On the girth of extremal graphs without shortest cycles

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Abstract

Let $E_X(\nu; \{C_3, \ldots, C_n\})$ denote the set of graphs $G$ of order $\nu$ that contain no cycles of length less than or equal to $n$ which have maximum number of edges. In this paper we consider a problem posed by several authors: does $G$ contain an $n+1$ cycle? We prove that the diameter of $G$ is at most $n-1$, and present several results concerning the above question: the girth of $G$ is $g = n+1$ if (i) $\nu \geq n+5$, diameter equal to $n-1$ and minimum degree at least 3; (ii) $\nu \geq 12$, $\nu \not\in \{15, 80, 170\}$ and $n = 6$. Moreover, if $\nu = 15$ we find an extremal graph of girth 8 obtained from a 3-regular complete bipartite graph subdividing its edges. (iii) We prove that if $\nu \geq 2n-3$ and $n \geq 7$ the girth is at most $2n-5$. We also show that the answer to the question is negative for $\nu \leq n+1 + \lfloor (n-2)/2 \rfloor$.

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1. Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [4] for terminology and definitions.

Let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of a graph $G$, respectively. The order of $G$ is denoted by $|V(G)| = \nu(G)$ and the size by $|E(G)| = e(G)$. The minimum length of a cycle contained in $G$ is the girth $g(G)$ of $G$. If $G$ does not contain a cycle we set $g(G) = \infty$. By $C_s$ we will denote the cycle of length $s$, $s \geq 3$. The distance $d_G(x, y)$ in $G$ between two vertices $x, y$ is the length of a shortest $x - y$ path in $G$. The greatest distance between any two vertices in $G$ is the diameter $D(G)$ of $G$. Diameter and girth are related by $g(G) \leq 2D(G) + 1$.

Let $\mathcal{F}$ be a family of graphs. The extremal number $ex(\nu, \mathcal{F})$ is the maximum number of edges in a graph of order $\nu$ that does not contain any graph of $\mathcal{F}$ as a subgraph. The graphs of order $\nu$ and size $ex(\nu, \mathcal{F})$ not containing any
$F \in \mathcal{F}$ as a subgraph are the extremal graphs and the set of all such graphs is denoted by $EX(v, F)$. We refer to graphs from $EX(v, F)$ as extremal $F$-free graphs of order $v$, or just extremal.

Let $ex(v; \{C_3, \ldots, C_n\})$ denote the maximum number of edges in a graph of order $v$ and girth at least $n + 1$, and $EX(v; \{C_3, \ldots, C_n\})$ denote the set of all graphs of order $v$, girth at least $n + 1$, and with $ex(v; \{C_3, \ldots, C_n\})$ edges. Erdős and Sachs [5] showed that an $r$-regular graph of girth $g$ at least $n + 1$ with the least possible number of vertices has girth $g = n + 1$. (A proof of this result can be found in Lovász [12], pp. 66, 384, 385, see also the references therein.) These graphs are called $(r; g)$-cages. In this paper we consider a similar problem:

**What is the girth of an extremal $\{C_3, \ldots, C_n\}$-free graph of order $v$? Is it always $n + 1$ or can it be greater?**

This problem has been studied in [2, 8, 9, 13]. Some of the results obtained in these references are listed below.

**Theorem 1.1.** Let $G \in EX(v; \{C_3, \ldots, C_n\})$. The following assertions hold:

(i) the girth is $n + 1$ provided that

- $n = 4$ and $v \geq 7$ [8, 9];
- $n = 5$ and $v \geq 8$ [13];
- $n = 6$ and $v \geq 171$ [2];
- $n \geq 7$ and $v > (2(n - 2)^{n - 2} + n - 5)/(n - 3)$ [2];
- the maximum degree $\Delta$ of $G$ is $\Delta \geq n$ [13];

(ii) the girth is at most $n + 2$ provided that the minimum degree is $\delta(G) \geq 2$ and

- the maximum degree $\Delta$ of $G$ is $\Delta \geq [(n + 1)/2]$ [2].
- $n \geq 7$ and $v > (2(t - 2)^{n - 2} + t - 5)/(t - 3)$, where $t = [(n + 1)/2]$ [2];

(iii) if $v \geq 2n - 2$, then $g \leq 2n - 4$ [2].

The same kind of structural properties as contained in points (i), (ii) of the above theorem for bipartite graphs are stated in [3].

This paper contributes to the study of this problem by proving the following results for a graph $G$ belonging to the extremal family $EX(v, \{C_3, \ldots, C_n\})$:

(i) If $v$ is such that $n + 2 \leq v \leq n + 1 + [(n - 2)/2]$, then there exist extremal graphs of girth $g \geq n + 2$.

(ii) The diameter $D(G)$ is at most $n - 1$. Furthermore, if $D(G) = n - 1$ and the minimum degree is $\delta(G) \geq 3$, then the girth is $g = n + 1$, and for $n = 6, 7$ the requirement on the minimum degree is not needed.

(iii) If $n = 6$, $v \geq 12$ and $v \not\in \{15, 80, 170\}$, then the girth is $g = 7$. For $v = 15$ there exists an extremal graph of girth 8, which is a subdivision of a complete 3-regular bipartite graph.

(iv) If $n \geq 7$ and $v \geq 2n - 3$, then the girth $g$ satisfies $g \leq 2n - 5$.

(v) If $n = 7$ and $v \geq 11$, then the girth $g$ satisfies $8 \leq g \leq 9$.

Furthermore, as an immediate consequence of our results we obtain that the girth of an extremal graph is $n + 1$ for $n = 3, 4, 5$, which are the mentioned known results [8, 9, 13].

## 2. Results

Throughout the paper $n \geq 3$ is an integer. We begin with a theorem providing a negative answer to the question pointed in the introduction for small values of $v$.

**Theorem 2.1.** Let $n \geq 4$ and $v$ be integers such that $v \geq n + 1$. Let $G \in EX(v; \{C_3, \ldots, C_n\})$. Then $G$ contains just one cycle if and only if $n + 1 \leq v \leq n + 1 + [(n - 2)/2]$. Hence for this range of $v$ we have $ex(v; \{C_3, \ldots, C_n\}) = v$, and if $v > n + 1$ there exist graphs $G \in EX(v; \{C_3, \ldots, C_n\})$ with girth $g(G) \geq n + 2$.

**Proof.** Let $G \in EX(v; \{C_3, \ldots, C_n\})$ be with $v \geq n + 1$. First, assume that $v \leq n + 1 + [(n - 2)/2]$. Then $G$ contains a cycle $C_s$ of length $s$, and notice that $s \geq g(G) \geq n + 1$. Moreover, we may assume that $s \leq v - 1$, because otherwise $G = C_s$ and we are done. If $C_s$ is the only cycle contained in $G$, then $G$ consists of $C_s$ joined to one or several paths or trees, which implies that $e(G) = v$ and we have finished. Then suppose that $G$ contains two cycles, $C_s$ and $C'$ and we will arrive at a contradiction. As $g(G) \geq n + 1$ and $v < 2n + 2$, it follows that $C_s$ and $C'$ must share a path $II$, hence

$$n + 1 \leq |V(C')| \leq |V(II)| + v - s.$$
Then $|V(II)| \geq n + 1 + s - v \geq s - [(n - 2)/2]$. Let $x$, $y$ be the end vertices of the path $II$. Then the cycle induced by $(C' - II) \cup (C_2 - II) \cup \{x, y\}$ has length at most $v - |V(II)| - 2 \leq n + 1 + [(n - 2)/2] - s + [(n - 2)/2] + 2 \leq (1 + [(n - 2)/2]) \leq n$, which is a contradiction. Therefore $G$ contains just one cycle.

Conversely, an extremal graph $G$ with $v \geq n + 2 + [(n - 2)/2]$ contains two cycles. It suffices to take a cycle of length $n + 1$ and two vertices $x$, $y$ of the cycle at distance $[(n + 1)/2]$ apart; and take a vertex-disjoint path with the cycle starting in $x$ and finishing in $y$ of length $1 + [(n - 2)/2]$. In this way we obtain that $G$ contains at least two cycles and $g(G) = n + 1$. In order to finish the proof it is enough to observe that any graph resulting from joining a cycle of length at least $n + 2$ and a vertex-disjoint path of order at most $[(n - 2)/2] - 1$ is an extremal $\{C_3, \ldots, C_n\}$-free graph with girth $g(G) \geq n + 2$.

In what follows the set of neighbors of $u \in V(G)$ is denoted by $N_G(u)$. The number of neighbors of $u$ is the degree $d_G(u)$ of $u$ in $G$, or briefly $d(u)$ when it is clear which graph is meant. Let $G - u$ denote the graph formed from $G$ by deleting $u$. Next, we will make use of the following property.

**Definition 2.2.** Let $G$ be a connected graph with diameter $D$. We say that $G$ has the property $Q$ if the following assertion holds:

For each pair of vertices $u$, $v \in V(G)$ such that $d_G(u, v) = D$, we have

$$d_G(u, x) + d_G(x, v) = D,$$

for all $x \in V(G)$.

**Theorem 2.1** allows us to deduce that there exist extremal $\{C_3, \ldots, C_n\}$-free graphs with and without the property $Q$. For instance, if $n$ is odd, we may consider the graph $G$ resulting from joining a cycle of length $n + 1$ and a path of $(n - 1)/2$ vertices, both intersecting in one unique vertex. By **Theorem 2.1** it is clear that $G$ has diameter $D(G) = n + 1$ and it is an extremal $\{C_3, \ldots, C_n\}$-free graph satisfying the property $Q$. Analogously, if $n$ is even an extremal $\{C_3, \ldots, C_n\}$-free graph of diameter $D(G) = n - 1$ without property $Q$ results from joining a cycle of length $n + 1$ and a path of $n/2$ vertices both intersecting in one unique vertex. Furthermore, notice that any extremal graph of order $n + 2 \leq v \leq n + 1 + [(n - 2)/2]$ having girth $g \geq n + 2$ must have diameter $D \leq n - 1$ if $n$ is even, and $D(G) \leq n - 2$ if $n$ is odd. As the next theorem indicates, the diameter of an extremal $\{C_3, \ldots, C_n\}$-free graph $G$ is at most $n - 1$, which will be very helpful to study the girth of $G$.

**Theorem 2.3.** Every graph $G \in EX(v; \{C_3, \ldots, C_n\})$ satisfies:

(i) The diameter is $D(G) \leq n - 1$.

(ii) If there exists $u \in V(G)$ with $d(u) = 1$, then $D(G - u) \leq n - 2$, this inequality being an equality if $D(G) = n - 1$.

(iii) If $D(G) = n - 1$ and $G$ does not satisfy the property $Q$, then the girth is $g(G) = n + 1$.

**Proof.** (i) Let $D = D(G)$ be the diameter of $G \in EX(n; \{C_3, \ldots, C_n\})$, and let us take two vertices $u$, $v$ at distance $d_G(u, v) = D$. Then $D \leq n - 1$ because otherwise adding the edge $uv$ to $G$, we would obtain a graph $G'$ of order $n$ having girth $g(G') \geq n + 1$ and more edges than $G$ which contradicts the maximality of $G$.

(ii) Assume that there exists $u \in V(G)$ with degree 1. Let $G' = G - u$ and $v, w \in V(G')$ be two vertices such that $d_G(v, w) = D(G')$. We consider the new graph $G^*$ obtained by adding to $G'$ a new vertex $u^*$ and a new edge $u^*v$. Clearly, $D(G^*) = 1 + D(G')$ and $G^* \in EX(v; \{C_3, \ldots, C_n\})$, hence from (i) it follows that $D(G^*) \leq n - 2$, which implies that $D(G^*) \leq n - 2$. As $D(G^*) \geq D(G) - 1$, then $D(G') = n - 2$ if $D(G) = n - 1$.

(iii) Assume by way of contradiction that $g(G) \geq n + 2$. Since $G$ has no property $Q$, then there exist $u, v \in V(G)$ such that $d_G(u, v) = n - 1$ and there exists $x \in V(G) \setminus \{u, v\}$ for which $d_G(u, x) + d_G(x, v) \geq n$. This means that all the possible paths passing through $x$ that connect $u$ with $v$ have length at least $n$. Take any vertex $y \in N_G(x)$ and consider the graph $G'$ resulting by contracting the edge $xy$ in $G$. The girth of this new graph is $g(G') \geq n + 1$ and the diameter is $D(G') = n - 1$. So let $u', v' \in V(G')$ be such that $d_G(u', v') = n - 1$, and denote by $G^*$ the graph obtained from $G'$ by adding a new vertex $u^*$ and the edges $u'u^*$ and $u^*v'$. Clearly, $v(G^*) = v(G') + 1 = v$ and $g(G^*) = n + 1$, but $e(G^*) = e(G') + 2 = e(G) + 1$. This contradicts the extremality of $G$ and therefore $g(G) = n + 1$.

Clearly, every extremal graph $G \in EX(v; \{C_3, \ldots, C_n\})$ must be connected. As a consequence of the preceding theorem we may compute the connectivities of an extremal graph.
Recall that a graph $G$ is called connected if every pair of vertices is joined by a path; that is, if $D(G) < \infty$. If $S \subset V$ and $G-S$ is not connected, then $S$ is said to be a cut set. A (non-complete) connected graph is called $k$-connected if every cut set has cardinality at least $k$. The connectivity $\kappa(G)$ of a (non-complete) connected graph $G$ is defined as the maximum integer $k$ such that $G$ is $k$-connected. The connectivity of a complete graph $K_{n+1}$ on $\delta + 1$ vertices is defined as $\kappa(K_{\delta+1}) = \delta$. Connectivity has an edge-analogue. An edge-cut in a graph $G$ is a set $W$ of edges of $G$ such that $G-W$ is non-connected. The edge-connectivity $\lambda(G)$ of a graph $G$ is the minimum cardinality of an edge-cut of $G$. A classic result due to Whitney is that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for every graph $G$ of minimum degree $\delta(G)$. A graph is maximally connected if $\kappa(G) = \delta(G)$, and maximally edge-connected if $\lambda(G) = \delta(G)$. Sufficient conditions for a graph $G$ of minimum degree $\delta(G)$ to be maximally connected have been given in terms of its diameter and its girth. In this regard, the following result is contained in [6,7,11]:

$$\lambda(G) = \delta(G) \quad \text{if} \quad D(G) \leq 2[(g(G) - 1)/2];$$
$$\kappa(G) = \delta(G) \quad \text{if} \quad D(G) \leq 2[(g(G) - 1)/2] - 1.$$

(1)

**Corollary 1.** Every graph $G \in EX(v; \{C_3, \ldots, C_n\})$ has $\lambda(G) = \delta(G)$. Furthermore, if $D(G) \leq n - 2$, then $\kappa(G) = \delta(G)$.

**Proof.** By Theorem 2.3, we have $D(G) \leq n - 1 \leq g(G) - 2$, because $g(G) \geq n + 1$. Therefore, from (1) it follows that $\lambda(G) = \delta(G)$. Moreover, if $D(G) \leq n - 2$, then $D(G) \leq g - 3$, and hence $\kappa(G) = \delta(G)$ by (1).

The following lemmas describe the structure of a graph satisfying the property $Q$ under certain hypothesis.

**Lemma 1.** Let $G$ be a graph satisfying the property $Q$. Then each vertex $u \in V(G)$ has at most one vertex $v$ at maximum distance. Moreover, $G$ has at most two vertices of degree one in $G$, which must be at maximum distance.

**Proof.** Let us denote by $D$ the diameter of $G$ and take three vertices $u, v, w$ such that $d_G(u, v) = D$ and $d_G(u, w) = D$. Property $Q$ implies that $d_G(u, w) + d_G(w, v) = D$, which gives $d_G(w, v) = 0$ or in other words, $w = v$. Therefore $u$ has at most one vertex at maximum distance.

Now take two vertices $u, v$ such that $d_G(u, v) = D$. Suppose that there exists $u' \in V(G) \setminus \{u, v\}$ of degree one, and consider the vertex $u'_1 \in N_G(u')$. Then $d_G(u, u') + d_G(u', v) = D$ and $d_G(u, u'_1) + d_G(u'_1, v) = D$, because of the property $Q$. But taking into account that $d_G(u, u') = 1 + d_G(u, u'_1)$ and $d_G(u', v) = 1 + d_G(u'_1, v)$ we obtain that $D = D + 2$, which is a contradiction. Therefore the only vertices which may have degree one are $u$ and $v$.

**Lemma 2.** Let $G$ be a connected graph with diameter $D$, girth $g$ and satisfying the property $Q$. Let $u, v \in V(G)$ be two vertices such that $d_G(u, v) = D$, and let $I : u = u_0, u_1, \ldots, u_D = v$ be any shortest path joining $u$ with $v$. Then

(i) if $g \geq D + 2$, then $d(u_i) = 2$ for any $i \in \lfloor D/2 \rfloor, \lceil D/2 \rceil$;
(ii) if $g \geq 2D - 3$, then $d(u_i) = 2$ for any $i = 2, \ldots, D - 2$;
(iii) if $\delta(G) \geq 2$ and $g \geq \max\{2D - 3, D + 3\}$, then $G$ is a cycle of length $2D$.

**Proof.** (i) Suppose that $d(u_i) \geq 3$ for some $i \in \lfloor D/2 \rfloor, \lceil D/2 \rceil$ and let $y \in N_G(u_i) \setminus \{u_{i-1}, u_{i+1}\}$. Notice that $d_G(u, u_i) = i$ and $d_G(u_i, v) = D - i$. Then we have $d_G(u, y) \leq d_G(u, u_i) + 1 = i + 1$. If $d_G(u, y) = i + 1$, then $d_G(y, v) = D - i - 1$ because $G$ satisfies the property $Q$. As $y, u_i, u_{i+1}, \ldots, v$ is a path of length $D - i + 1$, it follows that $G$ contains a cycle of length $2D - 2i \leq 2D - 2\lfloor D/2 \rfloor \leq D + 1 < g$, which is impossible. Therefore $d_G(u, y) \leq i$, which means that there is a cycle in $G$ of length at most

$$d_G(u, y) + i + 1 \leq 2i + 1 \leq 2\lfloor D/2 \rfloor + 1 \leq 2(\lfloor g - 2/2 \rfloor + 1).$$

(2)

This is only possible if $g$ is odd in which case all the inequalities in (2) become equalities; that is, $d_G(u, y) = i = \lfloor D/2 \rfloor$, and $g = D + 2$ with $D$ odd (because $g$ must be odd). Since $G$ satisfies the property $Q$, then $d_G(y, v) = D - d_G(u, y) = D - (D + 1)/2 = (D - 1)/2$. However, $y, u_i, u_{i+1}, \ldots, v$ is a path of length $D - i + 1 = (D + 1)/2$, hence, there is a cycle in $G$ of length at most $(D - 1)/2 + (D + 1)/2 = D < g$ which is a contradiction.

(ii) Notice that if $D \leq 5$, then the result is true by (i). Thus, assume that $D \geq 5$ and suppose that $d(u_i) \geq 3$ for some $i \in \{2, \ldots, D - 2\}$ and let $y \in N_G(u_i) \setminus \{u_{i-1}, u_{i+1}\}$. As in the above case we have $d_G(u, y) \leq i + 1$. If
Fig. 1. An extremal graph for \( n = 7, v = 12, e = 13, g = 8 \) satisfying property \( Q \).

d_{G}(u, y) = i + 1 \text{ then } d_{G}(y, v) = D - i - 1 \text{ because of the property } Q. \text{ As } y, u_{i}, u_{i+1}, \ldots, v \text{ is a path of length } D - i + 1, \text{ then there is a cycle in } G \text{ of length at most } 2D - 2i \leq 2D - 4 < g \text{ which is impossible. Thus } d_{G}(u, y) \leq i \text{ which implies that there is a cycle in } G \text{ of length at most }
\begin{equation}
d_{G}(u, y) + i + 1 \leq 2i + 1 \leq 2D - 3.
\end{equation}

Since \( g \geq 2D - 3 \) we have that all the above inequalities of (3) are equalities; that is, \( g = 2D - 3, d_{G}(u, y) = i = D - 2 \). As \( G \) satisfies the property \( Q \), then \( d_{G}(y, v) = 2 \), say \( v, x, y, \) the shortest path. But \( v, x, y, u_{D-2}, u_{D-1}, v \) is a cycle of length 5, which implies that \( g = 2D - 3 \leq 5 \), hence \( D \leq 4 \) against the assumption \( D \geq 5 \).

(iii) By hypothesis \( D \leq g - 3 \leq 2\{(g - 1)/2\} - 1 \), which means by (1) that \( G \) is 2-connected because \( \delta(G) \geq 2 \). Then by Menger’s Theorem, there exist two vertex-disjoint paths between any two vertices. Thus we find a path \( P_{x} : u, x_{1}, x_{2}, \ldots, x_{r}, v \) that joins \( u \) with \( v \), verifying \( V(II) \cap V(P_{x}) = \{u, v\} \). Moreover, \( P_{x} \) must be of length \( D \) since \( G \) satisfies the property \( Q \). Hence these two paths form a cycle \( C \) of length \( 2D \). By (ii) we obtain \( d(u_{i}) = d(x_{i}) = 2 \) for \( i \in \{2, \ldots, D - 2\} \). Moreover, \( d_{G}(u_{1}, x_{D-1}) \geq D - 2 \), and \( d_{G}(x_{1}, u_{D-1}) \geq D - 2 \), since \( d_{G}(u, v) = D \). All these facts imply that \( d_{G}(u_{2}, x_{D-2}) = \min\{1 + d_{G}(u_{1}, x_{D-1}) + 1, D - 2 + d_{G}(u_{D-1}, x_{D-1})\} = D \) and analogously, \( d_{G}(x_{2}, u_{D-2}) = D \). Thus, (ii) applied to the path \( u_{2}, u_{1}, u_{0}, x_{1}, \ldots, x_{D-2} \) it follows that \( d(u_{0}) = d(x_{1}) = 2 \); and finally considering the path \( x_{2}, \ldots, x_{D-1}, v, u_{D-1}, u_{D-2} \) and the path \( x_{2}, x_{1}, u, u_{1}, \ldots, u_{D-2} \) we obtain that \( d(x_{D-1}) = d(u_{1}) = 2 \). Therefore, \( G \) must be a cycle of length \( 2D \).

An immediate consequence of Lemma 2 is that every graph satisfying the property \( Q \) has minimum degree at most two. Next we obtain a theorem in which any extremal \( \{C_{3}, \ldots, C_{n}\} \)-free graph satisfying the property \( Q \) for small values of \( n \) and with \( g \geq n + 2 \) is shown to have diameter at most \( n - 2 \) or to be \( C_{6} \) or a \( C_{6} \) joined to a pendant edge. In [1] some extremal numbers have been calculated. Among other results, it is proved that
\begin{equation}
ex(v;\{C_{3}, \ldots, C_{n}\}) = v + 1 \quad \text{provided that } [(3n + 1)/2] \leq v \leq 2n - 1.
\end{equation}

For instance, an extremal graph for \( n = 7 \) and \( v = 12 \) is obtained from a bipartite complete \( K_{2,3} \) by subdividing its edges and adding a pendant edge to a vertex of degree 3, see Fig. 1. Notice that this graph is an example of extremal graph satisfying property \( Q \) of order \( v \geq n + 2 \) for \( n = 7 \).

**Theorem 2.4.** Let \( n, v \) be two integers such that \( 3 \leq n \leq 7 \) and \( v \geq n + 2 \). Let \( G \in EX(v;\{C_{3}, \ldots, C_{n}\}) \) satisfying the property \( Q \) and suppose that \( g \geq n + 2 \). Then \( n \geq 4 \) and either the diameter \( D \leq n - 2 \), or \( n = 4 \) and the graph is \( G = C_{6} \), or \( n = 6 \) and the graph consists of a \( C_{8} \) joined to a pendant edge.

**Proof.** Let \( G \) be a graph satisfying the hypothesis of the theorem. From Lemma 2, it follows that \( \delta(G) \leq 2 \), and from Theorem 2.3 we have \( D \leq n - 1 \). Notice that if \( n = 3 \) then \( D = 2 \). Assume that \( D = n - 1 \), otherwise the theorem is true. Hence we have \( g \geq n + 2 = D + 3 \geq 2D - 3 \) because \( n \leq 7 \). Thus if \( \delta(G) = 2 \) from Lemma 2, it follows that \( G \) must be a cycle of length \( 2D = 2n - 2 \). This implies that \( n = 3 \) is impossible. For \( n = 4 \) the graph is \( C_{6} \). For \( n = 5 \) the possible graph is \( C_{7} \), which does not satisfies the property \( Q \). For \( n = 6, 7 \), the graph \( G = C_{2n-2} \) is not an extremal graph, because the graph \( G^{*} \) resulting by gluing two cycles of length \( n + 1 \) by a common path of length 3, has \( 2n - 2 \) vertices and one more edge than \( G \) (see Theorem 2.1.) Hence if \( \delta(G) = 2 \) the only possible graph is \( C_{6} \) corresponding to \( n = 4 \).

Therefore assume that \( \delta(G) = 1 \) and let \( u \) be a vertex with \( d(u) = 1 \). By Lemma 1 there is only one vertex \( v \) such that \( d_{G}(u, v) = D, \) and \( d(x) \geq 2 \) for every vertex \( x \) different from \( u \) and \( v \). Let us consider a path \( II : u = u_{0}, u_{1}, \ldots, u_{D-1}, u_{D} = v \) of length \( D \). By Lemma 2 we have \( d(u_{i}) = 2 \) for \( i = 2, \ldots, D - 2 \) because \( g \geq 2D - 3 \). As \( D = n - 1 \) and \( v \geq n + 1, \) \( d(u_{j}) \geq 3 \) for some \( j \in \{1, D - 1, D\} \). Let us see that \( d(u_{1}) \geq 3 \). Otherwise \( d(u_{1}) = 2 \), which implies that \( d(v) = 1 \), because if \( x \in \mathbb{N}_{G}(v) - u_{D-1}, \)
then \( d_G(u, x) = D - 1 + d_G(u_{D-1}, x) \geq D + 1 \), which is impossible. Therefore \( d(u_{D-1}) \geq 3 \), and for all \( y \in N_G(u_{D-1}) \setminus \{u_{D-2}, v\} \), \( d_G(u, y) = d_G(u, u_{D-1}) + 1 = D \), contradicting Lemma 1 because \( y \neq v \). Hence \( d(u_1) \geq 3 \) as claimed. Take \( x \in N_G(u_1) \setminus \{u_2, u\} \). As \( G \) satisfies the property \( Q \) we have \( d_G(x, v) = D - 2 \). Notice that \( d(v) = d(u_1) - 1 \) because if \( d(v) \neq d(u_1) - 1 \), then \( d_G(x, u_{D-1}) = D - 3 \), and this shortest path together with the path \( x, u_1, u_2, \ldots, u_{D-1} \) of length \( D - 1 \) form a cycle of length \( 2D - 4 \) which contradicts \( g \geq 2D - 3 \). Therefore we have obtained that \( G \) has a cycle of length \( 2D - 2 \), that is, \( n + 2 \leq g \leq 2D - 2 = 2n - 4 \) which is only possible if \( n = 6, 7 \), thus item (i) is valid. Observe that for \( n = 6, 7 \) the obtained graph \( G \) has \( v = (n - 3)(d(u_1) - 1) + 3 \) and \( e = (n - 2)(d(u_1) - 1) + 1 \). From Theorem 1.1(i) we may suppose that \( \Delta = \Delta(G) \leq n - 1 \), where \( \Delta \) is the maximum degree of \( G \), because by hypothesis \( g \geq n + 2 \). Then we have the following cases:

(a) For \( n = 6 \) we obtain \( (v, e) = \{(9, 9), (12, 13), (15, 17)\} \). A graph of order \( v = 12, e = 14 \) and girth \( g = 7 \) is shown in Fig. 2. Then the pair \( (12, 13) \) cannot be the parameters of an extremal graph \( G \in EX(v; \{C_3, \ldots, C_6\}) \).

Also, the graph resulting from the subdivision of the complete bipartite \( K_{3,3} \) has \( v = 15, e = 18 \), and girth \( g = 8 \), see graph \( G \) of Fig. 3. Thus, the pair \( (15, 17) \) cannot be the parameters of an extremal graph. Therefore we conclude that when \( n = 6 \) the graph \( G \) is formed by a cycle of length 8 joined to a pendant edge which has \( v = e = 9 \).

(b) For \( n = 7 \) we obtain \( (v, e) = \{(11, 11), (15, 16), (19, 21), (23, 26)\} \). The pair \( (11, 11) \) cannot be the parameter of an extremal graph, because the subdivision of \( K_{2,3} \) has \( v = 11 \) and \( e = 12 \). See also (4). We already has shown that the pair \( (15, 16) \) cannot be the parameter of an extremal graph. And \( (19, 21) \) and \( (23, 26) \) cannot be the parameters of an extremal graph, because the subdivision of \( K_{3,4} \) has \( v = 19 \) and \( e = 24 \), and the subdivision of \( K_{3,5} \) has \( v = 23 \) and \( e = 30 \). Therefore we conclude that when \( n = 7 \) there is no extremal graph of degree 1 with diameter \( n - 1 \) satisfying property \( Q \).

Combining both Theorems 2.3 and 2.4 we obtain the following result, which contains the known results for the cases \( n = 3, 4, 5 \), see [8,9,13].

**Theorem 2.5.** Let \( G \in EX(v; \{C_3, \ldots, C_n\}) \). Then \( g(G) = n + 1 \) if one of the following assertions holds:

(i) The minimum degree is \( \delta(G) \geq 3 \) and the diameter is \( D(G) = n - 1 \).

(ii) \( n = 3, 4, 5, 6, 7, v \geq n + 4 \) and the diameter is \( D(G) = n - 1 \).

(iii) \( n = 3, 4, 5 \) and \( v \geq n + 4 \).

**Proof.** (i) Since \( \delta(G) \geq 3 \), then \( G \) does not satisfy the property \( Q \) by Lemma 2. Therefore from Theorem 2.3(iii), it follows that \( g = n + 1 \).
(ii) Since \( v \geq n + 4, n \leq 7 \) and \( D(G) = n - 1 \), then from Theorem 2.4, it follows that \( g = n + 1 \) and the result is true, or \( G \) does not satisfy the property \( Q \). In this latter case we also obtain that \( g = n + 1 \) by applying again Theorem 2.3(iii).

(iii) For \( n = 3, 4 \) the result follows from item (ii), because otherwise we would have \( (n + 2)/2 \leq D \leq n - 2 \) which is impossible. Moreover, item (iii) of Theorem 1.1 implies that \( g = 6 \) if \( n = 5 \). Hence the result is valid. ■

Theorem 2.5(ii) guarantees that every graph \( G \in \mathcal{D}(v; \{C_3, \ldots, C_n\}) \) has girth \( g(G) = n + 1 \) for \( n = 6, 7 \) whenever \( v \geq n + 4 \) and \( D(G) = n - 1 \). In what follows the objective is to prove that the girth is \( g(G) = n + 1 \) for the cases \( n = 6, 7 \) assuming \( D(G) \leq n - 2 \). Let \( e \) be an edge of \( G \). As usual we will denote by \( G/e \) the graph obtained from \( G \) by contracting the edge \( e \). We begin by proving a technical lemma.

Lemma 3. Let \( G \in \mathcal{D}(v; \{C_3, \ldots, C_n\}) \) be with \( v \geq 2n - 3 \) and \( n \geq 6 \). If \( G \) contains a cycle of length \( 2n - 3 \), then the girth is \( g \leq 2n - 5 \).

Proof. Assume by way of contradiction that \( g \geq 2n - 4 \). Then the diameter is \( D \geq \lfloor g/2 \rfloor \geq n - 2 \). Let \( C_{2n-3} = u_0, u_1, \ldots, u_{n-2}, v_{n-2}, \ldots, v_1, u_0 \) be a cycle of length \( 2n - 3 \) contained in \( G \). We may assume that there is \( x \in \mathcal{N}_G(u_0) - C_{2n-3} \) because otherwise \( G = C_{2n-3} \) which is not an extremal graph by Theorem 2.1. Taking into account that in a graph with girth \( g \) the paths of length at most \( \lfloor g/2 \rfloor \) are shortest paths, we deduce that \( d_G(u_0, u_{n-2}) = d_G(u_0, v_{n-2}) = 2n - 2 \). Let us see that every cycle \( C \) in \( G \) containing both edges \( u_0x \) and \( u_{n-2}v_{n-2} \) must have length at least \( 2n - 3 \). Indeed, we have \( d_G(x, u_{n-2}) \geq n - 3 \) and \( d_G(x, v_{n-2}) \geq n - 3 \). Furthermore, if \( d_G(x, u_{n-2}) = n - 3 \) then \( d_G(x, v_{n-2}) \geq n - 2 \) because \( g \geq 2n - 4 \). Thus every cycle \( C \) in \( G \) containing both edges \( u_0x \) and \( u_{n-2}v_{n-2} \) must have length at least \( 2n - 3 \). Let us consider the graph \( G' = G/\langle u_0x, u_{n-2}v_{n-2} \rangle \), i.e., \( G' \) is the resulting graph from \( G \) by contracting the edges \( u_0x \) and \( u_{n-2}v_{n-2} \). Clearly, \( g(G') \geq 2n - 5 \geq n + 1 \), because \( n \geq 6 \). Moreover, this operation transforms the cycle \( C_{2n-3} \) of \( G \) in a cycle \( C' \) in \( G' \) of length \( 2n - 4 \), and hence \( D(G') \geq n - 2 \). Observe that \( v(G') = v(G) - 2 \) and \( e(G') = e(G) - 2 \). Let \( u', v' \) be two vertices of \( G' \) such that \( d_G(u', v') = D(G') \) and consider the graph \( G^* \) obtained by adding to \( G' \) the path \( u', x_1, x_2, v', \) where \( x_1, x_2 \notin V(G') \). Clearly, \( g(G^*) = D(G') + 3 \geq n + 1 \), \( v(G^*) = v(G') + 2 = v(G) \) and \( e(G^*) = e(G') + 3 = e(G) + 1 \). This is a contradiction because \( G \) is extremal. Therefore \( g \leq 2n - 5 \). ■

The incidence graph of an incidence structure \( \mathcal{I} \subseteq \mathcal{P} \times \mathcal{L} \) is a bipartite graph \( G = (\mathcal{P}, \mathcal{L}) \) in which two vertices \( a, b \) are adjacent if and only if they are incident, i.e., \( \langle a, b \rangle \in \mathcal{I} \) (See Chapter 5 of the book by Godsil and Royle [10]). The vertices of \( \mathcal{P} \) are called points and the vertices of \( \mathcal{L} \) are called lines. Two points being incident to a common line are called collinear and two lines being incident to a common point are called concurrent. A partial linear space is an incidence structure in which any two points are incident with at most one line. This implies that any two lines are incident with at most one point.

A generalized quadrangle is a partial linear space containing non-collinear points and non-concurrent lines satisfying that given any line \( L \) and a point \( p \) not on \( L \) there is a unique point \( p' \) on \( L \) such that \( p \) and \( p' \) are collinear.

Each of the known \((r;8)\)-cages for \( r \) odd prime is the incidence graph of a generalized quadrangle, see the survey by Wong [15] or the book [10].

Lemma 4. Let \( G \) be a graph of diameter \( 4 \) and girth \( 8 \) free of cycles of length \( 9 \). Then \( G \) is the incidence graph of a generalized quadrangle.

Proof. First let us see that \( G \) is a bipartite graph. Otherwise suppose that \( G \) contains odd cycles and let \( l \geq 11 \) be chosen as the minimum number such \( C_l \) is an odd cycle of \( G \). Then \( C_l \) has two vertices \( x \) and \( y \) joined by two disjoint paths, the shortest one of length \( \lfloor l/2 \rfloor \geq 5 \) denoted by \( C^1_l \) and the other one of length \( \lfloor l/2 \rfloor \geq 6 \) denoted by \( C^2_l \). As the diameter is \( 4 \), we have a shortest \( P_{xy} \) path of length \( d_G(x, y) \leq 4 \) between \( x \) and \( y \) different from \( C^i_l, i = 1, 2 \). Thus these two semi-cycles together with the path \( P_{xy} \) form two cycles of length at most \( \lfloor l/2 \rfloor + d_G(x, y) < l \) and \( \lfloor l/2 \rfloor + d_G(x, y) < l \), respectively. By the way of \( l \) has been chosen both numbers \( \lfloor l/2 \rfloor + d_G(x, y) \) and \( \lfloor l/2 \rfloor + d_G(x, y) \) must be even, which is a contradiction because \( \lfloor l/2 \rfloor + \lfloor l/2 \rfloor + 1 \). Therefore, \( G \) is a bipartite graph with classes \( \mathcal{P} \) and \( \mathcal{L} \). Let us call the vertices of \( \mathcal{P} \) points and the vertices of \( \mathcal{L} \) lines. Consider a line \( L \) and point \( p \) at distance 3. Since the girth is 8, there is a unique path \( L, p', L', p \) from \( L \) to \( p \). This provides the unique point \( p' \) satisfying the condition defining a generalized quadrangle. ■
Theorem 2.3

By Theorems 2.3 and 2.5 and Lemmas 3 and 4 we obtain the following theorem.

Theorem 2.5

Let \( g \) be a graph on \( \nu \) vertices. By Theorem 1.1 (ii). If \( 3 \leq \alpha, \beta \leq 5 \) we may conclude that \( g \) has no cycles of length 9, which leads to a graph \( G \) with \( g(G) = 8 \).

Moreover, if \( D(G) = 4 \) then \( g(G) = 8 \) by applying Theorem 2.5(ii). As \( D(G) = 4 \), by Lemma 4 we may conclude that \( G \) must be the incidence graph of a generalized quadrangle and thus all the vertices of one class have degree \( \alpha \geq 2 \) and all the vertices of the other class have degree \( \beta \geq 2 \) (see [10] pp. 235). Moreover, from Theorem 1.1(iii) we may suppose that \( \Delta = \Delta(G) \leq 5 \), where \( \Delta \) is the maximum degree of \( G \), hence \( 2 \leq \alpha, \beta \leq 5 \). By algebraic methods it has been proved that \( (\alpha, \beta) \in \{(3, 5), (4, 5)\} \) cannot exist (see [10] pp. 236–238). Consequently if \( 3 \leq \alpha, \beta \leq 5 \) we obtain

\[
(v, e) = \{(30, 45), (72, 135), (80, 160), (170, 425)\};
\]

or \( G \) is a subdivision of a complete bipartite graph \( K_{\alpha, \beta} \) with \( 2 \leq \alpha, \beta \leq 5 \) giving the following pairs of values:

\[
(v, e) = \{(14, 16), (15, 18), (17, 20), (19, 24), (23, 30), (24, 32), (29, 40), (35, 50)\}.
\]

Let us study all these values. Fig. 3 shows a graph \( G \) of girth 8, which is a subdivision of a 3-regular complete bipartite graph on \( \nu = 15 \) and \( e = 18 \). Contracting an edge (for instance \( ab \) in Fig. 3) we obtain a graph on \( \nu = 14 \) and \( e = 17 \) of girth \( g = 7 \), thus the pair \((v, e) = (14, 16)\) cannot be the parameters of an extremal graph. Moreover, adding to \( G \) two new vertices \( x, y \) and the path \( vxyw \) where \( d_G(v, w) = 4 \) we obtain a graph \( H \) (see Fig. 3) on \( \nu = 17 \) and \( e = 21 \) of girth \( g = 7 \), thus the pair \((v, e) = (17, 20)\) cannot be the parameters of an extremal graph. Furthermore, these graphs are extremal because in [1], it has been proved that \( ex(14; \{C_3, \ldots, C_6\}) = 17 \) and \( ex(17; \{C_3, \ldots, C_6\}) = 22 \).

Let \( \Gamma \) be the (3, 7)-cage which has \( v(\Gamma) = 24 \) and \( e(\Gamma) = 36 \), hence \( ex(24; \{C_3, \ldots, C_6\}) \geq 36 \). Then the pairs \((v, e) = (23, 30), (24, 32)\) are not valid. Moreover, removing from \( \Gamma \) a set of vertices formed by a vertex \( u \), its neighbors \( u_1, u_2, u_3 \) and one neighbor of \( u_1 \) we obtain a new graph \( \Gamma' \) on \( \nu = 19 \) and \( e = 25 \), hence the pair \((v, e) = (19, 24)\) is not valid.

Let \( \Gamma \) be the (3, 8)-cage which has \( v(\Gamma) = 30 \) and \( e(\Gamma) = 45 \), hence the pair \((29, 40)\) is not valid. Further, in [14] has been constructed a graph on 30 vertices and 47 edges having \( g = 7 \). Then the (3, 8)-cage is not an extremal graph for \( n = 6 \). Observe that the (3, 8)-cage has three vertices \( u, v, w \) such that \( d(u, v) = d(u, w) = d(v, w) = 4 \). Let \( \Gamma' \) be the graph obtained from \( \Gamma \) by adding two new vertex-disjoint paths, \( P_1 \) of length 3 connecting vertex \( u \) to vertex \( v \) and \( P_2 \) of length 4 connecting \( u \) and \( w \), see Fig. 4. In this way we obtain a graph \( \Gamma'' \) on 35 vertices, 52 edges and girth 7. So the pair \((v, e) = (35, 50)\) is not valid.

Let \( \Gamma \) be the (4, 8)-cage which has \( v(\Gamma) = 80 \) and \( e(\Gamma) = 160 \). Removing from \( \Gamma \) a set of vertices formed by a vertex \( u \), its neighbors \( u_1, u_2, u_3, u_4 \) and all the neighbors of \( u_1 \), and adding an additional edge \( u_4u_4 \) between two neighbors of \( u_4 \) we obtain a new graph \( \Gamma'' \) on \( \nu = 72 \) and \( e = 160 - 25 + 1 = 136 \) of girth 7, hence the pair \((v, e) = (72, 135)\) is not valid. 

\[\square\]
In [1], it has been proved that \( ex(15; \{C_3, C_4, C_5, C_6\}) = 18 \). For \( v = 15 \) Fig. 3 shows an extremal \( \{C_3, C_4, C_5, C_6\}\)-free graph of girth 8, which is a subdivision of a 3-regular complete bipartite graph. The problem of knowing if for \( n = 6 \) the \((4; 8)\)-cage and \((5; 8)\)-cage of order 80, 170, respectively, are extremal graphs is still opened.

As mentioned in the introduction, in [2] was proved that every graph \( G \in EX(v; \{C_3, \ldots, C_n\}) \) of minimum degree \( \delta(G) \geq 2 \) and maximum degree \( \Delta \geq \lceil (n + 1)/2 \rceil \) has girth \( g \leq n + 2 \). As a consequence it was derived in the same paper that the girth of an extremal graph \( G \in EX(v; \{C_3, \ldots, C_7\}) \) of minimum degree \( \delta(G) \geq 2 \) is \( g(G) \leq 9 \) if \( v \geq 64 \). Next, we present a theorem from which we can derive an improvement of this result.

**Theorem 2.7.** Let \( G \in EX(v; \{C_3, \ldots, C_n\}) \) be with \( v \geq 2n - 3 \) for \( n \geq 7 \). Then the girth is \( g(G) \leq 2n - 5 \).

**Proof.** We reason by contradiction assuming that \( g = g(G) \geq 2n - 4 \). Hence, from Lemma 3, it follows that \( G \) does not contain a cycle of length \( 2n - 3 \). Let \( C : u_0, u_1, \ldots, u_{g-1}, u_0 \) be a shortest cycle in \( G \), and consider the graph \( G' = G/[u_0u_1, u_1u_2] \), which has \( g(G') \geq 2n - 6 \geq n + 1 \) because \( n \geq 7 \). Hence the diameter \( D(G') \geq \lceil g(G')/2 \rceil \geq (2n - 6)/2 = n - 3 \). Assume first that \( D(G') \geq n - 2 \) and let \( u', v' \) be two vertices of \( G' \) such that \( d_{G'}(u', v') = D(G') \). Let us consider the graph \( G^* \) obtained from \( G' \) by adding two new vertices \( x_1, x_2 \) and the three edges \( u'x_1, x_1x_2, x_2v' \). We have \( g(G^*) = \min\{g(G'), D(G') + 3\} \geq n + 1 \); \( |V(G^*)| = |V(G')| + 2 = v \), and \( e(G^*) = e(G) + 1 \), which contradicts the maximality of \( G \). Consequently, \( D(G') = \lceil g(G')/2 \rceil = (2n - 6)/2 = n - 3 \), which implies that \( g(G') = 2n - 6 \), as otherwise \( g(G') = 2n - 5 \), yielding \( g(G) = 2n - 3 \), contradicting the fact that \( G \) does not contain cycles of length \( 2n - 3 \).

Therefore, every path in \( G \) of length \( n - 2 \) is a shortest one because \( g = g(G) = 2n - 4 \). Let \( x \in N_G(u_0) \setminus \{u_1, u_{g-1}\} \), and let \( \hat{u}_{012} \) denote the new vertex in \( G' \) resulting from the contraction. Clearly we have \( d_G(x, u_{n-1}) = n - 2 \) because \( x, u_0, u_{2n-5}, \ldots, u_{n-1} \) is a shortest path; and the paths passing through the edges \( u_0u_1 \) or \( u_1u_2 \) have length at least \( n \). Then \( d_{G'}(x, u_{n-1}) = 1 + d_{G'}(\hat{u}_{012}, u_{n-1}) = 1 + n - 3 = n - 2 \) which is a contradiction with the fact that \( D(G') = n - 3 \). Therefore the girth is \( g \leq 2n - 5 \). \( \blacksquare \)

**Corollary 2.** The girth of every graph \( G \in EX(v; \{C_3, C_4, C_5, C_6, C_7\}) \) with \( v \geq 11 \) is \( 8 \leq g(G) \leq 9 \).

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**References**


