Fast-Decodable Space–Time Codes for the $N$-Relay and Multiple-Access MIMO Channel

Amaro Barreal, Camilla Hollanti, *Member, IEEE*,
and Nadya Markin, *Member, IEEE*,

Abstract

In this article, the first general constructions of fast-decodable, more specifically (conditionally) $g$-group decodable, space–time block codes for the Nonorthogonal Amplify and Forward (NAF) Multiple-Input Multiple-Output (MIMO) relay channel under the half-duplex constraint are proposed. In this scenario, the source and the intermediate relays used for data amplification are allowed to employ multiple antennas for data transmission and reception. The worst-case decoding complexity of the obtained codes is reduced by up to 75%. In addition to being fast-decodable, the proposed codes achieve full-diversity and have nonvanishing determinants, which has been shown to be useful for achieving the optimal Diversity-Multiplexing Tradeoff (DMT) of the NAF channel.

Further, it is shown that the same techniques as in the cooperative scenario can be utilized to achieve fast-decodability for $K$-user MIMO Multiple-Access Channel (MAC) space–time block codes. The resulting codes in addition exhibit the conditional nonvanishing determinant property which, for its part, has been shown to be useful for achieving the optimal MAC-DMT.

Index Terms

Central simple algebras, distributed space–time block codes, fading channels, fast-decodability, lattices, multiple-access channel (MAC), multiple-input multiple-output (MIMO), relay channel.

A. Barreal and C. Hollanti are with the Department of Mathematics and Systems Analysis, Aalto University, Finland (e-mail: firstname.lastname@aalto.fi). They are financially supported by the Academy of Finland grants #276031, #282938, and #283262, and by a grant from Magnus Ehrnrooth Foundation, Finland. The support from the European Science Foundation under the COST Action IC1104 is also gratefully acknowledged.

N. Markin is with the School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore (e-mail: nmarkin@ntu.edu). She is financially supported by the Singapore National Research Foundation under Research Grant NRF-RF2009-07. Part of this research was carried out while N. Markin was visiting Aalto University, January 2012.

Parts of this paper were presented in MTNS12 [1], ISIT12 [2] and 4ICMCTA.
I. INTRODUCTION

The amount of data stored and the data traffic worldwide has reached incredible numbers. It was estimated that in 2011, $1800 \cdot 10^{18}$ bytes of data needed to be stored worldwide, and astonishing $5200 \cdot 10^{18}$ bytes of information have been created between January and November 1st 2014 [3]. The availability of such an astronomical amount of data and rapid progress in communications engineering and wireless communications explain the observed growth of mobile data traffic, which increased from $0.82 \cdot 10^{18}$ bytes at the end of 2012 to $1.5 \cdot 10^{18}$ bytes at the end of 2013, whereof 56% of the traffic was mobile video traffic. In addition, about 526 million mobile devices and connections were added globally in 2013, and the number of mobile-connected devices will exceed the number of people on earth by the end of 2014 [4].

These facts illustrate that networks will soon need to accommodate many new types of devices and be up to the enormous load while still living up to the expectations of exigent future users, which will be accessing any type of data from different devices at any time and from any corner of the world, demanding high reliability, reasonable speed, low energy consumption, etc.

With this goal in mind, a tremendous effort is being made by both academic and industrial researchers focusing on the future 5th Generation (5G) wireless systems. Although many aspects still need to be discussed, as of today it is clear that 5G will consist of an integration of different techniques rather than being a single new technology, including distributed antenna systems and massive Multiple-Input Multiple-Output (MIMO) systems [5]. Yet the various radio-access technologies to be included in 5G are only one side of the coin, as the overall performance of future networks will highly depend on the channel coding techniques employed.

As a second motivating aspect, the issue of reliably storing the worldwide available data led to considering distributed storage systems, and a plethora of research has been done in the last few years regarding optimal storage codes, thus focusing on the network layer. However, an important aspect that needs consideration is data repair and reconstruction over wireless channels to provide flexibility and user mobility, even if the storage cloud itself would be wired. This feature is related to the more general concept of wireless edge [6]–[8]. Many of the known physical layer communications techniques are however futile in this scenario due to the high decoding complexity they require. This calls for less complex coding techniques and transmission protocols, for instance introducing helping relays, as proposed in [9] or subsequent work [10].
A. Related Work and Contributions

Fast-Decodable (FD) codes are codes enjoying reduced complexity of Maximum-Likelihood (ML) decoding due to a smart inner structure allowing for parallelization in the ML search. First introduced in [11], FD codes have been the subject of much interest [12]–[18]. Fast-decodability of a code is achieved precisely when a subset of the generating matrices of the code satisfies certain mutual orthogonality conditions [18], [19], which will be made explicit later on.

Recent results [16]–[18] show that fast-decodability imposes constraints on the rate of the code on one hand, and on the algebraic parameters of the code, on the other. In particular, codes arising from division algebras (division being the criterion resulting in full-diversity) can enjoy a reduction in complexity order by a factor of no more than 4, i.e., a complexity reduction of 75%. In this paper, the best possible complexity reduction by a factor of 4 is in fact achieved.

On the other hand, the increasing interest in cooperative diversity techniques motivates the study of distributed codes. Since the introduction of the multiple-access relay channel in [20], many protocols have been considered for data exchange in this scenario, such as the amplify-and-forward [21] or compute-and-forward [22] protocol. Several distributed codes have been proposed [23], [24], and it was in [25] where the issue of the high decoding complexity of distributed codes was firstly addressed. Codes with low decoding complexity have then been constructed e.g., in [26], using Clifford algebras as the underlying structure. As far as the authors’ are aware, all attempts to construct FD distributed codes, however, assume a single antenna at the source and the relays and furthermore do not achieve the Nonvanishing Determinant (NVD) property.

The aim of this article is to illustrate how to construct codes with desirable properties for good performance and reduced decoding complexity for flexible distributed and noncooperative multiuser physical layer MIMO communications. The main contributions of this article are:

- Theorem 1, which provides a method for constructing an infinite family of FD distributed Space–Time (ST) codes having code rate 4 real symbols per channel use (rscu) and satisfying the NVD property for any number $N$ of relays. The theorem assumes a single antenna at the source and each of the relays. The resulting codes exhibit a worst-case decoding complexity $|S|^{5N}$ as opposed to $|S|^{8N}$ of a non-FD code of the same rank. The codes achieve the optimal Diversity-Multiplexing gain Tradeoff (DMT) for the relay channel when the destination has two antennas.
• Theorem 2 resulting in a construction of an infinite family of FD distributed ST codes with code rate $2\frac{r}{r_c}$ achieving the NVD property for a number $N = \frac{p-1}{2}$ of relays for any prime $p \geq 5$. Here, we assume that the number of antennas $n_s$ at the source and the number of antennas $n_r$ at each relay satisfy $n_s + n_r = 4$. One antenna suffices at the destination. The resulting codes have worst-case decoding complexity $|S|^{4N}$ or $|S|^{2N}$, respectively corresponding to a 50% or 75% reduction from the complexity $|S|^{8N}$ of a non-FD code of the same rank. According to a recent result by Berhuy et al., 75% is the best possible complexity reduction for any division algebra based code.

To the best of the authors’ knowledge, the obtained codes are the first distributed ST codes for multiple antennas which are FD and have the NVD property.

• Experimental results (Fig. 1) providing evidence that using the iterative construction (cf. Sec. III-C) or forcing the property of fast-decodability on the codes do not have a negative effect on their performance.

• Extension of the results on FD distributed ST codes to the noncooperative MIMO Multiple-Access Channel (MAC) for an arbitrary number $K$ of users, resulting in FD codes for this scenario which achieve the conditional NVD property.

• As a nontechnical contribution, the paper is written in a self-contained way, also providing a concise overview of algebraic FD ST-codes.

The paper is organized as follows: We start with a recapitulation of ST codes in Section II and how they are constructed from cyclic division algebras. We also briefly study fast-decodability and the notions of conditional $g$-group and $g$-group decodability. Further, we recall the iterative ST code construction from cyclic algebras. In Section III we propose two constructive methods to obtain FD ST codes with the NVD property for an $N$-relay channel with the half-duplex constraint, where the source and each of the relays are equipped with either one or multiple antennas. We then show in Section IV how these constructions can help obtain FD ST codes in the $K$-user MIMO-MAC. Section V concludes the paper.

II. SPACE–TIME CODES

The increasing demand for user mobility observed during the past decades has motivated a plethora of research in the area of wireless communications. The change from the well-studied case of wired communications to data transmission over wireless networks called for novel coding
techniques that were able to deal with the fading effects of wireless channels. From the start, algebraic and number theoretical tools have been proven useful for constructing well-performing codes, at first considering only devices at both ends of the channel equipped with a single antenna each. The rapid progress in communications engineering quickly led to considering multiple antennas at both ends of the channel for data rate increase. Considering this type of channels, known as multiple-input multiple-output channels, space–time coding was introduced as a promising technique for error prevention when transmitting information in the MIMO scenario, a process which can be modeled as

\[
Y_{n_d \times T} = H_{n_d \times n_s} X_{n_s \times T} + N_{n_d \times T},
\]

(1)

where the subscripts \(n_s\), \(n_d\) and \(T\) denote the number of antennas at the source, number of antennas at the destination, and the decoding delay (i.e., the number of channel uses) respectively. We have chosen the notation to be compatible with the subsequent section on distributed codes. In the above equation, \(Y\) and \(X\) are the received and transmitted codewords, \(H\) is the random complex channel matrix modeling fading, typically assumed to be Rayleigh distributed, and \(N\) is a noise matrix whose entries are complex Gaussian with zero mean. We assume the channel is quasi-static, that is, \(H\) stays fixed during the transmission of the whole ST block, and then changes independently of its previous state. The destination is assumed to have perfect channel state information (CSI-D).

In order to avoid accumulation of the received signals, forcing a discrete (e.g. a lattice) structure on the code is helpful. We will only consider linear ST block codes:

**Definition 1.** Let \(\{B_i\}_{i=1}^k\) be an independent set of fixed \(n_s \times T\) complex matrices. A linear space–time block code of rank \(k\) is a set of the form

\[
\mathcal{X} = \left\{ \sum_{i=1}^k s_i B_i \middle| s_i \in S \subset \mathbb{Z} \right\}
\]

where \(S \subset \mathbb{Z}\) is the finite signaling alphabet used.

**Definition 2.** The code rate of \(\mathcal{X}\) is defined as \(R = k/T\) real symbol per channel use (rscu), and the code is said to be full-rate (for \(n_d\) destination antennas) if \(k = 2n_d T\), that is, \(R = 2n_d\).

**Remark 1.** In literature, the code rate is commonly defined in complex symbols. However, since we connect the rate to the lattice dimension, it is more convenient to define it over the real alphabet, for not every lattice has a \(\mathbb{Z}[i]\)-basis.
We also want to point out that, here, the channel may be asymmetric \((n_s \neq n_d)\), and hence full rate is more meaningfully defined as the maximum rate that still maintains the discrete structure at the receiver and allows for linear detection methods such as sphere-decoding. Due to asymmetry, this does not necessarily imply \(R_{\text{full}} = 2n_s\), which would be the full rate in the symmetric case. If the code matrix carries more than \(k = 2n_dT\) symbols, the received signals will accumulate, and hence it is not desirable to go over the rate \(2n_d\).

Henceforth, we will refer to a linear ST block code simply as a ST code, and to its defining matrices \(B_i\) as weight matrices. Throughout the paper, \(S\) will denote the signaling alphabet accompanying the considered ST code \(X\).

**Definition 3.** A ST code \(X\) as above whose weight matrices \(\{B_i\}_{i=1}^k\) form a basis of a lattice \(\Lambda \subset \text{Mat}(n_s \times T, \mathbb{C})\) is called a ST lattice code of rank \(k = \text{rank}(\Lambda)\). A lattice, for us, is a discrete abelian subgroup of the ambient space, which in our case will be a complex matrix space. In other words, \(B_i\) above are basis matrices of \(\text{Mat}(n_s \times T, \mathbb{C})\), and thus \(k \leq 2n_sT\). In case of equality, \(\Lambda\) is called a full-rank lattice.

Let now \(X, X'\) denote distinct code matrices ranging over a code \(X\). We briefly recall the most important design criteria for ensuring a reliable performance, which are:

- **Diversity gain criterion:** \(\min_{X \neq X'} \text{rank}(X - X') = \min\{n_s, T\}\). A ST code satisfying this criterion is called a full-diversity code.

- **Coding gain criterion:** \(\Delta_{\text{min}} := \min_{X \neq X'} \det[(X - X')(X - X')^\dagger]\) should be (after normalization to unit volume, see [27]) as big as possible. If \(\inf \Delta_{\text{min}} > 0\) for the infinite code

\[
X_\infty = \left\{ \sum_{i=1}^k s_i B_i \bigg| s_i \in \mathbb{Z} \right\},
\]

i.e., the determinants do not vanish when the code size increases, the ST code is said to have the Nonvanishing Determinant (NVD) property.

Meeting these criteria can be ensured by choosing the algebraic structure from which the codes are constructed in a smart way. Indeed, Algebraic Number Theory and Class Field Theory, in particular central simple algebras and their orders, have shown to be useful for constructing good ST codes, see [28]–[31] among many others.
A. Space–Time Codes from Division Algebras

For the rest of this paper, the assumption that the number \( n_s \) of transmit antennas and the decoding delay \( T \) coincide will remain valid unless stated otherwise.

Division algebras were first considered in \([32]\) as a tool for ST coding, leading to fully-diverse ST codes. The NVD property was first achieved in \([33]\) for the Golden code, and the results were generalized to other Perfect codes in \([31]\), where the lattices used for code construction were additionally forced to be orthogonal. The orthogonality requirement was later sacrificed in \([28]\)–\([30]\) for improved performance, and the use of maximal orders was proposed to get denser lattices and higher coding gains. Finally, it was noted in \([27]\), \([30]\) that the comparison of different ST codes requires meaningful normalization. We will now revise some of these notions.

Let \( D \) be a division \( \mathbb{F} \)-algebra for a field \( \mathbb{F} \). The use of division algebras for the construction of general ST codes is motivated by the following result.

**Proposition 1.** \([32\text{ Prop. 1}]\) Let \( \phi : D \to \text{Mat}(n, \mathbb{F}) \) be a ring homomorphism and \( X \subset \phi(D) \) a finite subset. Then for any \( X, X' \in X \) with \( X \neq X' \), \( \text{rank}(X - X') = n \).

Full-diversity can thus be guaranteed by choosing a division algebra as the underlying algebraic structure, while imposing a further algebraic restriction will also ensure the NVD property. Among different types of division algebras, Cyclic Division Algebras (CDAs) arising from number field extensions have been proposed in \([32]\) and heavily used for ST coding ever since.

**Definition 4.** Let \( K/\mathbb{F} \) be a cyclic Galois extension of degree \( n \) of number fields, and fix a generator \( \sigma \) of its cyclic Galois group \( \Gamma(K/\mathbb{F}) \). The triple

\[
\mathcal{C} = (K/\mathbb{F}, \sigma, \gamma) := \bigoplus_{i=0}^{n-1} u^iK,
\]

where \( u^n = \gamma \in K^\times \) and \( \kappa u = u\sigma(\kappa) \) for all \( \kappa \in K \), is called a cyclic algebra of degree \( n \). The algebra \( \mathcal{C} \) is division, if every nonzero element of \( \mathcal{C} \) is invertible.

**Remark 2.** It is useful to mention the case \( n = 2 \) separately. In this case, \( K = \mathbb{F}(\sqrt{a}) \) for some square-free \( a \in \mathbb{Z} \), and the cyclic algebra \( \mathcal{C} = (K/\mathbb{F}, \sigma, \gamma) \) is known as a quaternion algebra. It can equivalently be denoted as \( \mathcal{C} = (a, \gamma)_F \cong K \oplus jK \cong F \oplus iF \oplus jF \oplus kF \), where the basis elements satisfy \( i^2 = a, j^2 = \gamma, ij = -ji = k \). In the case of \( a = \gamma = -1 \) this gives rise to the famous Hamiltonian quaternions and the well-known Alamouti code.
The following lemma is a straightforward generalization of an original result due to A. Albert involving the field norm $\text{Nm}_{K/F} (\cdot)$ of $K$ over $F$. The lemma gives us a simpler way to determine whether a cyclic algebra $C$ is division in contrast to the original result involving all the powers $\gamma^i$, $i = 1, 2, \ldots, n - 1$. For instance, for $n = 8$, the original result would involve seven different powers of $\gamma$, whereas the lemma below only calls for $\gamma^4$.

**Lemma 1.** [34, Prop. 2.4.5] Let $C = (K/F, \sigma, \gamma)$ be a cyclic algebra of degree $n$. If $\gamma$ is chosen such that $\gamma^{n/p} \notin \text{Nm}_{K/F} (K^\times)$ for all primes $p | n$, then $C$ is a division algebra.

The advantage of using CDAs for ST coding is that a lattice structure (and thus discrete structure) is easily ensured by restricting the choice of elements to a ring within the CDA known as an order.

**Definition 5.** Let $K$ be a number field, $F \subseteq K$ a subfield with ring of integers $O_F$. Let further $C$ be a $K$-central algebra. An $O_F$-order $O$ in $C$ is a subring of $C$ that shares the same identity element as $C$ and is a full $O_F$-lattice in $C$, i.e., $O \cdot F = C$. Further, maximality is defined with respect to inclusion.

Moreover, by using the left-regular representation of the CDA we can easily construct a ST code (cf. Def. [1]). To that end, let $O$ be an order within a CDA $C = (K/F, \sigma, \gamma)$ of degree $n$ and $\alpha = \sum_{i=0}^{n-1} \alpha_i u^i \in O$. The representation of $\alpha$ over the maximal subfield $K$ is given by

$$\lambda : \alpha \mapsto \begin{bmatrix} \alpha_0 & \gamma \sigma(\alpha_{n-1}) & \cdots & \gamma \sigma^{n-1}(\alpha_1) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \sigma(\alpha_{n-2}) & \cdots & \sigma^{n-1}(\alpha_0) \end{bmatrix}.$$ (2)

The map $\lambda$ is an isomorphism, allowing us to identify an element $\alpha$ with its matrix representation.

Let $k$ be the absolute degree of $C$ over $\mathbb{Q}$ and $\{B_i\}_{i=1}^k$ a matrix basis of $O$ over $\mathbb{Q}$. A ST code constructed from the order $O$ for a fixed signaling alphabet $S \subseteq \mathbb{Z}$ is of the form

$$\mathcal{X} = \left\{ \sum_{i=1}^k s_i B_i \mid s_i \in S \right\}.$$ 

By choosing $C$ to be division, and since $\lambda : O \to \text{Mat}(n, K)$ is a ring homomorphism, the code $\mathcal{X} \subseteq \lambda(O)$ is a finite subset of a lattice, and by Prop. [1] is fully diverse. Moreover, the restriction of the elements to an $O_F$-order $O$ ensures that for any matrix $\lambda(a)$, $\det(\lambda(a)) \in O_F$, thus guaranteeing the NVD property for $F = \mathbb{Q}$ or $F$ imaginary quadratic (cf. [30]).
B. Fast-Decodable Space–Time Codes

The use of multiple antennas for data transmission and reception across wireless channels has its many advantages, but also increases the complexity of the coding schemes; especially when it comes to the decoding of the received signal. The considered linear ST block codes as in Def. 1 allow for a decoding technique known as Maximum-Likelihood (ML) decoding. Given a ST code $\mathcal{X}$ and a transmission model (1), and recalling that the noise involved has zero mean, ML decoding amounts to finding the codeword $X \in \mathcal{X}$ that minimizes the expression

$$\delta(X) := ||Y - HX||_F^2,$$  \hspace{1cm} (3)

where $||\cdot||_F$ denotes the Frobenius norm.

**Definition 6.** The ML decoding complexity of a rank-$k$ ST code $\mathcal{X}$ is defined as the minimum number of values that have to be computed for finding the solution to (3). It is upper bounded by the worst-case (ML) complexity $|S|^k$ corresponding to an exhaustive search, where $S$ is the real signaling alphabet used. Note further that the corresponding sphere-decoding complexity is also upper bounded by $|S|^k$, though on average sphere-decoding is much faster.

A ST code $\mathcal{X}$ is said to be Fast-Decodable (FD) if its worst-case ML decoding complexity is of the form $|S|^{k'}$ for $k' < k - 2$.

**Remark 3.** It is not sufficient to demand $k' < k$, as the ML decoding complexity of any code can be reduced to $|S|^{k-2}$ due to Gram-Schmidt orthogonalization.

**Remark 4.** In the rest of this article, when we say that a code has complexity $|S|^{k'}$, we mean the aforementioned worst-case complexity. By using, e.g., sphere-decoding the complexity can be of course reduced, both for FD and non-FD codes, but the search dimension will be determined by $|S|^{k'}$ as we need to jointly decode $k'$ symbols. Hence $|S|^{k'}$ gives us a way to compare complexities independently of the decoding method we would finally choose to use.

The property of fast-decodability can best be examined by considering the metric $\delta(\cdot)$ (3). Let $H$ be the channel matrix and $\text{vec}(\cdot) : \text{Mat}(m \times n, \mathbb{C}) \rightarrow \mathbb{R}^{2mn}$ be the map which stacks the columns of a matrix followed by separating the real and imaginary parts of the obtained vector components.
Next, we define \( B := \left[ \text{vec}(HB_1) \ldots \text{vec}(HB_k) \right] \in \text{Mat}(2Tn_d \times k, \mathbb{R}) \), so that every received codeword \( HX \) can be represented as \( Bs \) for a coefficient vector \( s = (s_1, \ldots, s_k)^T \in S^k \). The problem of decoding now reads

\[
\arg \min_{X \in \mathcal{X}} \{ ||Y - HX||^2_F \} \sim \arg \min_{s \in S^k} \{ ||\text{vec}(Y) - Bs||^2_E \},
\]

where \( ||\cdot||_E \) denotes the Euclidean norm, and a real sphere-decoder can be employed to perform the latter search \[11\].

To further simplify this expression, let \( B = QR \) be the QR-decomposition of \( B \), where

\[
Q = \left[ q_1 \ldots q_k \right] \in \text{Mat}(2n_s n_d \times k, \mathbb{R}) \text{ is an orthonormal matrix, i.e., } Q^T Q = I,
\]

and \( R \in \text{Mat}(k, \mathbb{R}) \) is upper triangular,

\[
R = \begin{bmatrix}
||r_1|| q_{12} & \cdots & q_{1k} \\
\vdots & \ddots & \vdots \\
0 & \cdots & ||r_k||
\end{bmatrix},
\]

with \( r_1 = b_1, q_1 = \frac{r_1}{||r_1||} \), and for \( i = 2, \ldots, k, q_i = \frac{r_i}{||r_i||}, \) where \( r_i = b_i - \sum_{j=1}^{i-1} \langle q_j, b_i \rangle q_j \).

This decomposition further simplifies the decoding problem to finding \( s \in S^k \) that minimizes

\[
||\text{vec}(Y) - QRs||^2_E = ||Q^T \text{vec}(Y) - Rs||^2_E.
\]

The introduction of this \( R \)-matrix in the decoding process makes it possible to directly read out the decoding complexity of a particular code. For this purpose we will make use of a specific quadratic form, which has been introduced in \[13\] as a tool for studying fast-decodability independently of the channel matrix \( H \).

**Definition 7.** The Hurwitz-Radon Quadratic Form (HRQF) is the map

\[
Q : \mathcal{X} \rightarrow \mathbb{R}; \quad X \mapsto \sum_{1 \leq i \leq j \leq k} s_is_jm_{ij},
\]

where \( m_{ij} := ||B_iB_j^\dagger + B_jB_i^\dagger||^2_F \) and \( s_i \in S \). Associating the matrix \( M = (m_{ij}) \) with \( Q \), the HRQF can be written as \( Q(X) = sMs^T \), where \( s = [s_1 \ldots s_k] \).

**Remark 5.** Note that \( m_{ij} = ||B_iB_j^\dagger + B_jB_i^\dagger||^2_F = 0 \) if and only if \( B_iB_j^\dagger + B_jB_i^\dagger = 0 \), that is, \( B_i \) and \( B_j \) are mutually orthogonal. Moreover, premultiplication of the weight matrices by a channel matrix \( H \) does not affect the zero structure of \( M \), whereas it does affect that of \( R \). Nevertheless, the zero structure of the \( R \) and \( M \) matrices are conveniently related to each other (see Prop. \[2\] Cor. \[7\] or \[13\]).
With the help of the HRQF we can further specify two important families of FD codes, for which an explicit decoding complexity expression can be given.

1) Conditional $g$-group decodability:

**Definition 8.** $X$ is conditionally $g$-Group Decodable ($g$-GD) if there exists a partition of \{1, ..., $k$\} into $g + 1$ nonempty subsets $\Gamma_1, \ldots, \Gamma_g, \Gamma^X$, $g \geq 2$, such that $B_iB_j^\dagger + B_jB_i^\dagger = 0$ for $i \in \Gamma_u, j \in \Gamma_v$ and $1 \leq u < v \leq g$. Equivalently, $\langle q_i, b_j \rangle = 0$ for $i < j$, $i \in \Gamma_u, j \in \Gamma_v, u \neq v$.

**Proposition 2.** [17, Thm. 2] There exists an ordering of the weight matrices such that the $R$-matrix obtained for conditionally $g$-GD ST codes has the particular form

\[
R = \begin{bmatrix}
D_1 & N_1' \\
\vdots & \ddots \\
D_g & N_g'
\end{bmatrix}
\]

where the empty spaces are filled with zeros, $D$ is a $(k - |\Gamma^X|) \times (k - |\Gamma^X|)$ block-diagonal matrix whose blocks are of size $|\Gamma_i| \times |\Gamma_i|$, $1 \leq i \leq g$, $N$ is a square upper-triangular $|\Gamma^X| \times |\Gamma^X|$ matrix, and $N'$ is a rectangular matrix.

**Remark 6.** We intentionally refer to [17] for this result although the authors do not use the term conditional $g$-group decodable. However, Def. 8 above coincides with the codes considered in the referred source. A similar observation was already made in [14].

2) $g$-group decodability:

**Definition 9.** $X$ is $g$-group decodable if there exists a partition of \{1, ..., $k$\} into $g$ nonempty subsets $\Gamma_1, \ldots, \Gamma_g$ such that $B_iB_j^\dagger + B_jB_i^\dagger = 0$ for $i \in \Gamma_u, j \in \Gamma_v, u \neq v$.

**Remark 7.** A $g$-GD ST code $X$ becomes conditionally $g$-GD by conditioning $|\Gamma^X|$ variables. Conversely, $X$ is $g$-GD if it is conditionally $g$-GD and its corresponding subset $\Gamma^X$ is empty.

From this remark and Prop. 2 we immediately get the following corollary.

**Corollary 1.** There exists an ordering of the weight matrices such that the $R$-matrix obtained for $g$-GD ST codes is of the form

\[
R = \begin{bmatrix}
D_1 \\
\vdots \\
D_g
\end{bmatrix}
\]

where $D_i$ is a $|\Gamma_i| \times |\Gamma_i|$ upper-triangular matrix, $1 \leq i \leq g$.

\(^1\)An algorithm for finding the optimal ordering is given in [13].
By how fast-decodability was introduced in Def. 6, both conditionally $g$-GD and $g$-GD ST codes are FD. The latter definitions, however, allow to deduce the exact decoding complexity reduction with the help of the HRQF. For this purpose, note that having a (conditionally) $g$-GD ST code, decoding the last $|\Gamma^X| \geq 0$ variables gives a complexity of $|S|^{|\Gamma^X|}$. The remaining variables can be decoded in $g$ parallel steps, where step $i$ involves $|\Gamma_i|$ variables, $1 \leq i \leq g$. This observation leads to the following result.

**Proposition 3.** Given a (conditionally) $g$-GD ST code $\mathcal{X}$ with possibly empty subset $\Gamma^X$, the decoding complexity of $\mathcal{X}$ is

$$|S|^{|\Gamma^X|} + \max_{1 \leq i \leq g} |\Gamma_i|.$$

**C. Iterative Construction from Cyclic Algebras**

The construction of ST codes exhibiting desirable properties requires a well-chosen underlying algebraic framework. Constructing codes for a larger number of antennas means dealing with higher degree field extensions and algebras, which are harder to handle. Recently, [15] proposed an iterative ST code construction that, starting with an $n \times n$ ST code, results in a new $2n \times 2n$ ST code with the same rate and double rank. The advantage of this construction is that when applied carefully, the resulting codes inherit good properties from the original ST codes. Next, we will briefly recall this construction.

**Definition 10.** Let $F$ be a finite Galois extension of $\mathbb{Q}$ and $C = (K/F, \sigma, \gamma)$ be a CDA of degree $n$. Fix $\theta \in C$ and $\tau \in \text{Aut}_\mathbb{Q}(K)$.

(a) The iterative code construction will be performed using the function

$$\alpha_{r,\theta} : \text{Mat}(n, K) \times \text{Mat}(n, K) \to \text{Mat}(2n, K)$$

$$(X, Y) \mapsto \begin{bmatrix} X^{\theta \tau(Y)} \\ Y^{\tau(X)} \end{bmatrix}.$$

(b) If $\theta = \zeta \theta'$ for $\zeta \in \{\pm 1, \pm i\}$ and $\theta' \in \mathbb{R}_{>0}$ is totally real or imaginary, define the alike function

$$\hat{\alpha}_{r,\theta} : \text{Mat}(n, K) \times \text{Mat}(n, K) \to \text{Mat}(2n, K)$$

$$(X, Y) \mapsto \begin{bmatrix} X^{\sqrt{\theta'} Y} \\ \sqrt{\theta'} Y^{\tau(X)} \end{bmatrix}.$$
Before giving a series of criteria for $\theta$ and $\tau$ to be met, we briefly make use of the above functions to construct ST codes. Suppose that $C$ gives rise to a ST code $\mathcal{X}$ of rank $k$ defined via matrices $\{B_i\}_{i=1}^k$. Then, the matrices $\{\alpha_{\tau,\theta}(B_i,0), \alpha_{\tau,\theta}(0,B_i)\}_{i=1}^k$ define a rank-2k ST code $X_k = \left\{ \sum_{i=1}^k [s_i \alpha_{\tau,\theta}(B_i,0) + s_{k+i} \alpha_{\tau,\theta}(0,B_i)] \middle| s_i \in S \subset \mathbb{Z} \right\}$.

Proposition 4. [15, Thm. 1, Thm. 2] Let $C = (K/F, \sigma, \gamma)$ be a CDA giving rise to a ST code $X$ defined by the matrices $\{B_i\}_{i=1}^k$. Assume that $\tau \in \text{Aut}_Q(K)$ commutes with $\sigma$ and complex conjugation, $\tau(\gamma) = \gamma$, $\tau^2 = \text{id}$, and fix $\theta \in F^{(\tau)}$, where $F^{(\tau)}$ is the subfield of $F$ fixed by $\tau$. Identifying an element of $C$ with its matrix representation, we have:

(i) The image $I = \alpha_{\tau,\theta}(C,C)$ is an algebra and is division if and only if $\theta \neq z\tau(z)$ for all $z \in C$. Moreover, $\det(\alpha_{\tau,\theta}(x,y)) \in F^{(\tau)}$ for any $\alpha_{\tau,\theta}(x,y) \in I$.

(ii) If in addition $\theta = z\theta'$ is totally real or totally imaginary, the image $\tilde{I} = \tilde{\alpha}_\theta(C,C)$ retains both the full-diversity and NVD property. Let $\tilde{X}_k$ be generated by the weight matrices $\{\tilde{\alpha}_{\tau,\theta}(B_i,0), \tilde{\alpha}_{\tau,\theta}(0,B_i)\}_{i=1}^k$. Then, if for some $i, j$, $B_i B_j^\dagger + B_j B_i^\dagger = 0$, we have

$$\tilde{\alpha}_{\tau,\theta}(B_i,0)\tilde{\alpha}_{\tau,\theta}(B_j,0)^\dagger + \tilde{\alpha}_{\tau,\theta}(B_j,0)\tilde{\alpha}_{\tau,\theta}(B_i,0)^\dagger = 0,$$

$$\tilde{\alpha}_{\tau,\theta}(0,B_i)\tilde{\alpha}_{\tau,\theta}(0,B_j)^\dagger + \tilde{\alpha}_{\tau,\theta}(0,B_j)\tilde{\alpha}_{\tau,\theta}(0,B_i)^\dagger = 0.$$

III. Fast-Decodable Space–Time Codes for Distributed Communications

In the following scenario, we consider the situation of a single user (e.g., a base station) communicating to a single destination over a wireless network, being assisted by $N$ other users acting as intermediate relays, in principle either amplifying and forwarding or decoding and forwarding the received signal. Henceforth we will assume the former and, more specifically, the Nonorthogonal Amplify-and-Forward (NAF) scheme introduced in [35], as it is known to outperform the previously proposed AF schemes. Here, in contrast to the orthogonal schemes, the source is allowed to transmit at the same time as the relays. In addition, we assume the half-duplex constraint, i.e., the relays can only either receive or transmit a signal at a given time instance.

A. MIMO Relay Channel

Denote by $n_s$, $n_d$ and $n_r$ the number of antennas at the source, destination, and each of the $N$ relays, respectively. A superframe consisting of $N$ consecutive cooperation frames of length
Each composed of two partitions of \( T/2 \) symbols, is defined, and all channels are assumed to be static during the transmission of the entire superframe \([36]\).

The matrices \( F, H_i \) and \( G_i, 1 \leq i \leq N \) denote the Rayleigh distributed channels from the source to the destination, relays, and from the relays to the destination, respectively.

Below, we present the relay channel model in detail for the case \( n_r \leq n_s \), and refer to \([36]\) for the case when \( n_r > n_s \). The transmission process can be modeled as

\[
Y_{i,1} = \gamma_{i,1}FX_{i,1} + V_{i,1}, \quad i = 1, \ldots, N
\]

\[
Y_{i,2} = \gamma_{i,2}FX_{i,2} + V_{i,2} + \gamma_{R_i}G_iB_i(\gamma'_{R_i}H_iX_{i,1} + W_i), \quad i = 1, \ldots, N
\]

where \( Y_{i,j} \) and \( X_{i,j} \) are the received and transmitted matrices, \( V_{i,j} \) and \( W_i \) represent additive white Gaussian noise, the matrices \( U_i \) are needed for normalization and \( \gamma_{i,j}, \gamma_{R_i}, \gamma'_{R_i} \) are Signal-to-Noise Ratio (SNR) related scalars. For more details, the reader may consult \([36]\).

From the destination’s point of view, the above transmission model can be viewed as a virtual single-user MIMO channel modeled as

\[
Y_{n_d \times n} = H_{n_d \times n}X_{n \times n} + V_{n_d \times n},
\]

where \( n = N(n_s + n_r) \), \( X \) and \( Y \) are the (overall) transmitted and received signals, and the structure of the channel matrix \( H \) is determined by the different relay paths. Thus this virtual antenna array created by allowing cooperation can be used to exploit spatial diversity even when a local antenna array may not be available. We have also made the assumption that \( T = n \).

It was shown in \([36]\) that given a rate-\( 4n_s \) block-diagonal ST code \( \mathcal{X} \), where \( n_s = n_r \) and each \( X \in \mathcal{X} \) takes the form

\[
X = \text{diag}\{\Xi_i\}_{i=1}^N = \begin{bmatrix} \Xi_1 & \cdots & \Xi_N \end{bmatrix}
\]

with \( \Xi_i \in \text{Mat}(2n_s, \mathbb{C}) \) and such that \( \mathcal{X} \) is NVD, the equivalent code

\[
C = \begin{bmatrix} C_1 & \cdots & C_{N-1} \end{bmatrix}, \quad \text{where} \quad C_i = \begin{bmatrix} \Xi_i[n_s:1:2n_s] & \Xi_i[n_s+1:2n_s,1:2n_s] \end{bmatrix}
\]
achieves the optimal DMT for the channel, transmitting $C_i$ in the $i^{th}$ cooperation frame. It would thus be desirable to have ST codes of this block-diagonal form for which we have:

1) Full rate $2n_d$, that is, the number of independent real symbols (e.g., Pulse Amplitude Modulation (PAM)) per codeword equals $2n_d N(n_s + n_r)$.
2) Full rank $N(n_s + n_r)$.
3) NVD.

**Remark 8.** The last condition can be safely abandoned at the low SNR regime without compromising the performance. For very low SNR, even relaxing on the full-diversity condition does not have adverse effect, since the determinant criterion is asymptotic in nature. However, to guarantee robustness regardless of the SNR, we will design our codes so that they will possess the NVD property.

In what follows, we will also see that our codes are of rate $R = 4$ in the Single-Input Multiple-Output (SIMO) case, i.e., they are full-rate in the case $n_d = 2$. In the MIMO case, the codes will have $R = 2$, which is full-rate for $n_d = 1$. In exchange to relatively low rate, one or two antennas will suffice for linear detection such as sphere-decoding.

**B. Fast-Decodable Distributed Space–Time Codes**

We will now introduce flexible methods for constructing FD ST codes for the MIMO-NAF channel for any number of relays. The following function is crucial for the proposed constructions.

**Definition 11.** Consider an $N$-relay MIMO-NAF channel. Given a ST code $\mathcal{X} \subset \text{Mat}(n_s + n_r, \mathbb{C})$ and a suitable function $\eta$ of order $N$ (i.e., $\eta^N(X) = X$), to be specified, define the function

$$
\Psi_{\eta,N} : \mathcal{X} \rightarrow \text{Mat}(nN, \mathbb{C})
$$

$$
X \mapsto \text{diag}\{\eta^i(X)\}_{i=0}^{N-1} = \begin{bmatrix}
X \\
\vdots \\
\eta^{N-1}(X)
\end{bmatrix}.
$$

$^2$DMT is an asymptotic performance measure, indicating a tradeoff between the transmission rate and decoding error probability as a function of SNR. Here, our primary goal is fast-decodability with NVD, so instead of giving a detailed definition of the DMT, we refer the reader to [45], [36].
Remark 9. In the following and for the remaining of this section, we will use techniques from Algebraic Number Theory to prove certain properties about the structures used for code construction. An interested reader is referred to [37] as a good source to review these techniques.

1) SIMO: Assume a single source cooperative communications scenario where the transmitter and each of the \( N \) intermediate relays are equipped with a single antenna, \( n_s = n_r = 1 \). Assume further a single destination with \( n_d \geq 2 \) antennas. Consider the following tower of extensions, where \( \xi \) is taken to be totally real and \( m \in \mathbb{Z}_{>1} \) is square-free.

\[
\mathcal{C} = (a, \gamma)_{\mathbb{K}} \cong (L/K, \sigma : \sqrt{a} \mapsto -\sqrt{a}, \gamma)
\]

\[
\begin{array}{c}
\mathbb{L} = \mathbb{K}(\sqrt{a}) \\
\mathbb{L} = \mathbb{K}(\sqrt{a}) \\
\mathbb{K} = \mathbb{F}(\xi) \\
\mathbb{F} = \mathbb{Q}(\sqrt{-m}) \\
\mathbb{Q}(\sqrt{a})
\end{array}
\]

Assume that \( \mathcal{C} \) is division (see Lemma [1] and recall the notation from (2)). Let \( \sigma : \sqrt{a} \mapsto -\sqrt{a} \) be the generator of \( \Gamma(L/K) \), and fix a generator \( \eta \) of \( \Gamma(K/F) \).

To have balanced energy and good decodability, it is necessary to slightly modify the matrix representation of the elements in \( C \), thus for \( c, d \in O_L, O \subset C \) an order, instead of representing \( x = c + \sqrt{\gamma}d \in O \) by its left-regular representation \( \lambda(x) \), we define the following similar and commonly used function that maintains the original determinant,

\[
\tilde{\lambda} : x \mapsto \left[ \begin{array}{c} c \\ \sqrt{\gamma}d \\ -\sqrt{\gamma}\sigma(d) \\ \sigma(c) \end{array} \right].
\] (4)

Theorem 1. Arising from the algebraic setup above with \( a < 0, \gamma < 0 \), define

\[
\mathcal{X} = \{ \Psi_{\eta,N}(X) \}_{X \in \tilde{\lambda}(O)} = \left\{ \text{diag}\{\eta^j(X)\}_{j=0}^{N-1} \mid X \in \tilde{\lambda}(O) \right\}.
\]

The code \( \mathcal{X} \) is of rank \( 8N \), rate \( R = 4 \) rscu and has the NVD property. Moreover, it is conditionally 4-GD, and its decoding complexity can be reduced from \( |S|^{8N} \) to \( |S|^{5N} \), where \( S \) is the PAM constellation used. The code is full-rate if \( n_d = 2 \).

Proof. Let \( \{ \beta_1, \ldots, \beta_N \} \) be a basis of \( \mathbb{K} \) over \( \mathbb{F} \) and consider the following \( \mathbb{K} \)-basis of \( C \):

\[
\left\{ \Gamma_{1,1} = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right], \Gamma_{2,1} = \left[ \begin{array}{c} \sqrt{\gamma} \\ 0 \end{array} \right], \Gamma_{3,1} = \left[ \begin{array}{c} 0 \\ \sqrt{\gamma} \sqrt{\alpha} \end{array} \right], \Gamma_{4,1} = \left[ \begin{array}{c} 0 \\ -\sqrt{\gamma} \sqrt{\alpha} \end{array} \right] \right\}.
\]
This basis can first be extended to an $F$-basis of $C$ as $\{\Gamma_{i,j} = \Gamma_{i,1}\beta_j\}_{1 \leq i \leq 4}$, and further to a $Q$-basis of $C$ by complementing with $\Gamma_{i,j} = \sqrt{-m}\Gamma_{i-4,j}$ for $i = 5, \ldots, 8$, $j = 1, \ldots, N$. From this set we get a $\mathbb{Z}$-basis for the code $\mathcal{X}$ as

$$B = \{\Psi_{\eta,N}(\Gamma_{i,j})\}_{1 \leq i \leq 8, 1 \leq j \leq 4},$$

which is of length $8N = \text{rank}(\mathcal{X})$. Thus by Def. 2, the code has rate $R = 4$.

Let now $\Psi_{\eta,N}(X) \in \mathcal{X}$ be a codeword, where $\eta$ denotes a generator of $\Gamma(K/F)$. As assumed, the coefficients of $X$ are taken from the ring of integers $\mathcal{O}_L$ of $L$, thus $\det(X) \in \mathcal{O}_K$ and hence

$$\det(\Psi_{\eta,N}(X)) = \prod_{i=0}^{N-1} \det(\eta^i(X)) = \prod_{i=0}^{N-1} \eta^i(\det(X)) = N\text{m}_K/F(\det(X)) \in \mathcal{O}_F.$$

Since $F$ is imaginary quadratic, it follows that $\det(\Psi_{\eta,N}(X)) \geq 1$, giving the NVD property.

For deriving the decoding complexity, a direct computation shows that $\Gamma_{i,j}\Gamma_{u,v}^\dagger + \Gamma_{u,v}\Gamma_{i,j}^\dagger = 0$ for $1 \leq i,u \leq 4$, $i \neq u$ and $1 \leq j,v \leq N$, hence $\Psi_{\eta,N}(\Gamma_{i,j})\Psi_{\eta,N}(\Gamma_{u,v})^\dagger + \Psi_{\eta,N}(\Gamma_{u,v})\Psi_{\eta,N}(\Gamma_{i,j})^\dagger = 0$. Consequently, the matrix $M = (m_{i,j})$ associated with the HRQF is of the form

$$M = \begin{bmatrix}
\star & 0 & 0 & 0 & \star & \star & \star & \star \\
0 & 0 & 0 & \star & \star & \star & \star & \star \\
0 & 0 & 0 & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star & \star & \star \\
\end{bmatrix},$$

where each of the entries is an $N \times N$ matrix. By Def. 8, $\mathcal{X}$ is conditionally 4-GD and exhibits decoding complexity $|S|^{4N+N} = |S|^{5N}$.

**Remark 10.** ST codes constructed using the method from Thm. 7 will achieve the optimal DMT of the channel, according to [36, Thm. 4].

**Example 1.** For $N = 2$ relays and $\xi = \sqrt{31}$, consider the following tower of extensions.

$$\mathcal{C} = (-5, -2)_K \cong (L/K, \sigma : \sqrt{-5} \mapsto -\sqrt{-5}, -2)$$

$$\begin{align*}
L &= K(\sqrt{-5}) \\
K &= \mathbb{Q}(t, \xi) \\
F &= \mathbb{Q}(t)
\end{align*}$$
To see that \( C \) is division, note that the polynomials \( x^2 + 1 = 0 \) and \( x^2 - 31 = 0 \) split completely mod 5. This implies that the ideal \( I = (-5)\mathcal{O}_K \) splits completely into \( I = p_1 p_2 p_3 p_4 \) distinct prime ideals \( p_i \) of \( \mathcal{O}_K \). Let \( p \) be any of these prime ideals. As \( |\mathcal{O}_K/p| = 5 \) we have \( \mathcal{O}_K/p \cong \mathbb{F}_5 \). Note that \(-2\) is not a square in \( \mathbb{F}_5 \), thus not a square mod \( p \). By [38, Thm. 7.1], \( C \) is division.

Let \( x = c + \sqrt{-2}d \) with \( c, d \in \mathcal{O}_L \) and for \( \sigma \) as above, \( X = \bar{\lambda}(x) = \begin{bmatrix} c & \sqrt{2}d \\ \sqrt{-\sigma(d)} & \sigma(c) \end{bmatrix} \).

For \( \langle \eta \rangle = \Gamma(K/F) \), define the 2-relay code
\[
\mathcal{X} = \{ \Psi_{\eta,2}(X) \}_{X \in \bar{\lambda}(\mathcal{O}_L)} = \left\{ \text{diag} \{ \eta^i(X) \}_{i=0} \right\} X \in \bar{\lambda}(\mathcal{O}_L) \}
\]
The resulting code is a fully diverse NVD code of rank 16, which is conditionally 4-GD having decoding complexity \(|S|^{10}\) in contrast to \(|S|^{16}\).

**Example 2.** Let \( N = 3 \) be the number of relays and \( \zeta_7 \) the 7th root of unity. For \( \xi = \zeta_7 + \zeta_7^{-1} \) totally real, consider the following tower of extensions.

\[
C = (-11, -1)K \cong (L/K, \sigma : \sqrt{-11} \mapsto -\sqrt{-11}, -1)
\]

\[
L = \mathbb{Q}(\sqrt{-11}, \zeta_7)
\]

\[
K = \mathbb{Q}(\zeta_7)
\]

\[
\mathbb{Q}(\sqrt{-7}) F = \mathbb{Q}(\sqrt{-7})
\]

Note that \( K = F(\xi) \). The quaternion algebra \( C \) is division. To see this, note that the ideal \((-11)\mathcal{O}_K \) splits as \( I = p_1 p_2 \) for distinct prime ideals \( p_i \) of \( \mathcal{O}_K \). Let \( p \) be either of these prime ideals. We know that \( |\mathcal{O}_K/p| = 11^3 \), hence \( \mathcal{O}_K/p \cong \mathbb{F}_{11^3} \). Next we establish that \(-1\) is not a square in \( \mathbb{F}_{11^3} \), which follows from the fact that \( 4 \nmid |\mathbb{F}_{11^3}^\times| \). Hence, \(-1\) is not a square mod \( p \).

By [38, Thm. 7.1], \( C \) is division.

Note that \( C \cong (-1, -11)_K \) and let \( x = c + \sqrt{-11}d \) with \( c, d \in \mathbb{Z}[i, \zeta_7] \), \( \sigma \) as above and \( \langle \eta : \zeta_7 \mapsto \zeta_7^2 \rangle = \Gamma(K/F) \), so that \( X = \bar{\lambda}(x) = \begin{bmatrix} c & \sqrt{11}d \\ \sqrt{-\sigma(d)} & \sigma(c) \end{bmatrix} \). Define the 3-relay code
\[
\mathcal{X} = \{ \Psi_{\eta,3}(X) \}_{X \in \bar{\lambda}(\mathcal{O}_L)} = \left\{ \text{diag} \{ \eta^i(X) \}_{i=0} \right\} X \in \bar{\lambda}(\mathcal{O}_L) \}
\]
The constructed full-diversity code \( \mathcal{X} \) is of rank 24, satisfies the NVD property and has decoding complexity \(|S|^{15}\) in contrast to \(|S|^{24}\).
2) MIMO: Before we give a general result for this setting, we are interested in how the iterative construction introduced in Sec. [33-C] and forcing fast-decodability reflects on the performance of the constructed codes. For this purpose, we construct three simple example codes with different properties and analyze their performance.

Consider \( N = 2 \) relays, each equipped with \( n_r \geq 1 \) antennas, a single source with \( n_s \geq 1 \) transmit antennas and such that \( n_s + n_r = 4 \), and a single destination with \( n_d \geq 1 \) antennas. We construct three different rate-two codes for this scenario, each of them with different characteristics, arising from the following towers of extensions.

\[
\begin{align*}
\mathcal{C}_s &= (\mathbf{K}_s/F_s, \sigma_s, -1) \\
[2] \\
\mathbf{K}_s &= F_s(i) \\
\mathbf{F}_s &= \mathbb{Q}(\sqrt{-7}) \\
\mathcal{C}_g &= (\mathbf{K}_g/F_g, \sigma_g, i) \\
[2] \\
\mathbf{K}_g &= F_g(\sqrt{5}) \\
\mathbf{F}_g &= \mathbb{Q}(i) \\
\mathcal{C}_m &= (\mathbf{K}_m/F_m, \sigma_m, -\frac{5}{2}) \\
[2] \\
\mathbf{K}_m &= \mathbb{Q}(\zeta_5) \\
\mathbf{F}_m &= \mathbb{Q}
\end{align*}
\]

\( \sigma_s : i \mapsto -i \)

\( \sigma_g : \sqrt{5} \mapsto -\sqrt{5} \)

\( \sigma_m : \zeta_5 \mapsto \zeta_5^3 \)

1. The Silver code, well known to be FD [39], [40], is constructed from the CDA \( \mathcal{C}_s \) and is a finite subset

\[
\mathcal{S} \subset \left\{ \frac{1}{\sqrt{7}} \left[ x_1 \sqrt{7} + (1+i)x_2 + (1-1/2)x_3 - x_1 \sqrt{7} + (1-1/2)x_3 - x_1 \sqrt{7} + (1-1/2)x_3 \right] | x_i \in \mathbb{Z}, 1 \leq i \leq 4 \right\}.
\]

Using \( \theta_s = -17 \) [15], Prop. 9, \( \tau_s = \sigma_s \), and given two elements \( X, Y \in \mathcal{S} \), we construct a distributed iterated Silver code by first applying the iterative construction in order to serve two relays, and further adapting it to the relay scenario as follows.

\[
\mathcal{X}_s = \left\{ \Psi_{id,2}(\bar{a}_{\tau_s,\theta_s}(X, Y)) = \left[ \begin{array}{c}
\bar{a}_{\tau_s,\theta_s}(X, Y) \\
\bar{a}_{\tau_s,\theta_s}(X, Y)
\end{array} \right] | X, Y \in \mathcal{S} \right\},
\]

where the choice \( \eta = id \) is made as the original code is already NVD, and

\[
\bar{a}_{\tau_s,\theta_s}(X, Y) = \begin{bmatrix}
X & -\sqrt{\theta_s} \tau_s(Y) \\
\sqrt{\theta_s} Y & \tau_s(X)
\end{bmatrix}.
\]

The resulting code \( \mathcal{X}_s \) is NVD and FD. Indeed, it is conditionally 4-GD with decoding complexity \( |S|^{10} \) as opposed to \( |S|^{16} \).

2. The Golden code, a well-performing ST code introduced in [33], is constructed from \( \mathcal{C}_g \) and consists of codewords taken from the set

\[
\mathcal{S} \subset \left\{ \frac{1}{\sqrt{5}} \left[ \nu(x_1 + x_2 \omega) \sigma_g(\nu)(x_3 + x_4 \sigma_g(\omega)) \sigma_g(\nu)(x_1 + x_2 \sigma_g(\omega)) \right] | x_i \in \mathbb{Z}, 1 \leq i \leq 4 \right\},
\]

where \( \nu(x_1 + x_2 \omega) \sigma_g(\nu)(x_3 + x_4 \sigma_g(\omega)) \sigma_g(\nu)(x_1 + x_2 \sigma_g(\omega)) \) are the codewords.
where $\omega = (1 + \sqrt{5}) / 2$ and $\nu = 1 + i - i \omega$. The Golden code, although very good in performance, does not exhibit an inner structure as described above that would lead to fast-decodability. Note, however, that it is possible to improve the decoding complexity of the Golden code by other means, e.g., by using a simple algorithm introduced in [41].

Set $\theta_g = 1 - i$ [15] Prop. 12], $\tau_g = \sigma_g$. Then, for two elements $X, Y \in \mathcal{G}$, we first iterate the original code and then adapt it to the relay scenario as follows.

$$
\mathcal{X}_g = \left\{ \Psi_{id,2}(\alpha_{\tau_g,\theta_g}(X, Y)) = [\alpha_{\tau_g,\theta_g}(X, Y) \, \alpha_{\tau_g,\theta_g}(X, Y)] \mid X, Y \in \mathcal{G} \right\}.
$$

As in the previous example, the original code is already NVD, thus the choice $\eta = \text{id}$ maintains this property. We have

$$
\alpha_{\tau_g,\theta_g}(X, Y) = \begin{bmatrix} X \, \theta_g \tau_g(Y) \\ Y \, \tau_g(X) \end{bmatrix}.
$$

The original code is not directly FD, hence, the decoding complexity of $\mathcal{X}_g$ is $|S|^{14}$.

3. We also consider the FD MIDO_{A4} code constructed in [16], using $C_m$ as the algebraic structure. Let $\zeta_5$ be the $5^{th}$ root of unity and choose $\{1 - \zeta_5, \zeta_5 - \zeta_5^2, \zeta_5^2 - \zeta_5^3, \zeta_5^3 - \zeta_5^4\}$ a basis of $\mathbb{Z}[\zeta_5]$. Setting $r = | - 8 / 9 |^{1 / 4}$, codewords are taken from

$$
\mathcal{M} \subset \left\{ \begin{bmatrix} x_1 & -r^2 x_2 & -r^3 \sigma_m(x_4) & -r \sigma_m(x_3) \\ r^2 x_2 & x_1 & r \sigma_m(x_3) & -r^2 \sigma_m(x_4) \\ r x_3 & -r^3 x_4 & \sigma_m(x_1) & -r^2 \sigma_m(x_2) \\ r^3 x_4 & r x_3 & r^2 \sigma_m(x_2) & \sigma_m(x_1) \end{bmatrix} \mid x_i \in \mathbb{Z}[\zeta_5], 1 \leq i \leq 4 \right\}.
$$

Given an element $X$ from the set, the adaptation to the relay channel does not require iterating the original code due to its size, and as again it is already NVD, the relay version is simply

$$
\mathcal{X}_m = \left\{ \Psi_{id,2}(X) = [X \, 0 \, X] \mid X \in \mathcal{M} \right\}.
$$

The resulting code is fully diverse, has the NVD property, and is conditionally 2-GD with decoding complexity $|S|^{12}$, as opposed to $|S|^{16}$.

**Remark 11.** The decoding complexity of $\mathcal{X}_m$ can be further reduced to $|S|^{10}$ by choosing $\{1, \zeta_5 + \zeta_5^{-1}, \zeta_5 - \zeta_5^{-1}, \zeta_5^2 - \zeta_5^{-2}\}$ as a basis of $\mathbb{Z}[\zeta_5]$. This, however, slightly worsens its performance.

More generally, we can build FD distributed ST codes having code rate $R = 2$ or $R = 4$ by using the codes in [16] as building blocks. Using block repetition, we get $R = 2$ for any even $n_s + n_r$. By using independent blocks, $R = 4$ can be achieved. The codes in [16] already have NVD, so also the distributed code will have NVD. The resulting distributed codes will have
the same complexity reduction $25 - 37.5\%$ as the codes in [16]. However, here we aim at even bigger complexity reduction, so we will not elaborate on this beyond this remark.

We are interested in how iterating ST codes and fast-decodability influence the performance of the codes. To that end, we compare in Fig. 1 the constructed example codes, along with another version of the distributed Silver code constructed using $\alpha_{\tau, \theta}(\cdot, \cdot)$ instead of $\tilde{\alpha}_{\tau, \theta}(\cdot, \cdot)$, and a third version using $\theta_s = -1$. This choice does not satisfy the requirements from Prop. 4 and thus generally results in ST codes that are not fully diverse. In our simulations, however, we have used a 2-PAM signaling alphabet and, in this case, the choice $\theta_s = -1$ still ensures that the resulting code will be fully diverse. It is obvious from Fig. 1 that the choice of the involved $\theta$ is utterly important. This, in fact, is known to be the case for general ST coding, and both versions of the distributed Silver code with $\theta_s = -17$ have a poor performance due to the fact that $|17|$ respectively $|\sqrt{17}|$ are not close to 1. From this simulation it is clear that, at least for the example codes, neither the iteration nor imposing fast-decodability on the codes has a negative effect on the performance of the codes.

Motivated by the observations above, we now propose a construction for obtaining FD distributed ST codes for multiple antennas at the source and relays. In the following, assume a single source, now equipped with $n_s \geq 1$ antennas, and for $p \geq 5$ prime, let $N = (p - 1)/2$ be the number of relays, each equipped with $n_r \geq 1$ antennas and such that $n_r + n_s = 4$. Assume further a single destination with $n_d \geq 1$ antennas. Consider the following tower of extensions,
\[ C = (a, \gamma)_K \cong (L/K, \sigma : \sqrt{a} \mapsto -\sqrt{a}, \gamma) \]

\[ L = K(\sqrt{a}) \]

\[ K = Q(\xi) \]

\[ F = Q(\sqrt{a}) \]

\[ \mathcal{C} = (a, \gamma)_K \cong (L/K, \sigma : \sqrt{a} \mapsto -\sqrt{a}, \gamma) \]

where \( K = Q(\xi) \subset Q(\zeta_p) \) is the maximal real subfield of the \( p \)th cyclotomic field, that is, \( \xi = \zeta_p + \zeta_p^{-1} \), and \( a \in \mathbb{Z} \) square-free. Let \( \langle \sigma \rangle = \Gamma(L/K) \) and \( \langle \eta \rangle = \Gamma(L/F) \).

**Theorem 2.** In the setup above, choose \( a \in \mathbb{Z}_{<0} \) such that \( p = aO_K \) is a prime ideal. Fix further \( \gamma \in \mathbb{Z}_{<0} \) and \( \theta \in O_K = \mathbb{Z}[\xi] \) such that

- \( \gamma \) and \( \theta \) are both nonsquare mod \( p \),
- the quadratic form \( \langle \gamma, -\theta \rangle_L \) is anisotropic,
- \( \theta = \zeta \theta' \) where \( \zeta \in \{ \pm 1 \} \) and \( \theta' \in \mathbb{R}_{>0} \),

and further let \( \tau = \sigma \). Then, if \( \mathcal{O} \subset \mathcal{C} \) is an order, the distributed ST code

\[ \mathcal{X} = \left\{ \Psi_{\eta,N}(\tilde{\alpha}_{r,\theta}(X, Y)) = \text{diag} \left\{ \eta^{i}(\tilde{\alpha}_{r,\theta}(X, Y)) \right\}^{N-1} \bigg| X, Y \in \tilde{\lambda}(\mathcal{O}) \right\} \]

is a full-diversity ST code of rank \( 8N \), rate \( R = 2 \) rscu, exhibits the NVD property and is FD.

Its decoding complexity is \( |S|^{k'} \), where \( k' = \begin{cases} 4N & \text{if } a \equiv 1 \mod 4, \\ 2N & \text{if } a \not\equiv 1 \mod 4. \end{cases} \)

**Proof.** We first show that \( \mathcal{X} \) is fully diverse, wherefore it is enough to show that \( \mathcal{C} \) is division. By Lem. [1] it suffices to show that \( \gamma \notin \text{Nm}_{L/K}(L^\times) \). Let \( \alpha = \alpha_0 + \sqrt{a}\alpha_1 \in L^\times \). Then \( \text{Nm}_{L/K}(\alpha) = \alpha\sigma(\alpha) = \alpha_0^2 - a\alpha_1^2 \). Thus \( \gamma = \text{Nm}_{L/K}(L^\times) \iff \alpha_0^2 - a\alpha_1^2 - \gamma = 0 \) has nontrivial solutions in \( K \). But \( a < 0 \), \( \gamma < 0 \) and \( K \) is totally real, thus there can’t be any solutions.

Let now \( \{ b_i \}_{i=1}^N = \{ 1, \xi, \ldots, \xi^{N-1} \} \) be a power basis of \( \mathcal{O}_K \) and consider

\[ X = \tilde{\lambda}(x) = \left[ \begin{array}{c} x_1 + x_2 \omega \\\ \sqrt{-\gamma}(x_1 + x_2 \omega) \\frac{-\sqrt{-\gamma}(x_1 + x_2 \sigma(\omega))}{x_1 + x_2 \sigma(\omega)} \end{array} \right], \]

where \( \omega = \sqrt{a} \) if \( a \not\equiv 1 \mod 4 \) and \( \omega = \frac{1 + \sqrt{a}}{2} \) otherwise, so that \( \mathcal{O}_F = \mathbb{Z}[\omega] \). Let

\[ \mathcal{X}_0 = \left\{ X \big| X \in \tilde{\lambda}(\mathcal{O}) \right\} = \left\{ \sum_{i=1}^k s_i B_i^0 \bigg| s_i \in S \right\}, \]
where a set of weight matrices is given by $B_0 = \{B_i^0\}_{i=1}^k = \{X(b_i, 0, 0, 0), \ldots, X(0, 0, 0, b_i)\}_{i=1}^N$, thus $k = 4N$. Originating from this code, construct a set of weight matrices $B_i^{it} = \{\tilde{\alpha}_{\tau, \theta}(B_i^0, 0), \tilde{\alpha}_{\tau, \theta}(0, B_i^0)\}_{i=1}^{4N}$ defining the iterated code $X_0^{it} = \{\tilde{\alpha}_{\tau, \theta}(X, Y)\mid X, Y \in X_0\}$.

From $X_0^{it}$, the distributed ST code is constructed as

$$\mathcal{X} = \left\{ \Psi_{\eta, N}(X) \mid X \in X_0^{it} \right\},$$

and a defining set of weight matrices is given by $B = \{B_i\}_{i=1}^{8N} = \{\Psi_{\eta, N}(B_i^0)\}_{i=1}^{8N}$. $\mathcal{X}$ is thus a ST code of length $8N$ and by Def. [2] has rate $R = 8N/4N = 2$ rscu.

To see that $\mathcal{X}$ is NVD, note that by the restrictions imposed on the entries of elements in $X_0$, it is $\det(\tilde{\alpha}_{\tau, \theta}(X, Y)) \in \mathcal{O}_L$. It hence holds

$$\det[\Psi_{\eta, N}(\tilde{\alpha}_{\tau, \theta}(X, Y))] = \prod_{i=0}^{N-1} \det[\psi'(\tilde{\alpha}_{\tau, \theta}(X, Y))] = \prod_{i=0}^{N-1} \eta^i[\det(\tilde{\alpha}_{\tau, \theta}(X, Y))]$$

$$= \text{Nm}_{L/F} [\det(\tilde{\alpha}_{\tau, \theta}(X, Y))] \in \mathcal{O}_F.$$

As $F$ is imaginary quadratic, $\det[\Psi_{\eta, N}(\tilde{\alpha}_{\tau, \theta}(X, Y))] \geq 1$.

It remains to show that $\mathcal{X}$ is FD. To that end, group the matrices $\{B_i^0\}_{i=1}^N$ as follows:

$$G_1 = \{X(b_i, 0, 0, 0)\}_{i=1}^N; \quad G_2 = \{X(0, b_i, 0, 0)\}_{i=1}^N;$$

$$G_3 = \{X(0, 0, b_i, 0)\}_{i=1}^N; \quad G_4 = \{X(0, 0, 0, b_i)\}_{i=1}^N.$$ 

Let $X_i \in G_i$. A direct computation shows that for $i = 1, 2$ and $j = 3, 4$, $X_iX_j^\dagger + X_jX_i^\dagger = 0$. Moreover, it holds

$$X_1X_2^\dagger + X_2X_1^\dagger = 0 \quad \text{if } a \not\equiv 1 \mod 4,$$

$$\neq 0 \quad \text{if } a \equiv 1 \mod 4,$$

and the same holds for $X_3, X_4$. We thus conclude that $\mathcal{X}$ is 2-GD if $a \equiv 1 \mod 4$, exhibiting a decoding complexity of $|S|^{2N}$ and is 4-GD otherwise, in which case its decoding complexity is $|S|^N$. By Prop. [4](ii) and since $\theta, \tau$ are chosen to satisfy the requirements of the iterative construction ( [15], Cor. 8)), the iterated code $X_0^{it}$ and consequently the distributed ST code $\mathcal{X}$ exhibit a decoding complexity of $|S|^{4N}$ in the former, and $|S|^{2N}$ in the latter case. 

\[\square\]

**Remark 12.** We remark that by the results obtained in [17], the decoding complexity of codes arising from division algebras can only be reduced by a factor of 4. The constructive method proposed in Thm. [2] results in codes whose decoding complexity is reduced by either 50% or
75% compared to non-FD codes of the same rank. Thus, in the latter case, our codes indeed achieve the maximal complexity reduction by a factor of 4.

In fairness, we should point out that even though the complexity reduction of the proposed codes is in some cases the best possible, this does not necessarily mean that the codes are very simple to decode. If the lattice dimension is large to start with, then even 25% of the original dimension may be too high for practical applications. This is one of the reasons why we have preferred to design codes with low rates, since the rate reflects to the lattice dimension when the number of relays and the number of antennas at the source and relays are fixed (cf. Def. [2]).

One promising way to further reduce the complexity of distributed ST codes is to utilize so-called less-than-minimum-delay codes [42], as noted in [2].

Example 3. We make use of the above result to construct a FD distributed ST code for $N = 5$ relays, $n_s + n_r = 4$, arising from the following tower of extensions,

$$
\begin{align*}
C &= (-3, -1)_K \cong (L/K, \sigma : \sqrt{-3} \mapsto -\sqrt{-3}, -1) \\
\text{L} &= Q(\sqrt{-3}, \xi) \\
K &= Q(\xi) \\
F &= Q(\sqrt{-3})
\end{align*}
$$

where $\xi = \zeta_{11} + \zeta_{11}^{-1}$. Let $\tau = \sigma$ and $\langle \eta : \xi \mapsto \xi^2 - 2 = \Gamma(L/F)$. Choose $\theta = 1 - \xi = \zeta \theta'$ with $\zeta = -1$ and $\theta' \in \mathbb{R}_{>0}$. It is $O_K = \mathbb{Z}[\xi]$. Note that $p = (-3)O_K$ is a prime ideal, and further $\text{ord}(\gamma) = 2$, $\text{ord}(\theta) = 242$ in $O_K/p$. Since there is no element of order 4 in $O_K/p$, both elements are nonsquare mod $p$. Moreover, the quadratic form $\langle \gamma, -\theta \rangle_L$ is anisotropic, as

$$-v_1^2 - v_2^2 \theta = 0 \text{ for } v_1, v_2 \in L \Leftrightarrow -\theta = v^2$$

for some $v \in L$. But $-\theta$ is prime in $L$ and can thus not be a square. By the above arguments, the conditions from Thm. [2] are satisfied.

Let $x \in O \subset C$, $\omega = \frac{1 + \sqrt{-3}}{2}$, and $\{b_i\}_{1 \leq i \leq 5} = \{1, \xi, \xi^2, \xi^3, \xi^4\}$ a basis of $O_K$. For $x_1, \ldots, x_4 \in O_K$, define a set of weight matrices for a ST code $X_0$ consisting of codewords of the form

$$X = \tilde{\lambda}(x) = \begin{bmatrix} x_1 + x_2 \omega & -(x_3 + x_1 \sigma(\omega)) \\ x_3 + x_4 \omega & x_1 + x_2 \sigma(\omega) \end{bmatrix},$$

as $\{B_i\}_{1 \leq i \leq 20} := \{X(b_i, 0, 0, 0), \ldots, X(0, 0, 0, b_i)\}_{1 \leq i \leq 5}$. 
To adapt this code to the proposed scenario, we first iterate $X_0$ to obtain the set

$$X_0^\eta = \left\{ \tilde{\alpha}_{r,\theta}(X, Y) = \left[ X \begin{array}{c} \sqrt{\eta(Y)} Y \end{array} \right] \middle| X, Y \in \tilde{\lambda}(O) \right\},$$

and for which a basis can be given as $\{B_i^\eta\}_{1 \leq i \leq 40} := \{\tilde{\alpha}_{r,\theta}(B_i, 0), \tilde{\alpha}_{r,\theta}(0, B_i)\}_{1 \leq i \leq 20}.$

The constructed code can finally be adapted to the relay channel by making use of the map $\eta$, resulting in the distributed ST code defined by the weight matrices $B = \{\Psi_{\eta,5}(B_i^\eta)\}_{1 \leq i \leq 40}$:

$$\mathcal{X} = \left\{ \Psi_{\eta,5}(\tilde{\alpha}_{r,\theta}(X, Y)) = \text{diag} \{\eta(\tilde{\alpha}_{r,\theta}(X, Y))\}_{0 \leq i \leq 4} \middle| X, Y \in \tilde{\lambda}(O) \right\}$$

The resulting relay code is fully diverse and moreover NVD. It is FD and more specifically 2-GD with decoding complexity $|S|^{20}$ as opposed to $|S|^{40}$, thus resulting in a reduction of 50%.

**Example 4.** We conclude this section with an example for $N = 4$ relays that demonstrates the importance of the conditions in Thm. 2. Let $\zeta_5$ be the 5th root of unity and $\xi = \zeta_5 + \zeta_5^{-1}$.

$$C = (-3, 1 - \zeta_5)K \cong (L/K, \sigma : \sqrt{-3} \mapsto -\sqrt{-3}, 1 - \zeta_5)$$

$$L = \mathbb{Q}(\zeta_5, \sqrt{-3})$$

$$K = \mathbb{Q}(\zeta_5)$$

$$F = \mathbb{Q}(\sqrt{-3})$$

$$\mathbb{Q}$$

Note that $K$ is not totally real. Let $\tau = \sigma$ and $\langle \eta \rangle = \Gamma(L/F)$. Choose further $\theta = \frac{\zeta_5 + 1}{\zeta_5 - 1}$. The quaternion algebra $C$ is division, and the choice of $\tau$ and $\theta$ certainly satisfy the criteria required in Thm. 2 (and hence in Prop. 4). To see this, note that $\mathfrak{p} = (-3)\mathcal{O}_K$ is a prime ideal with residue field $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_{25}$. The order of $\gamma$ and $\theta$ within the multiplicative group $\mathbb{F}_{25}^\times$ are $\text{ord}(\gamma) = 80$, $\text{ord}(\theta) = 16$. Since there is no element of order 32 in $\mathbb{F}_{25}^\times$, they are both nonsquare mod $\mathfrak{p}$.

Further, the quadratic form $\langle \gamma, -\theta \rangle_L$ is anisotropic. This is as for $v_1, v_2 \in L$,

$$v_1^2 \gamma - v_2^2 \theta = 0 \Leftrightarrow v^2 = \frac{1 + \zeta_5}{(1 - \zeta_5)(\zeta_5 - 1)} = -\frac{1}{5} \alpha$$

with $\alpha = 3\zeta_5^3 + 4\zeta_5^2 + 3\zeta_5$ and $v \in L$. It thus suffices to show that $\alpha$ is not a square in $L$. But it is $\alpha = p_1p_2$ with primes

$$p_1 = \left( \left( -\frac{1}{2}\zeta_5^3 + \zeta_5 + 1 \right) \sqrt{-3} + \frac{1}{2}\zeta_5^3 - \zeta_5 - 1 \right),$$

$$p_2 = \left( \left( \frac{1}{2}\zeta_5^3 + \frac{1}{2}\zeta_5^2 + \frac{1}{2}\zeta_5 + 1 \right) \sqrt{-3} - \frac{1}{2}\zeta_5^3 - \frac{1}{2}\zeta_5^2 - \frac{1}{2}\zeta_5 - 1 \right).$$
Let \( x = c + \sqrt{\gamma} d \), \( c, d \in \mathcal{O}_L \) and for \( X = \lambda(x) = \begin{bmatrix} c & \gamma \sigma(d) \\ d & \sigma(c) \end{bmatrix} \), define the distributed ST code

\[
\mathcal{X} = \left\{ \Psi_{\eta,\lambda}(\alpha_{\tau,\rho}(X,Y)) = \text{diag} \{ \eta_i(\alpha_{\tau,\rho}(X,Y)) \}_{i=0}^3 \bigm\vert X,Y \in \lambda(\mathcal{O}) \right\}.
\]

The choices of \( \gamma \) and \( K \) do not agree with Thm. 2 and the constructed ST code is not FD. In fact, the resulting code exhibits a decoding complexity of \( |S|^3 \) as opposed to \( |S|^2 \), where the reduction is merely due to the Gram-Schmidt orthogonalization.

IV. FAST-DECODABLE NONCOOPERATIVE SPACE–TIME CODES

In this section we consider the transmissions by \( N \) users to a joint destination, e.g., an uplink transmission to a base station. Both the users and the destination can be equipped with multiple antennas. In contrast to the previous section, however, no cooperation is allowed between the users.

A. Multiple-Access Channel

Assuming a noncooperative multi-user communications scenario, the channel is known as either a symmetric or asymmetric multiple-access channel (MAC), depending on whether all users are equipped with the same or a different number of transmit antennas.

The transmission of user \( K_i \), for any \( i = 1, \ldots, N \), is a point-to-point communication problem over a wireless MIMO channel modeled by a matrix \( H_i \) of corresponding size. The ST code matrices of user \( K_i \) are generated independently of those of the remaining users \( K_j \), \( j \neq i \). For the sake of exposure, we assume the symmetric case, namely that every user \( K_i \) is equipped with \( n_s \) antennas. The destination is assumed to have \( n_d \) antennas.

The disadvantage of having independent code matrices is that the overall ST code does not exhibit a lattice structure, but can be still represented via a set of linear dispersion matrices acting as weight matrices. An important consequence is the following result.
Proposition 5. [43 Thm. 3] For any $N > 1$ and $n_x \geq 1$, there do not exist any linear MIMO-MAC codes satisfying the NVD criterion.

Remark 13. The above proposition motivated the definition of the Conditional Nonvanishing Determinant (CNVD) property introduced in [44]. A ST code has the CNVD property if its minimum determinant is either zero or bounded from below. This property, accompanied with a suitable code rate, was shown to be sufficient for achieving the optimal MAC-DMT [43].

B. Fast-Decodable MAC Space–Time Codes

Let $K \geq 2$ be the number of users communicating with a single destination. Each of the users encodes his information independent of the remaining users, and might even use a different underlying algebraic structure for code construction. The symmetric scenario, however, ensures that the codewords from every user will be of the same dimensions.

For $k \in \{1, \ldots, K\}$ consider user $K_k$ employing a ST code $\mathcal{X}_k$ carved from a CDA $\mathcal{C}_k = (\mathbb{L}_k/\mathbb{K}_k, \sigma_k, \gamma_k)$ of degree $n$. A codeword $X_k \in \mathcal{X}_k$ is of the form

$$X_k = \lambda(x_k) = \sum_{i=1}^{n^2} s_{k,i} B_{k,i}$$

for some $x_k \in \mathcal{O}_k \subset \mathcal{C}_k$, where $\lambda$ is defined as in (2), $s_{k,i} \in \mathcal{S}_k$ are the signaling coefficients and $\{B_{k,i}\}_{1 \leq i \leq n^2}$ is the set of weight matrices. Let $\mathbb{F}_k \subset \mathbb{K}_k$ be an intermediate field so that $\mathbb{K}_k/\mathbb{F}_k$ is cyclic Galois of degree $m$ with $\langle \tau_k \rangle = \Gamma(\mathbb{K}_k/\mathbb{F}_k)$.

The overall codeword of user $k$ is of the form $U_k = [X_k \, \tau_k(X_k) \, \ldots \, \tau_k^{m-1}(X_k)]$, and the overall transmitted codeword by all users is hence

$$X = \begin{bmatrix} x_1 & \tau_1(x_1) & \ldots & \tau_1^{m-1}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_K & \tau_K(x_K) & \ldots & \tau_K^{m-1}(x_K) \end{bmatrix}.$$  

A set of linear dispersion matrices can be given for the code by complementing the corresponding lattice basis of each user with zero-matrices of suitable size, namely

$$\mathcal{B} = \left\{ \begin{bmatrix} 0_{(k-1)n} \\ B_{k,i} \\ 0_{(K-k)n} \end{bmatrix}_{Kn} \right\}_{1 \leq k \leq K} =: \{B_k\}_{1 \leq k \leq Kn^2}.$$  

The resulting code is not NVD due to Prop. 5, but is FD and exhibits the CNVD property if the algebraic structures are chosen properly, as a straightforward adaptation of the results about distributed ST codes of the previous section to this noncooperative scenario.
Example 5. Let \( K = 2 \) users be communicating with a single destination, each of them equipped with \( n_s = 2 \) antennas. Consider the tower of extensions used in Exp. 7. Both users carve their ST codes from the quaternion algebra \( \mathcal{C} \). Let \( \langle \tau : \sqrt{31} \mapsto -\sqrt{31} \rangle = \Gamma(\mathbf{K}/\mathbf{F}) \), thus for \( k = 1, 2 \), codewords are of the form \( U_k = [x_k \, \tau(x_k)] \) where for \( x_k \in \mathcal{O} \subset \mathcal{C} \),

\[
X_k = \tilde{\lambda}(x_k) = \begin{bmatrix}
-x_k,1 + x_k,2 \sqrt{-5} & -\sqrt{2}(x_k,3 + x_k,4 \sigma(\sqrt{-5}))
\sqrt{2}(x_k,3 + x_k,4 \sqrt{-5}) & x_k,1 + x_k,2 \sigma(\sqrt{-5})
\end{bmatrix}.
\]

The overall transmitted codewords are of the form

\[
X = \begin{bmatrix}
X_1 \, \tau(X_1) \\
X_2 \, \tau(X_2)
\end{bmatrix}.
\]

For a basis \( \{b_i\}_{1 \leq i \leq 4} = \{1, \tau, \sqrt{31}, \tau \sqrt{31}\} \) of \( \mathcal{O}_K \), a set of weight matrices for this code is

\[
\{B_i\}_{1 \leq i \leq 32} := \left\{ \begin{bmatrix}
X_1(b_i,0,0,0) \, \tau(X_1(b_i,0,0,0)) \\
X_2(0,0,0,0) \, \tau(X_2(0,0,0,0))
\end{bmatrix}, \ldots, \begin{bmatrix}
X_1(0,0,0,0) \, \tau(X_1(0,0,0,0)) \\
X_2(0,0,0,0) \, \tau(X_2(0,0,0,0))
\end{bmatrix} \right\},
\]

\[
\begin{bmatrix}
X_1(0,0,0,0) \, \tau(X_1(0,0,0,0)) \\
X_2(0,0,0,0) \, \tau(X_2(0,0,0,0))
\end{bmatrix}, \ldots, \begin{bmatrix}
X_1(0,0,0,0) \, \tau(X_1(0,0,0,0)) \\
X_2(0,0,0,0) \, \tau(X_2(0,0,0,0))
\end{bmatrix}\right\}_{1 \leq i \leq 4}.
\]

The resulting code is 2-GD with decoding complexity \( |S|^{16} \) as opposed to \( |S|^{32} \).

V. Conclusions

In this work, we have proposed two methods for constructing distributed space–time block codes for cooperative communications employing intermediate relays for signal amplification. We have separately considered the scenarios where both the source and each of the relays are equipped with a single or multiple antennas, resulting in flexible constructive methods to obtain fast-decodable, more specifically 2-group and (conditionally) 4-group decodable, full-diversity space–time codes which have nonvanishing determinants for both cases. The obtained codes can be decoded with a very low number of antennas, in the MIMO case even a single antenna suffices, and their worst-case decoding complexity is reduced by up to 75\%, which is known to be the best possible reduction. These are highly desirable properties for many applications related to the future 5G networks, such as device-to-device communications and proximity-based services on the wireless edge.

We have further shown how to use these methods to obtain fast-decodable space–time codes for the \( K \)-user MIMO-MAC scenario. Although codes for this channel cannot exhibit the non-vanishing determinant property due to the nature of the communications setting, the constructed codes using the methods introduced in this work exhibit the conditional nonvanishing determinant property, which is known to be useful for achieving the optimal MAC-DMT.
REFERENCES


