent (such as its private preferences, e.g., $\alpha$ in Assumption 1), denoted by $\theta_i \in \Theta_i$, where $\theta_i$ is referred as the type of the $i^{th}$ agent and $\Theta_i$ is the set of all possible types. Note that the private information $\theta_i$ is relevant to calculate the profit that each agent receives at a given outcome. Hence, the strategy of each agent might be a function of the type $\theta_i$.

Summarizing, mechanism design consists in designing a solution system to a decentralized optimization problem with private information [6]. A mechanism $\Gamma = \{\Sigma_1, \ldots, \Sigma_N, g(\cdot)\}$ defines a set of strategies $\Sigma_i$ for each player and an outcome rule $g : \Sigma_1 \times \ldots \times \Sigma_N \rightarrow O$ that maps from the set of possible strategies to the set of possible outcomes $O$ [22]. Note that the mechanism also defines the set of strategies and the method used to select the final outcome. In particular, a direct revelation mechanism (such as the Vickrey-Clarke-Groves mechanism [7]−[9]) defines the set of strategies as the private information of the agents, i.e., $\Sigma_i = \Theta_i$, for all $i \in P$. In such cases, the principal is in charge of deciding the outcome of the game based on the information sent by the agents. For example, in voting systems, the strategies are preferences reports made by agents, while the outcome (the selection of one candidate) might be decided using the Borda rule or plurality with elimination, among others. Likewise, in auctions the strategies are bids and the outcome (item allocation and its price) might be decided using a mechanisms such as the second price auction [20].

In this case, we propose an indirect revelation mechanism that uses a one dimension message space and can be implemented in a decentralized way (see Section IV). In our setting, the type of an agent is its electricity preferences along the day, denoted as $\theta_i = [\alpha^1_i, \ldots, \alpha^T_i]$. The strategy of each agent is the daily consumption profile $q_i$, and consequently, the set of all possible strategies for the $i^{th}$ agent is defined as $\Sigma_i \equiv \mathbb{R}_{\geq 0}^T$. In this case, the set of all possible outcomes $O$ is composed by all the possible electricity prices, i.e., $O \equiv \mathbb{R}_{\geq 0}$. Note that the outcome rule $g(\cdot)$ of the mechanism is the price scheme used in the electricity system. In this case, the mechanism objective is to achieve an optimal demand profile in a strategic setting, i.e., the Nash equilibrium should be equal to the optimal outcome. To this end, we modify the price scheme adding an incentive function $I_i(\cdot)$ designed to align the users welfare function with the population objective.
function. The incentive function models the externality imposed by an agent on the rest of the population. The externality is modeled as the impact in prices caused by the participation of a single individual. Thus, the incentives have the form

\[ I_i(q) = \|q_i\|_1 (h_i(q_{-i}) - p(\|q\|_1)) , \]  

where \( h_i(q_{-i}) \) is a term that estimates the electricity price when the \( i^{th} \) user does not take part in the electricity system. The form of this incentive is related to the price used in the Vickrey-Clarke-Groves mechanism [20] and some utility functions used in potential games [32]. Here, we model \( h_i(q_{-i}) \) as

\[ h_i(q_{-i}) = p\left(\sum_{j \neq i} q_j + f(q_{-i})\right) , \]  

where \( f(q_{-i}) \) is a function that represents the alternative behavior of the \( i^{th} \) agent. In this case, we consider linear functions of the form

\[ f(q_{-i}) = \sum_{j \neq i} \alpha_j q_j , \]  

where \( \alpha_i \in \mathbb{R} \), for all \( i \in \mathcal{P} \).

Along these lines, with the introduction of this incentives mechanism we obtain a new game defined as the 3-tuple \( G_I = \langle \mathcal{P}, \Sigma_i \in \mathcal{P}, (W_i)_{i \in \mathcal{P}} \rangle \), where \( \mathcal{P} \) is the set of players, \( \Sigma_i \) is the set of available strategies of each player, and \( W_i : \Sigma_1 \times \cdots \times \Sigma_N \rightarrow \mathbb{R} \) is the utility function of the \( i^{th} \) player, that is defined as

\[ W_i(q_i, q_{-i}) = U_i(q_i, q_{-i}) + I_i(q) \]

\[ = v_i(q_i) - q_i p\left(\|q\|_1\right) + I_i(q) . \]  

It can be proved that the Nash equilibrium of the game \( G_I \) is equal to the optimal equilibrium of the original game \( G \) defined in Section II-B2 [21].

B. Mechanism properties

First, we determine if it is possible to find some function \( f(\cdot) \) such that the mechanism is self-sustaining, i.e., a mechanism in which rewards introduced by the incentive function \( I_i(\cdot) \) can be found by the benefits obtained in the optimal outcome. In the next theorem, we prove
that under certain conditions, the incentives scheme does not satisfy the budget balance property. That is, the amount of rewards (price discounts) and penalties (price increment) are not balanced, and consequently, the mechanism requires either inflow or outflow of resources.

**Theorem 2.** Suppose that Assumptions 2 and 3 are satisfied. Also consider that \( p(z) = \beta z + b \), where \( z \in \mathbb{R} \), \( \beta > 0 \), and \( b \geq 0 \), and a population of two or more agents. Then, there does not exist a function \( f(\cdot) \) of the form in Eq. (9), such that the incentives mechanism described by Eqs. (7) and (8) satisfies the budget balance property.

This result is analogous to the Myerson-Satterthwaite impossibility theorem [33], which states the impossibility of designing a mechanisms with ex-post efficiency and without external subsidies in games between two parties. However, Theorem 2 considers a nonlinear price scheme and efficiency is defined as the maximization of the aggregated utility, rather than the maximization of the aggregated valuation.

Now, since it is not possible to find a budget balanced mechanism, we investigate the design of a mechanism that satisfies the following fairness conditions.

**Condition 1** (Fairness conditions).

i. Incentives for the \( i^{th} \) and \( j^{th} \) agents are equivalent whether their consumption is the same, i.e., if \( q_i = q_j \), then \( I_i(q) = I_j(q) \).

ii. If \( q_i = q_j \) for all \( i, j \in \mathcal{P} \), then \( I_i(q) = I_j(q) = 0 \).

iii. A higher power consumption deserves a lower incentive, i.e., if \( q_j > q_i \), then \( I_j(q) < I_i(q) \).

The following result shows the existence of a mechanism that satisfies the fairness conditions stated above.

**Proposition 1.** Assume a population with \( N \geq 2 \) agents, incentives defined by Eq. (7) and (8), and an affine price function \( p(z) = \beta z + b \) for some \( \beta > 0 \), and \( b \geq 0 \). If the function \( f(\cdot) \) has the form

\[
f(q_{-i}) = \frac{1}{N-1} \sum_{h \neq i} q_h,
\]

for all \( j, i \in \mathcal{P} \), then the incentives mechanism satisfies the fairness properties in Condition 1.

Henceforth, we are going to use the the following incentives that satisfy the Condition 1:

\[
I_i(q) = \|q_{-i}\|_1 \left( p \left( \frac{N}{N-1} \|q_{-i}\|_1 \right) - p(\|q\|_1) \right)
\]
Therefore, the utility function in Eq. (10) can be rewritten as
\[ W_i(q) = v_i(q_i) - \|q\|_1 p (\|q\|_1) + \|q_{-i}\|_1 p \left( \frac{N}{N-1} \|q_{-i}\|_1 \right). \] (13)

Remark 2. Note that each user only needs to know the aggregated demand $\|q\|$ and its last consumption to calculate the consumption $q_i$ that maximizes $W_i(q)$. Thus, the mechanism can be implemented using a one-dimensional message space that communicates the aggregated demand to each user. When considering multiple time periods, we can still use a one-dimensional message space if the consumption $q^k_i$ is calculated sequentially. Otherwise, the implementation might require a $T$-dimensional message space to calculate simultaneously the consumption along a day.

Note that with an affine price function the incentives can be rewritten as
\[ I_i(q) = \beta \left( \sum_{j \neq i} q_j \right) \left( \frac{1}{N-1} \sum_{j \neq i} q_j - q_i \right) \] (14)
for all $q_i \geq 0, i, j \in P$. Thus, the population incentives can be expressed as
\[ \sum_{i=1}^{N} I_i(q) = \beta q^T A q, \] (15)
where $A = \left( -\frac{1}{N-1} ee^T + \frac{N}{N-1} I \right)$ and $e$ is a vector in $\mathbb{R}^N$ with all its components equal to 1. Now, with this expression we can analyze some properties of incentives given by Eq. (14). In the next proposition we show that the system requires external subsidies to maintain the incentives scheme. In other words, the mechanism satisfies the weak budget balance property [22].

Proposition 2. Suppose that Assumptions 2 and 3 are satisfied. Given an incentives mechanism of the form in Eq. (14), then the incentives required by the population are positive, i.e., $\sum_{i=1}^{N} I_i(q) \geq 0$, for all $q \in \mathbb{R}^{N}_{\geq 0}$.

Based on the previous result, we can prove that the social aggregate surplus reached by the population is greater when the incentives scheme is introduced.

Proposition 3. Consider a population of agents with utility function of the form in Eq. (2) and incentives described by Eq. (7). Also, consider that Assumptions 2 and 3 are satisfied. Then, the aggregate surplus is improved by implementing the incentives mechanism.

Remark 3. Let us consider an homogeneous population, composed by agents with equal preferences. In such population, the energy consumed at the equilibrium is the same for every
agent. Thus, according to Condition 1, a homogeneous population requires null incentives at the equilibrium. In particular, incentives would be required only to shift the system from an inefficient outcome toward the optimal equilibrium. Hence, it is necessary to segment the population in subsets with similar preferences to reduce requirements of external subsidies.

Next, we prove that the incentives mechanism is individual rational.

**Theorem 3.** The mechanisms described by Eq. (12) is individual rational, that is,

\[ W_i(0, \mu_{-i}) \geq 0, \]

for all \( \mu_{-i} \).

So far, we have verified that in a strategic environment the aggregate surplus is improved with the adoption of incentives (see Proposition 3). Also, the mechanism guarantees that the utility of an individual is always positive. However, an individual that is enrolled in an inefficient system might migrate toward a system that implements incentives only if there are guarantees that its profit will not be reduced after the change. The following result guarantees that the agents that has a lower consumption with respect to the average can expect a greater utility in the system with incentives.

**Theorem 4.** Every agent that consumes less resources than the average in the optimal outcome of the game \( G \), defined by \( (U_1, \ldots, U_n) \), can expect a greater profit in the Nash equilibrium of the game with incentives \( G_I \), defined by \( (W_1, \ldots, W_N) \). That is, given the optimal outcome \( \mu \) of \( G_I \), if \( \mu_i < \frac{1}{N} \sum_{h \in P} \mu_h \), then \( U_i(\mu) < W_i(\mu) \). Otherwise, \( U_i(\mu) \geq W_i(\mu) \).

Theorem 4 shows that some users can expect a higher utility in the system (with incentives \( G_I \)) than the optimal outcome of \( G \). This happens because the low consumption is rewarded in \( G_I \). The extent to which an agent can expect major utility in \( G_I \), with respect to the inefficient outcome \( \xi \) is an open problem.

**IV. Decentralized Implementation of the Mechanism**

In previous sections, we have analyzed the characteristics of the electricity system market at different equilibrium points. However, the equilibrium analysis is meaningful only if we can guarantee that strategic players can reach such outcomes. In this section, we are concerned with the behavioral modeling (dynamics) of rational individuals that are involved in a game. Particularly, we show that the Nash equilibrium can be learned in a decentralized manner.
We propose an indirect revelation mechanism that does not rely on a central agent to assign resources, but allows each agent to use the resources without any restriction. In this way it distributes the computation tasks among the population. Specifically, each individual is responsible for optimizing its own consumption based on the aggregated energy consumption of the society, that can be broadcasted by the utility (see Fig. 1).

While it might be unrealistic to assume that real consumers would regulate their consumption along the day, we can think on automation devices that optimize the energy consumption (following some dynamics), based on particular preferences (local information) and reports from the central entity. In general, users might respond to higher prices by purchasing more efficient appliances [31]. Therefore, we assume that each customer’s automation device carries out a learning process to adjust its consumption to the price signals sent by the utility. The learning process is seen as a method to solve a resource allocation problem, in which each individual finds the amount of resources that should be used in a given time period. Here, we assume that the daily consumption of each user is bounded by $Q_i$, which can be interpreted as the maximum consumption capacity of the $i^{th}$ agent. Accordingly, the optimization problem that each agent solves (see Eqs. (5) and (13)) can be rewritten as

$$\begin{align*}
&\text{maximize} & W_i(q_i, q_{-i}) \\
&\text{subject to} & \sum_{k=1}^{T} q_i^k + q_{i}^{T+1} = Q_i \\
& & q_i^k \geq 0, i = \{1, \ldots, N\}, k = \{1, \ldots, T\}.
\end{align*}$$

Note that if $Q_i$ is large enough, we can assure that the solution to the optimization problems in (5) and (16) is the same. This formulation is convenient to define the fitness and strategies in the
population game defined below. We introduce some notation to be consistent with the literature in population games [34].

A. Population Games Approach

Population games is a tool that can model externalities imposed by the actions (strategy) of an agent on the payoff of the other players. In this case, we assume that each user implements some evolutionary dynamics to maximize its own utility (see problem in (16)). Thus, the electricity game can be seen as a multi-population game, in which each customer represents a population in the society $P$. From the perspective of population games [34], each population is composed by a large number of agents, who change their actions (or strategies) following some protocols. The aggregated actions of agents that follow a given protocol can be modeled by means of a differential equations (called evolutionary dynamics), that describes changes in the strategies adopted in the population.

Thus, each population is composed by a large number of agents who can adopt different strategies from the set $S^i = \{1, 2, \ldots, T+1\}$, where $i \in P$. In this case, we denote the state of the population $i \in P$ with the vector $x^i = [x^i_1, \ldots, x^i_{T+1}]^T$, where $x^i_k$ is the proportion of the strategy $k \in S^i$ in the population $i$. Note that the set of possible population states is defined as the simplex $X^i = \{x^i \in \mathbb{R}_{\geq 0}^{T+1} : \sum_{k \in S^i} x^i_k = Q_i\}$. The population dynamics approach can be used to solve optimization problems with restrictions, as it is shown in [35].

Let us formulate the population game as follows. We consider a society composed by $N$ populations with $T+1$ possible strategies per each population. For a given population $i$, the $k^{th}$ strategy’s expected value is denoted by $x^i_k = q^i_k$, for $k \in \{1, \ldots, T\}$, i.e., the population’s strategy is the amount of resources consumed in each time period. Moreover, the strategy $x^i_{T+1}$ is a slack variable that represents the power not consumed in any time interval and it is modeled as a consumption in the fictitious $(T+1)^{th}$ time interval. The slack variable is defined as

$$x^i_{T+1} = Q_i - \sum_{k=1}^T q^i_k.$$ 

Now, let us define the fitness (or payoff) function $F^i_k : \mathbb{R}^N \rightarrow \mathbb{R}$ for the $k^{th}$ strategy in the $i^{th}$ population as the derivative of the profit function $W_i(q)$, i.e., the fitness is equal to the marginal...
utility of the $i^{th}$ population, defined as

$$F^i_k(x) = \frac{\partial W^k_i}{\partial q^k_i}(q^k),$$

(17)

for $k \in \{1, \ldots, T\}$ (recall that $I_i(q^k)$ does not depend on $q^k_i$). On the other hand, the fitness of the fictitious variable (consumption in $k = T + 1$) is defined as zero, i.e., $F^i_{T+1} = 0$.

**Remark 4.** Note that the population game is a potential game with potential function $\Psi(q) = \sum_{k=1}^{T} \sum_{i \in P} U^k_i(q^k)$, i.e.,

$$\frac{\partial \Psi}{\partial q^k_i}(q) = F^i_k(q^k).$$

In this case, the utility is independent with respect to the time period $k$. Hence, the fitness of the strategy $k$ in one population depends on the strategy implemented by each population in the time interval $k$.

Potential games are a class of population games that can be solved using some dynamics presented in the next section [34].

**B. Evolutionary Dynamics**

We implement four evolutionary dynamics, namely *logit dynamics* (Logit), *replicator dynamics* (RD), *Brown-von Neumann-Nash dynamics* (BNN), and *Smith dynamics* (Smith) that belong to the families of *perturbed optimization, imitative dynamics, excess payoff dynamics*, and *pairwise comparison dynamics*, respectively [34], [36]. Each family uses different agent behavior models. A priori, it is difficult to know which dynamic is most convenient for a particular problem. Now, let us denote the dynamics of a society by means of

$$\dot{x} = V_d(x),$$

where $d \in D = \{\text{Logit}, \text{RD}, \text{Smith}, \text{BNN}\}$ denotes the dynamic used. The following differential equations describe the evolution in time of each strategy according to each evolutionary dynamic:

1) **Logit Dynamics:**

$$\dot{x}^i_k = \frac{\exp (\eta^{-1}F^i_k(x))}{\sum_{\gamma \in S} \exp (\eta^{-1}F^i_{\gamma}(x))}, \eta > 0,$$

(18)
2) Replicator Dynamics:

\[ \dot{x}_i^k = x_i^k \hat{F}_i^k(x) . \]  

3) Brown-von Neumann-Nash Dynamics (BNN):

\[ \dot{x}_i^k = \left[ \hat{F}_i^k(x) \right]_+ - x_i^k \sum_{\gamma \in S} \left[ \hat{F}_\gamma^i(x) \right]_+ \]  

4) Smith Dynamics:

\[ \dot{x}_i^k = \sum_{\gamma \in S} x_\gamma^i \left[ F_i^k(x) - F_\gamma^i(x) \right]_+ - x_k^i \sum_{\gamma \in S} \left[ F_\gamma^i(x) - F_k^i(x) \right]_+ . \]

These dynamics are defined in function of the excess payoff to strategy \( k \) as \( \hat{F}_i^k = F_i^k(q^k) - \bar{F}_i^k(q_k) \), where \( \bar{F}_i^k(q_k) \) is the average payoff the population \( i \).

Since the potential function \( \Psi(\cdot) \) is a concave function, we know from Lemma 3.1.3 and Corollary 3.1.4 in [34] that the population game has a unique Nash equilibrium, that corresponds to the maximum of \( \Psi \). Furthermore, the dynamics in Eq. (19), (20), and (21) satisfies the positive correlation (PC) property. Hence, according to Lemma 7.1.1 [34], \( \Psi \) is a Lyapunov function for these differential equations. We can use these facts with the Theorem 7.1.2 [34] to show that the Nash equilibrium is globally asymptotically stable in the dynamics (20) (21). Moreover, (19) is locally stable because it does not necessarily converge to the Nash equilibrium. In particular, the solutions to (19) might not reach the Nash equilibrium if the initial conditions are in the border of the simplex \( X^i \) (replicator admits solutions/rest points that are not NE as well as closed orbits). Analog convergence results can be derived for the perturbed best response (logit) dynamics.

It is important to highlight that in this implementation we use two different time domains. On the one hand, we represent a daily time domain by means of \( k \in \{1, \ldots, T\} \). This time domain is discrete and represents different time intervals during a day. On the other hand, the evolutionary dynamics introduce a continuous time domain related to the evolution of the differential equations defined in Eq. (18)-(21). The scale of this continuous time domain can be considered much larger than the daily time domain, since adjustments in the consumption are
considered to be slow.

V. SIMULATION RESULTS

In this section, we illustrate some ideas of efficiency and the decentralized implementation of the incentives mechanism. First, let us illustrate the content of Section II-A by considering an heterogeneous population of agents. Later, we estimate the external subsidies (cost) required to implement an optimal outcome with each the evolutionary dynamic introduced in Section IV-B. In these experiments we select some functions used previously in the literature that satisfy Assumptions 1, 2, and 3 (see [18], [37]). On the one hand, we define the family of valuation functions as

\[ v(q^k, \alpha_i^k) = v_i^k(q_i^k) = \alpha_i^k \log(1 + q_i^k) \]

where \( \alpha_i^k > 0 \) is the parameter that characterizes the valuation of the \( i^{th} \) agent at the \( k^{th} \) time instant. On the other hand, the generation cost function is defined as

\[ C(\|q\|_1) = \beta(\|q\|_1)^2 + b\|q\|_1, \]

and the unitary price function is

\[ p(\|q\|_1) = \frac{C(\|q\|_1)}{\|q\|_1} = \beta\|q\|_1 + b. \]

Note that the generation cost only depends on the aggregated consumption, not on the time of the day. Furthermore, the fitness function of the system with incentives (see Eq. (17)) is

\[ F_i^k(q^k) = \frac{\alpha_i^k}{1 + q_i^k} - 2\beta \left( \sum_{j=1}^{N} q_j^k \right). \]

We define \( N = 5 \) users, \( Q_i = 30KWh \) for all \( i \), \( \beta = 1 \), \( b = 0 \), \( T = 24 \), and uniform initial conditions, i.e., \( x_i^k(0) = Q_i/(T + 1) \). Also, we define \( \eta = .005 \) for the Logit dynamics.

In order to model time varying valuations along a day, we assign to \( \alpha_i^k \) a value proportional to the actual consumption in an electrical system. In this case, we consider \( T = 24 \) time periods and define the valuations of each individual using consumption measurements provided by the Colombian electricity system administrator [38]. We define a heterogeneous society composed
by individuals with different valuations, such that

$$\alpha_i^k < \alpha_j^k,$$

for all $i, j \in \mathcal{P}$ with $i \neq j$ and $k \in \{1, \ldots, T\}$. Thus, the $i^{th}$ user has lower preferences, at any time interval, than the $i + 1^{th}$ user.

A. Inefficiency Example

In Fig. 2 we show the aggregated demand at the optimal solution and the Nash equilibrium of the game $G$ (considering users with heterogeneous preferences). We verify that the population’s demand profile is lower at the optimal solution. Furthermore, Fig. 3 shows that the utility of the society is greater at the optimal outcome. These properties are characteristics of the tragedy of the commons.

![Fig. 2: Daily power consumption of the society at both the optimal and the inefficient outcome.](image)

Now, let us analyze the ideas related with the dynamical systems. In this case, we consider both homogeneous and heterogeneous populations. In order to the analyze the response of the population to the economic incentives, we introduce incentives in the time period comprehended between 2 and 4 seconds (see Fig. 4). We observe that the introduction of the incentives produces an increment in population average utility, as well as a reduction in the average consumption. Also, note that the total incentives delivered to the homogeneous population is close to zero. This happens because the following conditions are satisfied: 1) the population is homogeneous;
2) agents’ initial conditions are identical; and 3) their consumption updates are synchronized. These conditions imply that the consumption adjustment of the agents are identical at each time instant. Hence, since Condition 1 holds, the total amount of incentives is null. In contrast, the heterogeneous population requires incentives different from zero, even in the equilibrium.

Fig. 5 shows the incentive given to each user in the optimal equilibrium. Note that the user with a lower valuation (user 1) is the one that receives more incentives, while the user with larger valuations (user 5) has the lower incentives. This happens because the user with lower valuations can reduce its consumption much more than a user with higher valuations, and consequently, receives more incentives.

B. Evolutionary Dynamics

The evolution of utility, demand, and incentives for different dynamics is shown in Figs. 6 and 7. Note that despite using the same initial condition, the evolution of the system is different with each dynamical model. In particular, BNN and Smith dynamics converge faster to the optimum, in contrast with the Logit and replicator dynamics. This is achieved by means of a fast decrease in the power consumption.

Incentives in Fig. 7 show that, in the long run, all dynamics converge to the same level of incentives. Particularly, Smith dynamics requires more incentives during all time, except for logit dynamics, which has a sudden increase in the incentives close to the equilibrium point.
(b) Dynamics of a heterogeneous population.

Fig. 4: Dynamics of two populations in the presence of incentives using BNN dynamics.

In Fig. 7 it is not clear which dynamical model moves the state of the system to the optimal equilibrium using less resources. To answer this question, we simulate the total amount of incentives used by each model. Thus, let us define the aggregated incentives in a society in a particular time $t$ as

$$I_d(t) = \sum_{i \in P} \frac{1}{|S|} \sum_{k \in S} I_i(q^k(t)).$$

Now, the total accumulated incentives from $t_0$ to $t$ is defined as

$$\Phi_d(t) = \int_{t_0}^{t} I_d(\tau) d\tau.$$

Thus, $\Phi_d(t)$ gives a measurement of the total amount subsidies required by the system with
dynamic $d$, in the time interval $[t_0, t]$. In this case we do not have a reference to compare the subsidies requirements of each evolutionary dynamic. Hence, we compare the subsidies requirements with the average requirements of all the dynamics implemented. In order to see which dynamic requires more resources, we plot the cumulative resources required by each dynamic relative to the average. Hence, we define the cumulative incentives as

$$CI_d = \frac{\Phi_d(t)}{\sum_{d\in D} \Phi_d(t)}.$$  

Fig. 8 shows the results of the simulation of the relative subsidies required by each model of
Incentives

Smith dynamics requires much more resources during all the time stamp, but is particularly high during the first stages, while logit has the lower incentives requirements. However, BNN has the lower incentives in long run.

Fig. 8: Accumulated incentives during the evolution of the algorithm.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we propose an indirect revelation mechanism to maximize the aggregated surplus of the population. The main feature of this mechanism is that it does not require private
information from users, and employs a one dimensional message space to coordinate the demand profile of agents. These properties facilitate the distributed implementation of the mechanism. The mechanism entrusts the computation tasks among users, who should maximize its own utility function based on the aggregated demand (that is calculated and broadcasted by a central agent). Thus, users avoid revelation of private information (e.g., preferences), but are required to report the aggregated consumption of their appliances during some time periods.

We show that most users of the electricity system would join the incentives program voluntarily, since they might have positive utility. Particularly, users that consume less resources than the average can expect a higher utility in the system with incentives, because the low consumption is rewarded in $G_I$. However, the extent to which an agent can expect major utility in $G_I$, with respect to the inefficient outcome $\xi$ is an open problem. The mechanism is weakly budget balanced and might require external subsidies to succeed. In particular, the implementation cost of the mechanism depends on preferences and dynamics implemented by users. The cost associated with external subsidies might be reduced by segmenting the population into subsets with similar preferences.

We introduce an approach based on evolutionary dynamics that might be used by each user to find the demand profile that maximizes its utility. Particularly, when implemented locally by each user, the evolutionary dynamics lead to the global efficient equilibrium. We implement four popular evolutionary dynamics, namely logit dynamics, replicator dynamics, Brown-von Neumann-Nash dynamics, and Smith dynamics. We find that the system might converge faster to the equilibrium at expense of a major external subsidies, e.g., Smith dynamics. We find that BNN dynamics has a relatively fast convergence and uses less resources in the long run.

Future work will be focused on analyzing the characteristics of the mechanism on large populations. Also, it is interesting to explore dynamics that lead to minimum accumulated incentives, as well as possible applications of fast dynamics in environments with random components, such as renewable generation. On the other hand, it would be interesting to analyze the effects of corrupted information of either consumption or price, due to noise or intended attacks, on the system performance, and the cost associated with incentives.
APPENDIX

Proof of Theorem 1. The proof of numeral numeral (i) is made by contradiction, by assuming that $\mu^k_i \geq \xi^k_i$ for an individual $i$ at some time instant $k$. Since $v^k_i(\cdot)$ is concave and non-decreasing, its derivative $\dot{v}^k_i(\cdot)$ is monotone decreasing. According to our initial hypothesis we have that $\dot{v}^k_i(\mu^k_i) \leq \dot{v}^k_i(\xi^k_i)$. From the FOC in Eq. (4) and Eq. (6) we can conclude that

$$p \left( \|\xi^k\|_1 \right) + \xi^k_i \dot{\mu}^k_i \geq p \left( \|\mu^k\|_1 \right) + \|\mu^k\|_1 \dot{\mu}^k_i.$$

(22)

Note that the unitary price $p(\cdot)$ is an increasing convex function and its derivative $\dot{p}(\cdot)$ is monotone increasing. Hence, the previous expression implies that $\|\mu^k\|_1 \leq \|\xi^k\|_1$, which is true if there exists at least an individual $j$ such that

$$\mu^k_j \leq \xi^k_j.$$

This implies that $\dot{v}^k_j(\mu^k_j) \geq \dot{v}^k_j(\xi^k_j)$ and from the FOC in Eq. (4) and Eq. (6) we deduce that

$$p \left( \|\xi^k\|_1 \right) + \xi^k_j \dot{\mu}^k_j \leq p \left( \|\mu^k\|_1 \right) + \|\mu^k\|_1 \dot{\mu}^k_j.$$

(23)

From Eq. (22) and Eq. (23) we have

$$p \left( \|\xi^k\|_1 \right) + \xi^k_i \dot{\mu}^k_i \geq p \left( \|\mu^k\|_1 \right) + \|\mu^k\|_1 \dot{\mu}^k_i \geq p \left( \|\xi^k\|_1 \right) + \xi^k_j \dot{\mu}^k_j.$$

Consequently, we find that $\xi^k_i \geq \xi^k_j$, which leads to the following inequality:

$$\mu^k_i \geq \xi^k_i \geq \xi^k_j \geq \mu^k_j.$$

(24)

By Assumption $\dot{v}^k_i(\cdot)$ is monotone decreasing, i.e.,

$$\dot{v}^k_i(z) \geq \dot{v}^k_i(z + \varepsilon),$$

(25)

for two real numbers $z$ and $\varepsilon$ greater than zero. Furthermore, note that the marginal valuation of the $i^{th}$ and $j^{th}$ agents is equal at the Pareto equilibrium (see Eq. (4)), that is,

$$\dot{v}^k_i(\mu^k_i) = \dot{v}^k_j(\mu^k_j).$$
Now, using the property of Eq. (25) and (24) we can show that

\[ \dot{v}_j^k(\xi_j^k) \geq \dot{v}_j^k(\mu_j^k) = \dot{v}_i^k(\mu_i^k) \geq \dot{v}_i^k(\xi_i^k). \]

Using the FOC in Eq. (6) we obtain the following inequality:

\[ p \left( \| \xi^k \|_1 \right) + \xi_j^k \dot{p} \left( \| \xi^k \|_1 \right) \geq p \left( \| \xi^k \|_1 \right) + \xi_i^k \dot{p} \left( \| \xi^k \|_1 \right), \]

leading to \( \xi_j^k \geq \xi_i^k \), that is in contradiction with the statement in Eq. (22). Therefore, we conclude that \( \xi_i^k > \mu_i^k \) for some time instant \( t \) and every individuals \( i, j \in V \) such that \( i \neq j \).

Now, the proof of numeral (ii) is made by direct proof. From Assumption 2 we know that the competitive equilibrium is unique, and corresponds to the best possible outcome for the population. Hence, the competitive equilibrium is efficient in the sense of Pareto. On the other hand, from numeral (i) we conclude that \( \| \mu \|_1 < \| \xi \|_1 \). Therefore, \( \xi \neq \mu \), which implies that the total consumption at the Nash equilibrium \( \xi \) is not efficient in the sense of Pareto.

Proof of Theorem 2. This proof is made by contradiction. First, we assume that there exists a function \( f(\cdot) \) such that the mechanism is budget balanced, i.e., \( \sum_{i=1}^{N} I_i(q) = 0 \). Now, we express the incentives in matrix form. On that purpose, we first define \( [f(q_{-1}), \ldots, f(q_{-N})]^T = Fq \), as a vector with the \( i \)th element equal to \( f(q_{-i}) \). In particular, \( F = (e\alpha^T - \text{diag}(\alpha_1, \ldots, \alpha_N)) \), \( \alpha = [\alpha_1, \ldots, \alpha_N]^T \), diag\((\alpha_1, \ldots, \alpha_N)\) is a diagonal matrix, and \( e \) is a vector in \( \mathbb{R}^N \) with all its components equal to 1.

Since \( p(\cdot) \) is an affine function, Eq. (7) can be expressed as \( \sum_{i=1}^{N} I_i(q) = \beta \sum_{i=1}^{N} \left( \sum_{j \neq i} q_j \right) \left( f(q_{-i}) - q_i \right) \). This can be rewritten in matrix form as \( \sum_{i=1}^{N} I_i(q) = \beta q^T \Phi(Fq - q) \), where \( \Phi = (ee^T - I) \) and \( I \) is the identity matrix in \( \mathbb{R}^{N \times N} \).

Now, considering the budget balance condition, we have \( q^T \Phi Fq = q^T \Phi q \). This equation is satisfied if either \( q_i = 0 \) for all \( i \in \mathcal{P} \), or if \( F = I \). Note that \( F \) is a matrix with zeros in the diagonal, therefore, \( F \neq I \). Accordingly, none of the aforementioned conditions are satisfied for all vector \( q \in \mathbb{R}^N_{\geq 0} \). Consequently, we conclude that the budget balance property cannot be achieved by means of the incentives mechanism described by Eq. (7), (8), and (9).

Proof of Proposition 1. Let us consider an arbitrary consumption profile \( \hat{q} \) in \( \mathbb{R}^N_{\geq 0} \) such that \( \hat{q}_i = \hat{q}_j \), for some \( i, j \in \mathcal{P} \). Note that since the average price signal is an affine function, the
incentives function in Eq. (7) can be rewritten as $I_i(\hat{q}) = \beta \left( \sum_{h \neq i} \hat{q}_h \right) \left( \frac{1}{N - 1} \sum_{h \neq i} \hat{q}_h - \hat{q}_i \right)$. If we use an incentives scheme with $f(\cdot)$ defined by Eq. (11), then the incentives assigned to the $i^{th}$ and $j^{th}$ agent are

$$I_i(\hat{q}) = \beta \left( \sum_{h \neq i} \hat{q}_h \right) \left( \frac{1}{N - 1} \sum_{h \neq i} \hat{q}_h - \hat{q}_i \right). \quad (26)$$

$$I_j(\hat{q}) = \beta \left( \sum_{h \neq j} \hat{q}_h \right) \left( \frac{1}{N - 1} \sum_{h \neq j} \hat{q}_h - \hat{q}_j \right). \quad (27)$$

Since $\hat{q}_i = \hat{q}_j$, then $\sum_{h \neq i} \hat{q}_h = \sum_{h \neq j} \hat{q}_h$. Hence, $I_i(\hat{q}) = I_j(\hat{q})$. Thus, condition (i) is satisfied.

Now, if the consumption profile $\hat{q}$ satisfies $\hat{q}_i = \hat{q}_j = \sigma$ for all $i, j \in P$, then

$$I_i(\hat{q}) = \beta ((N - 1)\sigma) \left( \frac{N - 1}{N - 1} \sigma - \sigma \right) = 0.$$ 

Consequently, condition (ii) is satisfied.

Finally, let us consider an arbitrary vector $\hat{q}$ such that $\hat{q}_i > \hat{q}_j$ for some $i, j \in P$. Then we know that

$$\sum_{h \neq i} \hat{q}_h < \sum_{h \neq j} \hat{q}_h. \quad (28)$$

Furthermore, $\frac{1}{N - 1} \sum_{h \neq i} \hat{q}_h + \hat{q}_j < \frac{1}{N - 1} \sum_{h \neq j} \hat{q}_h + \hat{q}_i$, that can be rewritten as

$$\frac{1}{N - 1} \sum_{h \neq i} \hat{q}_h - \hat{q}_i < \frac{1}{N - 1} \sum_{h \neq j} \hat{q}_h - \hat{q}_j. \quad (29)$$

Inequalities in Eq. (28) and (29) can be used with Eq. (26) and (27) to show that $I_i(\hat{q}) < I_j(\hat{q})$. Thus, property (iii) is satisfied.

\[ \Box \]

Proof of Proposition 2. First, consider $q_i^2 + q_j^2 - 2q_iq_j = (q_i - q_j)^2 \geq 0$ for all $q_i \in \mathbb{R}_{\geq 0}^T$. Hence, we have that $q_i^2 + q_j^2 \geq 2q_iq_j$. Now, summing in both sides of the previous equation we obtain $\sum_{i=1}^N \sum_{j \neq i}(q_i^2 + q_j^2) \geq \sum_{i=1}^N \sum_{j \neq i} 2q_iq_j$, which is equivalent to $(N - 1) \sum_{i=1}^N q_i^2 \geq \sum_{i=1}^N \sum_{j \neq i} 2q_iq_j$. Reordering results

$$\sum_{i=1}^N q_i^2 \geq \frac{2}{N - 1} \sum_{i=1}^N \sum_{j \neq i} q_iq_j. \quad (30)$$

Now, let $A_{j,i} = \frac{1}{N - 1}$ if $i \neq j$ and $A_{i,i} = 1$ for all $i, j \in P$. Therefore, the incentives in Eq. (15)
can be expressed as $\beta q^\top A q = \beta \sum_{i=1}^N q_j \left( \sum_{j=1}^N q_{j,i} A_{j,i} \right)$. This can be rewritten as

$$\beta q^\top A q = \beta \left( \sum_{i=1}^N q_i^2 + \frac{-1}{N-1} \sum_{i=1}^N q_i \left( \sum_{j \neq i} q_j \right) \right).$$

From Eq. (30), it can be seen that $q^\top A q \geq 0$, for all $q \in \mathbb{R}^N_{\geq 0}$.

**Proof of Proposition 3.** Recall from Theorem 1 that the aggregated utility at the optimal outcome $\mu$ is greater than aggregated utility at the Nash equilibrium $\xi$, that is $\sum_{i \in P} U_i(\mu) > \sum_{i \in P} U_i(\xi)$. Also, recall that the system with incentives achieves the optimum outcome, then the social welfare of the system with incentives is $\sum_{i \in P} W_i(\mu) = \sum_{i \in P} (U_i(\mu) + I_i(\mu))$. Since the total amount of incentives is greater than zero, then the following inequality is satisfied:

$$\sum_{i \in P} (U_i(\mu) + I_i(\mu)) > \sum_{i \in P} U_i(\mu) > \sum_{i \in P} U_i(\xi).$$

Consequently, the aggregate surplus is improved by implementing the incentives scheme.

**Proof of Theorem 3.** Let us consider a consumption profile in which the $i^{th}$ individual is not consuming energy, i.e., $q$ such that $q_i = 0$ for some $i \in P$. Note that $\|q\|_1 = 0 + \|q_{-i}\|_1$. Hence, from Eq. (13) we have that the utility with incentives is equal to

$$W_i(q_i = 0, q_{-i}) = v_i(0) + \|q_{-i}\|_1 \left( p \left( \frac{N}{N-1} \|q_{-i}\|_1 \right) - p \left( \|q_{-i}\|_1 \right) \right)$$

Since the price function is increasing, we know that $p \left( \frac{N}{N-1} \|q_{-i}\|_1 \right) \geq p \left( \|q_{-i}\|_1 \right)$. Consequently, the utility function of every agent $i$ is greater or equal than zero.

**Proof of Theorem 4.** Let us rewrite the utility of the $i^{th}$ agent (see Eq. (2)) at the equilibrium as

$$U_i(\mu) = v_i(\mu_i) - \|\mu\|_1 p(\|\mu\|_1) + \|\mu_{-i}\|_1 p(\|\mu\|_1).$$

If we evaluate the utility with incentives (see Eq. (10)) in the Pareto optimal outcome $\mu$, we
can find that
\[
W_i(\mu) - U_i(\mu) = \left\| \mu \right\|_1 \left( p \left( \frac{1}{N-1} \| \mu_{-i} \|_1 + \| \mu_i \|_1 \right) - p(\| \mu \|_1) \right).
\]

Note that \( \mu_i < \frac{1}{N} \sum_{h \neq i} \mu_h \) can be rewritten as \( \mu_i < \frac{1}{N} \sum_{h \in P} \mu_h \).

Now, if \( \mu_i < \frac{1}{N} \sum_{h \neq i} \mu_h \) (that can be rewritten as \( \mu_i < \frac{1}{N} \sum_{h \in P} \mu_h \)), then \( \frac{1}{N-1} \| \mu_{-i} \|_1 + \| \mu_i \|_1 > \| \mu \|_1 \) and consequently \( W_i(\mu) - U_i(\mu) > 0 \).

On the other hand, if \( \mu_i \geq \frac{1}{N} \sum_{h \neq i} \mu_h \) (that can be rewritten as \( \mu_i \geq \frac{1}{N} \sum_{h \in P} \mu_h \)), then
\[
\frac{1}{N-1} \| \mu_{-i} \|_1 + \| \mu_i \|_1 \leq \| \mu \|_1
\]
and thus \( W_i(\mu) - U_i(\mu) \leq 0 \).

\[ \square \]

REFERENCES


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