An asymptotic expansion for the error in a linear map that reproduces polynomials of a certain order

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Received 17 December 2003; accepted 16 February 2005

Communicated by Amos Ron
Available online 18 April 2005

Abstract

Han’s ‘multinode higher-order expansion’ in [H] is shown to be a special case of an asymptotic error expansion available for any bounded linear map on $C([a..b])$ that reproduces polynomials of a certain order. The key is the formula for the divided difference at a sequence containing just two distinct points.

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Keywords: Asymptotic error expansion; Polynomial reproduction; Divided difference

In [H], Han shows that, for linear maps on $C([a..b])$ of the form $L : f \mapsto \sum_i \phi_i f(x_i)$ that reproduce polynomials of degree $\leq m$, and for a specific choice of coefficients $a_j$, independent of $L$ and $f$ but depending on $m$ and $r$, the following asymptotic error expansion

$$f(x) = Lf(x) + \sum_{j=0}^{r} \frac{a_j}{j!} L \left( (x - \cdot)^j D^j f \right)(x) + E(f, x)$$

holds, with $E(f, x)$ explicitly given as an integral involving $D^{m+r+1} f$. Since, for his particular choice of $L$, the sum involves the derivatives of $f$ at the points or nodes $x_i$ associated with $L$, Han thinks of this as a ‘multinode’ expansion for $f$.

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It is the purpose of this note to point out that this asymptotic error expansion, properly interpreted, holds for any bounded linear map $L$ on $C([a..b])$, with the same formula for $E(f, x)$. The key is the formula for the divided difference at a sequence containing just two distinct points.

It is easy to verify, for example by induction on $r$ and $m$, particularly for the special case $x = 0, y = 1$, that, for any $x \neq y$,

$$(-1)^{m+1}(y - x)^{r+m+1} \Delta(x^{[r+1]}, y^{[m+1]})$$

$$= \sum_{j=0}^{r} \binom{m+r-j}{r-j} (y-x)^j \Delta(x^{[j+1]}) - \sum_{k=0}^{m} \binom{r+m-k}{m-k} (x-y)^k \Delta(y^{[k+1]}),$$

with $\Delta(x^{[r+1]}, y^{[m+1]})$ denoting the divided difference at the point sequence that contains $x$ exactly $r + 1$ times and $y$ exactly $m + 1$ times.

The Peano kernel for the divided difference $\Delta(t_0, \ldots, t_n)$ at the sequence $(t_0, \ldots, t_n)$ is well-known to be the B-spline with knot sequence $(t_0, \ldots, t_n)$ that is normalized to integrate to $1/n!$, hence (cf. (5) below), for arbitrary $x$ and $y$,

$$(y - x)^{r+m+1} \Delta(x^{[r+1]}, y^{[m+1]}) = \int_{x}^{y} \left[ (t - x)^{m} \right]^{r} D^{r+m+1} f(t) \, dt,$$

with $\left[ s \right]^{n} := s^{n}/n!$

a handy notation for the normalized power.

Consequently, for any smooth $f$ and any $x$ and $y$, and using the fact that $\Delta(z^{[n+1]}) f = D^{n} f(z)/n!$,

$$- \int_{x}^{y} \left[ x - t \right]^{m} \left[ y - t \right]^{r} D^{r+m+1} f(t) \, dt$$

$$= \sum_{j=0}^{r} \binom{m+r-j}{r-j} \left[ y - x \right]^{j} D^{j} f(x) - \sum_{k=0}^{m} \binom{r+m-k}{m-k} \left[ x - y \right]^{k} D^{k} f(y). \tag{1}$$

If now $L$ is any bounded linear map on $C([a..b])$ that reproduces polynomials of degree $\leq m$, then, on applying $1 - L$ to both sides of (1) as functions of $x$, we find, for arbitrary $y$,

$$\int_{a}^{b} \left( 1 - L \right) \left[ (x - t)_{+} \right]^{m} \left[ y - t \right]^{r} D^{r+m+1} f(t) \, dt$$

$$= \binom{m+r}{m} (f - Lf)(x) + (1 - L) \left( \sum_{j=1}^{r} \binom{m+r-j}{r-j} \left[ y - \cdot \right]^{j} D^{j} f \right) (x), \tag{2}$$

using the facts that (i) the second sum on the right of (1) is a polynomial of degree $\leq m$ in $x$, hence is annihilated by $1 - L$; that (ii) for any (integrable) $g$ and any $x, y \in [a..b]$,

$$- \int_{x}^{y} g(t) \, dt = \int_{a}^{b} ((x - t)_{+}^{0} - (y - t)_{+}^{0}) g(t) \, dt$$
(with $z_+$ equal to $z$ for positive $z$ and 0 otherwise), hence
\[- \int_x^y \|x - t\|^m \|y - t\|^r g(t) \, dt \]
\[= \int_a^b \left( \|x - t\|_+^m \|y - t\|^r - \|x - t\|^m \|y - t\|^r_+ \right) g(t) \, dt,\]
will replace $z$ in hand. Theorem 2 of [H], i.e., the truncated Taylor series with integral remainder.

\[x \in \mathbb{R},\]
properly, namely as the function $L f$ of one sign (as it is, for any $f \in C([a..b])$, then divide both sides by $(m + r)_+$ and rearrange to arrive at the sought-for expansion
\[f(x) - L f(x) = \sum_{j=1}^r \frac{(m + r - j)}{(m + r)_+} L \left( \|x - \cdot\|_+^j D^j f \right)(x) + E(f, x), \tag{3}\]
with
\[E(f, x) := \int_a^b (1 - L) \left( (\cdot - t)_+^m \right) (x - t)^r D^m f \, dt / (m + r)!, \tag{4}\]
in which $\frac{r!(m + r - j)}{(m + r)_+!}$ could be rewritten as $\frac{r!(m + r - j)}{(m + r)_+!}$. Thus, when $L$ takes the particular form $L f := \sum_{i} \varphi_i f(x_i)$ for some functions $\varphi_i$ and some points $x_i$ in $[a..b]$, we now have in hand Theorem 2 of [H].

As a check, for $L : f \mapsto f(a)$, hence $m = 0$, we obtain
\[f(x) - f(a) = \sum_{j=1}^r \|x - a\|^j D^j f(a) + \int_a^b (x - t)_+^r D^r f \, dt / r!, \]
i.e., the truncated Taylor series with integral remainder.

Consider now the error $E(f, x)$ in the asymptotic error expansion (3) for general $L$.

To be sure, (4) is correct offhand only for $m > 0$. Even when $m = 0$, it is correct in Han’s context, i.e., when $L$ is of the form $f \mapsto \sum_{i} \varphi_i f(x_i)$. For more general $L$, $t \mapsto (L(\cdot - t)_+^m)(x)$ is not defined (since $L(\cdot - t)_+^m$ is not defined) and so must be interpreted properly, namely as the function $k(\cdot, \cdot)$ of bounded variation that vanishes at $b$ and represents the linear functional $\hat{k} : g \mapsto - (L \int_a^b g(t) \, dt)(x)$ in the sense that $\hat{k} f = \int_a^b dk(x, \cdot)$ for all $f \in C([a..b])$, with the existence of such $k(x, \cdot)$ guaranteed by the Riesz Representation Theorem.

With that concern laid to rest, assume that $f \in C^{(r+m+1)}([a..b])$ and that, for a given $x \in [a..b],$
\[[a..b] \mapsto \mathbb{R} : t \mapsto (1 - L) \left( (\cdot - t)_+^m \right)(x)\]
is of one sign (as it is, for any $x \in [a..b]$, when $L f$ is the Bernstein polynomial for $f$, or the Lagrange polynomial interpolant). Then (see (4)) the Peano kernel for $E(\cdot, x)$ is of one sign on $[a..x]$ and on $[x..b]$. Correspondingly,
\[E(f, x) = c_1(x) D^{m+r+1} f(\xi_1) + c_2(x) D^{m+r+1} f(\xi_2), \]
for some $\xi_1 \in [a..x]$, $\xi_2 \in [x..b]$,
with
\[ c_1(x) := E((-1)^{m+r+1} \| (x - \cdot) \|^{m+r+1}, x) \quad \text{and} \]
\[ c_2(x) := E(\| (\cdot - x) \|^{m+r+1}, x) \]
readily computable by retracing the steps that brought us to (3) but choosing, specifically, 
\[ f = (-1)^{m+r+1} \| (x - \cdot) \|^{m+r+1}, \]
i.e., 
\[ D^{m+r+1} f = (x - \cdot)^0, \]
to get \( c_1(x) \) and choosing 
\[ f = \| (\cdot - x) \|^{m+r+1}, \]
i.e., 
\[ D^{m+r+1} f = (\cdot - x)^0, \]
to get \( c_2(x) \). For this, we note that
\[ -\int_x^y \| x - t \|^{m} \| y - t \|^r \, dt = (-1)^{m+1} \| y - x \|^{m+r+1}, \quad (5) \]
for arbitrary \( x \) and \( y \), hence, e.g.,
\[ -\int_x^y \| x - t \|^{m} \| y - t \|^r (x - t)^0 \, dt = (-1)^{m+1} (x - y)^0 \| y - x \|^{m+r+1}. \]
Recalling that we obtained from this the corresponding error term by applying \( 1 - L \) to it as a function of \( x \), then setting \( y = x \) and dividing by \( \binom{m+r}{m} \), we get
\[ c_1(x) = (-1)^{m+1} \left( 1 - L \right) (\| (x - \cdot) \|^{m+r+1})(x)/\binom{m+r}{m} \]
\[ = (-1)^m L (\| (x - \cdot) \|^{m+r+1})(x)/\binom{m+r}{m}. \]
In the same way, we find that
\[ c_2(x) = (-1)^m L (\| (x - \cdot) \|^{m+r+1})(x)/\binom{m+r}{m}. \]
If now \( r \) is even, then \( c_1(x) \) and \( c_2(x) \) are of the same sign and, in that case,
\[ E(f, x) = c(x) D^{m+r+1} f(\xi) \quad \text{some} \ \xi \in [a . b], \]
with
\[ c(x) := c_1(x) + c_2(x) = E(\| \cdot \|^{m+r+1}, x) = (-1)^m L (\| x - \cdot \|^{m+r+1})(x)/\binom{m+r}{m}. \]
Thus, when \( L \) takes the particular form 
\[ L f := \sum_i \varphi_i f(x_i) \]
for some functions \( \varphi_i \) and some points \( x_i \) in \([a . b]\), we now have in hand Theorem 3 of [H].

References