Symmetrical path-cycle covers of a graph and polygonal graphs

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Abstract

A near-polygonal graph is a graph \( \Gamma \) which has a set \( C \) of \( m \)-cycles for some positive integer \( m \) such that each 2-path of \( \Gamma \) is contained in exactly one cycle in \( C \). If \( m \) is the girth of \( \Gamma \) then the graph is called polygonal. We introduce a method for constructing near-polygonal graphs with 2-arc transitive automorphism groups. As special cases, we obtain several new infinite families of polygonal graphs.

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1. Introduction

Let \( \Gamma \) be a graph. For a positive integer \( l \), an \( l \)-walk of \( \Gamma \) is a sequence of vertices \((\alpha_0, \alpha_1, \ldots, \alpha_l)\) such that \( \alpha_i \) is adjacent to \( \alpha_{i+1} \) for \( 0 \leq i \leq l - 1 \). If in addition \( \alpha_{i-1} \neq \alpha_{i+1} \) for \( 1 \leq i \leq l - 1 \), then an \( l \)-walk is called an \( l \)-arc; while if further all the \( \alpha_i \) are distinct then the \( l \)-arc is called an \( l \)-dipath (directed path). The identification of an \( l \)-dipath \((\alpha_0, \alpha_1, \ldots, \alpha_l)\) and its reverse \((\alpha_l, \ldots, \alpha_1, \alpha_0)\) is called an \( l \)-path, and denoted by \([\alpha_0, \alpha_1, \ldots, \alpha_l]\). An \( m \)-cycle is an \((m - 1)\)-path \([\alpha_1, \ldots, \alpha_m]\) such that \( \alpha_m \) is adjacent to \( \alpha_1 \).
Following [4], we call $\Gamma$ a near-polygonal graph if there exists a number $m$ and a collection $\mathcal{C}$ of $m$-cycles in $\Gamma$ such that each 2-path of $\Gamma$ is contained in exactly one cycle in $\mathcal{C}$. If $m$ is the girth $g(\Gamma)$ of $\Gamma$ then the graph is called polygonal; furthermore, if $\mathcal{C}$ is the set of all cycles of length $l$ such that $\Gamma$ is called a strict polygonal.

The main purpose of this paper is to introduce a method for constructing near-polygonal graphs and, as an application, to obtain new infinite families of polygonal graphs. Our method covers a graph $\Gamma$ of a set $\mathcal{C}$ of $m$-cycles such that each $l$-path of $\Gamma$ is covered by at least one cycle in $\mathcal{C}$; sometimes an $(l, m)$-path-cycle cover is simply called an $(l, m)$-cover. If in addition every $l$-path of $\Gamma$ lies in a constant number $\lambda$ of cycles of $\mathcal{C}$, then $\mathcal{C}$ is called a regular $\lambda$-$(l, m)$-cover, or simply called a $\lambda$-$(l, m)$-cover. Hence near-polygonal graphs are the graphs that have a 1-(2, $m$) cover for some $m$.

For a graph $\Gamma$ and a group $G \leq \text{Aut}(\Gamma)$, an $(l, m)$-cover $\mathcal{C}$ is called $G$-symmetrical if $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ is such that

(i) the restriction $G|_{C_i}$ of $G$ to each $C_i$ contains all rotations of $C_i$;

(ii) $G$ induces a transitive action on $\mathcal{C}$.

There are two possibilities for $G|_{C_i}$ to contain all rotations of $C_i$, namely $G|_{C_i} \cong \mathbb{Z}_m$, or $D_{2m}$. The corresponding symmetrical covers will be called $G$-rotary, or $G$-dihedral, respectively. For a positive integer $l$, a graph $\Gamma$ is called $(G, l)$-arc transitive, $(G, l)$-dipath transitive, or $(G, l)$-path transitive if $G$ acts transitively on $l$-arcs, $l$-dips, or $l$-paths of $\Gamma$, respectively. In the case of dipath and path transitivity, we also require that $l$-dips or $l$-paths exist in $\Gamma$, respectively.

Our construction of near-polygonal graphs is based on the following lemma.

**Lemma 1.1.** Let $\Gamma$ be a regular graph of valency at least 3, let $G \leq \text{Aut}(\Gamma)$, and let $l \geq 1$ be an integer. Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) holds for the following four statements (a)–(d).

(a) $\Gamma$ has a $G$-dihedral $(l, m)$-cover for some $m \geq 3$.
(b) $\Gamma$ is $(G, l)$-dipath transitive.
(c) $\Gamma$ has a $G$-rotary $(l, m)$-cover for some $m \geq 3$.
(d) $\Gamma$ is $(G, l)$-path transitive.

Moreover, if $\Gamma$ has a $G$-dihedral $(l, m)$-cover $\mathcal{C}$ and $G$ acts sharply transitively on the $l$-dips in $\Gamma$ then $\mathcal{C}$ is a 1-$(l, m)$-cover.

The following corollary of Lemma 1.1 is easier to apply for the construction of near-polygonal graphs; however, it is harder to control the length $m$ of the special cycles.

**Corollary 1.2.** Let $\Gamma$ be a connected sharply $(G, 2)$-arc transitive graph. Assume that for an arc $(\alpha, \beta)$ of $\Gamma$ there exists an involution $g \in G$ such that $(\alpha, \beta)^g = (\beta, \alpha)$. Then $\Gamma$ is near-polygonal.

Our main results are as follows.

**Theorem 1.3.** Let $m \geq 5$ be an arbitrary integer, let $q = p^e$ be a prime power with $q \equiv \pm 1 \pmod{m}$, and let $G = \text{PGL}(2, q) \times \text{PGL}(2, q)$. Then there exists a sharply $(G, 2)$-arc transitive near-polygonal graph $\Gamma$ of valency $q$ on $q(q - 1)(q + 1)^2$ vertices with a 1-$(2, m)$-cover.
In the cases $m = 5, 6, \text{ and } 7$, we can further strengthen Theorem 1.3.

**Theorem 1.4.** If $q \equiv \pm 1 \pmod{5}$ then $\Gamma$ in Theorem 1.3 can be chosen such that

(i) if $p \neq 3$, then $\Gamma$ is a strict polygonal graph of girth 5;

(ii) if $p = 3$, then $\Gamma$ is polygonal of girth 5 and the set of all 5-cycles is a 2-(2, 5) cover.

**Theorem 1.5.** If $q \equiv \pm 1 \pmod{6}$ then $\Gamma$ in Theorem 1.3 can be chosen such that

(i) if $p \neq 11$, then $\Gamma$ is a strict polygonal graph of girth 6;

(ii) if $p = 11$, then $\Gamma$ is a strict polygonal graph of girth 5, and satisfies the additional property that the 6-cycles also cover each 2-path exactly once.

**Theorem 1.6.** If $q \equiv \pm 1 \pmod{7}$ then $\Gamma$ in Theorem 1.3 can be chosen such that

(i) if $p = 2$ then $\Gamma$ is a polygonal graph of girth 7;

(ii) if $p$ is odd and the equation $y^3 - 5y^2 + 6y - 1 = 0$ has two distinct solutions $y_1, y_2$ in $\text{GF}(q)$ such that

$$10y_1y_2 - 3(y_1^2 + y_2^2) \text{ is a nonsquare in } \text{GF}(q)$$

then $\Gamma$ is a polygonal graph of girth 7.

**Remark.** We shall prove (cf. Lemmas 3.4 and 3.5) that for all prime powers $q = p^e \equiv \pm 1 \pmod{7}$, the equation $y^3 - 5y^2 + 6y - 1 = 0$ has three distinct solutions in $\text{GF}(q)$. However, we cannot characterize those values of $q$ where two of these solutions satisfy (1). Computations in GAP [3] indicate that for $e$ odd, for solutions $y_1, y_2$ the polynomial $10y_1y_2 - 3(y_1^2 + y_2^2)$ takes square and nonsquare values with about equal frequency. Hence there exist solutions $y_1, y_2$ satisfying (1) for about 7/8 of the values $q = p^e \equiv \pm 1 \pmod{7}$, with $p, e$ odd. In particular, out of the 57 such prime powers $q < 1000$, there are two distinct solutions $y_1, y_2$ such that $10y_1y_2 - 3(y_1^2 + y_2^2)$ is a nonsquare for 50 values of $q$. Note that if $e$ is even then $10y_1y_2 - 3(y_1^2 + y_2^2)$ is always a square in $\text{GF}(q)$ but if $y_1, y_2$ satisfy (1) then $10y_1y_2 - 3(y_1^2 + y_2^2)$ is a nonsquare in $\text{GF}(q^k)$ for all odd integers $k$.

There is a scarce supply of polygonal graphs of valency at least 4 and girth at least 6. Archdeacon and Perkel [1] gave a general construction which, given a polygonal graph of valency $r$ and girth $m$, constructs a cover that is a polygonal graph of valency $r$ and girth $2m$. Two infinite families of polygonal graphs of girth 6 and increasing valency were described in [5]. The graphs in one of these families have 2-arc transitive automorphism groups. The graphs announced in Theorems 1.3–1.6 also have 2-arc transitive automorphism groups (and so do any further examples constructed in the future via Lemma 1.1). The graphs announced in Theorems 1.6 constitute the first infinite family of polygonal graphs with girth 7 and valency at least 4 (without any restriction on the automorphism group). The largest girth in a known polygonal graph with 2-arc transitive automorphism group is 14.

Finally, we start to investigate the relation between $l$-dipath transitivity and the more frequently studied notion of $l$-arc transitivity. The fact that every $l$-dipath is an $l$-arc implies that $l$-arc transitive graphs are $l$-dipath transitive, but there are graphs which are $l$-dipath transitive but not $l$-arc transitive.
Theorem 1.7. Let $\Gamma$ be a connected regular graph of valency at least 3, and let $l$ be a positive integer. Then $\Gamma$ is $s$-dipath transitive for all $s \leq l$ if and only if one of the following holds:

(i) $\Gamma$ is $l$-arc transitive;
(ii) $\Gamma \cong K_n$ for $n \geq 4$, and $3 \leq l \leq n - 1$;
(iii) $\Gamma \cong K_{n,n}$ for $n \geq 3$, and $4 \leq l \leq 2n - 1$.

The difficulty of characterizing $l$-dipath transitive but not $l$-arc transitive graphs is caused by the fact that $l$-dipath transitivity does not imply $s$-dipath transitivity for all $s \leq l$ (see Example 1.8). Note that in graphs of minimal valency at least 2, $l$-arc transitivity implies $s$-arc transitivity for all $s \leq l$ because any $s$-arc can be extended to an $l$-arc.

Example 1.8. Let $n$ be even, and let $\Gamma$ be the disjoint union of complete graphs $K_{n/2}$ plus an additional edge connecting two vertices in different copies of $K_{n/2}$. Then $\Gamma$ is $(n - 1)$-dipath transitive and $(n - 2)$-path transitive, but not $s$-dipath transitive for any $s < n - 1$ and not $s$-path transitive for any $s < n - 2$.

2. Construction of $(l, m)$-covers and proof of Lemma 1.1

In this section, we first introduce a method for constructing a set of $m$-cycles which, under suitable conditions, provide an $(l, m)$-cover of a graph, and then prove Lemma 1.1 and Corollary 1.2.

Construction 2.1. Let $(\alpha_0, \ldots, \alpha_l)$ and $(\alpha_1, \ldots, \alpha_l, \alpha_{l+1})$ be $l$-dips in a graph $\Gamma$ (allowing that $\alpha_0 = \alpha_{l+1}$), let $G \leq \text{Aut}(\Gamma)$, and suppose that there exists $g \in G$ such that $\alpha_i^g = \alpha_{i+1}$ holds for $0 \leq i \leq l$. Let $C$ be the cycle generated by the vertices $\alpha_0^{(g)}$, and let $\mathcal{C} = C^G$.

We shall refer to the method described in Construction 2.1 as spinning an $l$-dipath. The next lemma shows that all symmetrical covers of a graph may be constructed by spinning an $l$-dipath.

Lemma 2.2. Let $\Gamma$ be a graph, and let $G \leq \text{Aut}(\Gamma)$. Then each $G$-symmetrical $(l, m)$-cover of $\Gamma$ may be obtained as in Construction 2.1.

Proof. Let $\mathcal{C}$ be a $G$-symmetrical $(l, m)$-cover of $\Gamma$, and let $C = [\alpha_0, \ldots, \alpha_{m-1}] \in \mathcal{C}$. By the definition of symmetrical covers, $G|\mathcal{C}$ contains all rotations of $C$ and so there exists some $g \in G$ such that $\alpha_i^g = \alpha_{i+1}$ for $0 \leq i \leq m - 1$, and $\alpha_{m-1}^g = \alpha_0$. In particular, $(\alpha_0, \ldots, \alpha_l)^g = (\alpha_1, \ldots, \alpha_l, \alpha_{l+1})$. Since $\mathcal{C}$ is closed for the action of $G$ and $G$ is transitive on $\mathcal{C}$, all members of $\mathcal{C}$ are images of $C$ under $G$. So $\mathcal{C}$ may be obtained by spinning the $l$-dipath $(\alpha_0, \ldots, \alpha_l)$. □

In order to apply Lemma 1.1, we need to ensure that a target group $G$ occurs as a group of automorphisms of some graph $\Gamma$. That goal can be achieved by defining $\Gamma$ as a coset graph (also called orbital graph), as described below.

For a group $G$ and a subgroup $H < G$, denote by $[G : H]$ the set of right cosets of $H$ in $G$. For an element $g \in G \setminus H$ with $g^2 \in H$, the coset graph $\Gamma := \text{Cos}(G, H, HgH)$ is defined as the graph with vertex set $[G : H]$ such that two vertices $Hx$ and $Hy$ are adjacent if and only if $yx^{-1} \in HgH$. Observe that from the condition $g^2 \in H$ it follows that $HgH = Hg^{-1}H$ and so $Hx$ and $Hy$ are adjacent if and only if $Hy$ and $Hx$ are adjacent, implying that $\Gamma$ is undirected.
Denote by $\alpha$ the vertex $H$ of $\Gamma$ and by $\beta$ the vertex $\alpha^g = Hg$. Note that $\beta^g = \alpha^g \cdot \alpha = G \cdot H$, $G_H = H^g$ and $G_{\alpha\beta} = H \cap H^g$. The neighbor set $N_\Gamma(\alpha)$ of $\alpha$ consists of the cosets in $HgH$, and the valency of $\alpha$ is the index $|G_H : G_{\alpha\beta}|$.

For the construction of near-polygonal graphs, we want to spin a 2-dipath $(\gamma, \alpha, \beta)$. The spinning element can be described easily in terms of the coset graph.

**Lemma 2.3.** For a coset graph $\Gamma = \text{Cos}(G, H, HgH)$ with $g^2 \in H$, let $\alpha = H$ and $\beta = \alpha^g$. Then an element $f \in G$ maps $\alpha$ to $\beta$ if and only if $f \in G_{\alpha\beta}$.

**Proof.** Suppose that $f \in G$ is such that $\alpha^f = \beta$. Then $\alpha^f = \alpha^g$, and hence $\alpha^{fg^{-1}} = \alpha$, that is, $fg^{-1} \in G_{\alpha}$ and $f \in G_{\alpha\beta}$. Conversely, if $f \in G_{\alpha\beta}$, then $f = xg$ for some $x \in G_{\alpha}$, and $\alpha^f = \alpha^{xg} = \alpha^g = \beta$.

Assume now that $\alpha^f = \beta$. If $f \in G_{\alpha\beta}g$, then $f = xg$ for some $x \in G_{\alpha\beta}$, and hence $\beta^f = \beta^{xg} = \beta^g = \alpha$. Conversely, if $\beta^f = \alpha$ then $\beta^f = \beta^g$ and $\alpha^f = \alpha^g$, and so $f \in G_{\alpha\beta}$. \hfill $\Box$

We shall also use the following well-known lemma.

**Lemma 2.4.** Let $\Gamma$ be a graph, and let $G \leq \text{Aut}(\Gamma)$ be transitive on the vertex set of $\Gamma$. Then $\Gamma$ is $(G, 2)$-dipath transitive if and only if $G_{\alpha}$ acts 2-transitively on $N_\Gamma(\alpha)$; furthermore, $\Gamma$ is sharply $(G, 2)$-dipath transitive if and only if $G_{\alpha}$ acts sharply 2-transitively on $N_\Gamma(\alpha)$.

**Proof of Lemma 1.1.** (a) $\Rightarrow$ (b): Let $C$ be a $G$-dihedral $(l, m)$-cover of $\Gamma$, and let $(\alpha_0, \ldots, \alpha_l)$ and $(\beta_0, \ldots, \beta_l)$ be two arbitrary $l$-dipaths in $\Gamma$. By definition, there exist $C_1, C_2 \in C$ such that $[\alpha_0, \ldots, \alpha_l]$ is contained in $C_1$ and $[\beta_0, \ldots, \beta_l]$ is contained in $C_2$, and there exists $g \in G$ such that $C^{\alpha_0} = C_2$. Let $(\gamma_0, \ldots, \gamma_l) = (\alpha_0, \ldots, \alpha_l)^g$. Then $[\gamma_0, \ldots, \gamma_l]$ is contained in $C_2$ and, as $G|_{C_2}$ is a dihedral group, there exists an element $h \in G$ fixing $C_2$ and satisfying $(\gamma_0, \ldots, \gamma_l)^h = (\beta_0, \ldots, \beta_l)$. Hence $(\alpha_0, \ldots, \alpha_l)^{gh} = (\beta_0, \ldots, \beta_l)$ and so $G$ is $(G, l)$-dipath transitive.

(b) $\Rightarrow$ (c): Suppose that $\Gamma$ is $k$-regular, $(G, l)$-dipath transitive, and let $(\alpha_0, \ldots, \alpha_l)$ be an $l$-dipath in $\Gamma$. First, we claim that there exists a vertex $\alpha_{i+1}$ of $\Gamma$ such that $(\alpha_1, \ldots, \alpha_l, \alpha_{i+1})$ is an $l$-dipath. If $\alpha_l$ has a neighbor not contained in $(\alpha_0, \ldots, \alpha_l)$ then our claim is obviously true. Suppose that $\alpha_{j_1}, \ldots, \alpha_{j_k}$ are the neighbors of $\alpha_l$, with $0 \leq j_1 < j_2 < \cdots < j_k = l - 1$. For $1 \leq i \leq k$, the sequence $A_i := (\alpha_0, \alpha_1, \ldots, \alpha_{j_i}, \alpha_{j_i+1}, \ldots, \alpha_{j_i+1})$ is an $l$-dipath in $\Gamma$. Therefore, since $\Gamma$ is $(G, l)$-dipath transitive, there exists $g_i \in G$ such that $(\alpha_0, \ldots, \alpha_l)^{g_i} = A_i$. Thus, because $g_i$ is an automorphism of $\Gamma$, we have that $(\alpha_{j_i+1}, \alpha_{j_i+1})^{g_i} = (\alpha_{j_i}, \alpha_{j_i+1})$ is an edge in $\Gamma$. So, using that $\Gamma$ is $k$-regular, we conclude that $\alpha_{j_i+1}$, for $1 \leq i \leq k$, are the only neighbors of $\alpha_{j_i}$. This implies $j_1 = 0$, otherwise $\alpha_{j_1-1}$ would also be a neighbor. Hence we can choose $\alpha_0$ as $\alpha_{i+1}$, proving our claim.

Since $\Gamma$ is $(G, l)$-dipath transitive, there exists $g \in G$ such that $(\alpha_0, \ldots, \alpha_l)^g = (\alpha_1, \ldots, \alpha_l, \alpha_{l+1})$ and we can apply Construction 2.1 to obtain a cycle $C = \alpha_0^{g^0}$, and $C = C^G$. Then $C$ is an $(l, |C|)$-path-cycle cover. It is also a $G$-rotary cover, because $(g)$ rotates $C$ and for any other cycle $C' = C^h$ in $C$, $(g^h)$ rotates $C'$.

(c) $\Rightarrow$ (d): This proof is analogous to the proof of (a) $\Rightarrow$ (b).

To show the final assertion of the lemma, suppose that $\Gamma$ is sharply $(G, l)$-dipath transitive and has a $G$-dihedral $(l, m)$-cover $C$ for some $m$. Suppose that, contrary to the statement of the lemma, there is an $l$-path $(\alpha_0, \ldots, \alpha_l)$ which is contained in two cycles $C_1, C_2 \in C$. Then
there exists $g \in G$ with $C_1^g = C_2$, and let $(\beta_0, \ldots, \beta_l) = (\alpha_0, \ldots, \alpha_l)^g$. Now $(\alpha_0, \ldots, \alpha_l)$ and $(\beta_0, \ldots, \beta_l)$ are two $l$-dipaths contained in $C_2$, so there exists $h \in G$ fixing $C_2$ and satisfying $(\beta_0, \ldots, \beta_l)^h = (\alpha_0, \ldots, \alpha_l)$. This is a contradiction with the sharply transitive property, because $(\alpha_0, \ldots, \alpha_l)^{gh} = (\alpha_0, \ldots, \alpha_l)$ but $C_1^{gh} = C_2 \neq C_1$ implies $gh \neq 1$. \hfill \Box

**Proof of Corollary 1.2.** By assumption, $g$ normalizes $G_{\alpha \beta}$ and $\langle G_\alpha, g \rangle = G$. By Lemma 2.4, the group $G_\alpha^{G_\alpha(\alpha)}$ is sharply 2-transitive. It follows that $G_\alpha$ is of even order. Let $P$ be the subgroup generated by all the involutions of $G_\alpha$. Then $P$ is normal in $G_\alpha$. If $g$ centralizes all involutions of $G_\alpha$, then $g$ centralizes $P$, and so $P$ is normalized by $\langle G_\alpha, g \rangle = G$, which is a contradiction since $G_\alpha$ is core free in $G$. Thus there exists an involution $h \in G_\alpha$ such that $hg \neq gh$.

Let $f = hg$. Then $f^g = f^{-1}$, and the orbit $\alpha(f)$ has length $m \geq 3$. Spin the arc $(\alpha, \beta)$ by the subgroup $\langle f \rangle$, say

\begin{align*}
\alpha_0 &= \alpha, & \alpha_1 &= \beta = \alpha^f, & \alpha_2 &= \beta^f = \alpha^{f^2}, & \ldots, & \alpha_{m-1} &= \alpha^{f^{m-1}}.
\end{align*}

Then for each $i = 2, 3, \ldots, m - 1$ we have

\begin{align*}
\alpha_i^g &= (\alpha_0^f)^i = \alpha_0^{gg^{-1}f_i^g} = \beta f^{-i} = \alpha^{f^{m-i+1}} = \alpha_{m-i+1}.
\end{align*}

Therefore, the restriction of $G$ to the cycle $[\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{m-1}]$ is a dihedral group of order $2m$. Hence, by Lemma 1.1, the graph $\Gamma$ is near-polygonal. \hfill \Box

Using Corollary 1.2, we can easily construct near-polygonal graphs.

**Example 2.5.** Let $G = \text{PSL}(2, p)$ where $p$ is a prime, $p \equiv \pm 3 \pmod{8}$ and $p \neq \pm 1 \pmod{10}$. Then $G$ contains a maximal subgroup $H \cong A_4$. Let $b$ be an element of $H$ of order 3. Then there exists an involution $g \in G$ such that $b^g = b^{-1}$. Thus the coset graph $\Gamma = \text{Cos}(G, H, HgH)$ is connected and sharply $(G, 2)$-arc transitive of valency 4. The involution $g$ exchanges the vertices $\alpha = H$ and $\beta = Hg$ and so, by Corollary 1.2, $\Gamma$ is near-polygonal.

**Example 2.6.** Let $G = S_p$ where $p > 5$ is a prime. Then $G$ contains a maximal subgroup $H \cong \mathbb{Z}_p : \mathbb{Z}_{p-1}$, a Frobenius group. Let $b$ be an element of $H$ of order $p - 1$, and let $g$ be an involution in $G \setminus H$ such that $b^g = b^{-1}$. Then $\langle H, g \rangle = G$. Thus the coset graph $\Gamma = \text{Cos}(G, H, HgH)$ is connected and sharply $(G, 2)$-arc transitive of valency $p$. The involution $g$ exchanges the vertices $\alpha = H$ and $\beta = Hg$ and so, by Corollary 1.2, $\Gamma$ is near-polygonal.

These two examples arise from vertex primitive 2-arc regular graphs. Such graphs are classified in [2].

### 3. Families of near-polygonal graphs

Throughout this section, $G$ denotes the group $\text{PGL}(2, q) \times \text{PGL}(2, q)$ for some prime power $q$. The elements of $\text{PGL}(2, q)$ can be identified with equivalence classes of $2 \times 2$ invertible matrices over the field $\text{GF}(q)$, with two matrices equivalent if and only if they are scalar multiples of each other. With a slight abuse of notation, we shall write that the matrices themselves are elements of $\text{PGL}(2, q)$, and that the elements of $G$ are pairs of matrices.

We identify $H \cong \text{AGL}(1, q)$ with the set of (equivalence classes of) lower triangular matrices $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$: $a, b, c \in \text{GF}(q)$, $ac \neq 0$ and define $\hat{H} := \text{Diag}(\text{AGL}(1, q) \times \text{AGL}(1, q)) = \{(h, h) : h \in H\}$.
For $a \in GF(q)$, let $p(a) := \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ and $\bar{p}(a) := (p(a), p(a)) \in \bar{H}$. Moreover, if $a \neq 0$ then let $d(a) := \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \in H$ and $d(a) := (d(a), d(a)) \in H$. Let $P = \{p(a), a \in GF(q)\}$, $\bar{P} = \{\bar{p}(a), a \in GF(q)\}$, $D = \{d(a), a \in GF(q)^*\}$, and $\bar{D} = \{\bar{d}(a), a \in GF(q)^*\}$. Then $H = PD$, $\bar{H} = \bar{P}D$, $P \triangleleft H$, and $\bar{P} \triangleleft \bar{H}$.

For $y \in GF(q)^*$, let $g(y) := \begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}$. Then, for $g = g(y_1, y_2) := (g(y_1), g(y_2)) \in G$ we have $g^2 = 1 \in H$, so we can define the coset graph $\Gamma = \Gamma(y_1, y_2) := Cos(G, \bar{H}, \bar{H}g\bar{H})$. Let $\alpha$ denote the vertex $\bar{H}$ and let $\beta$ denote the vertex $\bar{H}g$.

First, we determine the number of vertices, the valency, and the number of components of $\Gamma$. The number of vertices is $|G : \bar{H}| = |G|/|\bar{H}| = q(q - 1)(q + 1)^2$. For any $\bar{d} \in \bar{D}$ we have $\bar{d}^2 = \bar{d}^{-1}$, so $G_{\alpha\beta} = H \cap \bar{H}^s \geq \bar{D}$. Since $\bar{D}$ is a maximal subgroup of $\bar{H}$, we must have equality here, and so the valency of $\Gamma$ is $|G_{\alpha} : G_{\alpha\beta}| = |\bar{H} : \bar{D}| = q$.

For a vertex $\delta$ of $\Gamma$, let $W^{(r)}(\delta)$ denote the set of vertices reachable by an $r$-long walk from $\delta$. Then we have the following.

**Lemma 3.1.** $W^{(r)}(\alpha) = \bar{H}g\bar{P}\bar{g}\bar{P}\cdots\bar{g}\bar{P}$ (r iterations of $g\bar{P}$). For any vertex $\delta$, if $\alpha^h = \delta$ for some $h \in G$ then $W^{(r)}(\delta) = W^{(r)}(\alpha)h$.

**Proof.** It is well known (see for example [6, Lemma 9.3.1]) that $W^{(r)}(\alpha) = G_{\alpha}gG_{\alpha}g^r$ holds for any coset graph $Cos(G, G_{\alpha}, G_{\alpha}gG_{\alpha})$. In $\Gamma$, we can assert the stronger result in the statement of the lemma because $G_{\alpha} = \bar{H} = \bar{P}D$ and $\bar{D}$ normalizes $\bar{P}$ and $g$ normalizes $\bar{D}$. Hence $\bar{H}g\bar{H} = \{\bar{H}g\bar{P}d : p \in \bar{P}, \bar{d} \in \bar{D}\} = \{\bar{H}g\bar{P}d\bar{P}^{-1}g\bar{P}\bar{d} : \bar{P} = \bar{P}^{-1}\} = \bar{H}g\bar{P}$.

The second assertion of the lemma follows from the fact that if $\alpha^h = \delta$ then $h$ maps the $r$-long walks starting at $\alpha$ to the $r$-long walks starting at $\delta$ and, in particular, the endpoints of the walks starting at $\alpha$ to the endpoints of the walks starting at $\delta$. □

**Lemma 3.2.** Let $y_1, y_2 \in GF(q)^*$. If $y_1 = y_2$ then $\Gamma(y_1, y_2)$ has $q^3 - q$ components. If $y_1 \neq y_2$ then $\Gamma(y_1, y_2)$ has at most 2 components; it has two components if and only if $q$ is odd and $y_1y_2$ is a square in $GF(q)$.

**Proof.** The component containing $\alpha$ consists of the cosets reachable by some walk in $\Gamma$, that is, the cosets in $\bigcup_{r \geq 0} \bar{H}(g\bar{H})^r = \langle \bar{H}, g \rangle$. The subgroup $G^\ast := \langle \bar{H}, g \rangle$ projects surjectively on both coordinates of $G = PGL(2, q) \times PGL(2, q)$. If $y_1 = y_2$ then $G^\ast$ is a diagonal subgroup of $G$, of index $|PGL(2, q)| = q^3 - q$. If $y_1 \neq y_2$ then $G^\ast$ contains a nondiagonal subgroup $G^{**}$ that projects surjectively on both coordinates of $PSL(2, q) \times PSL(2, q)$, which can happen only when $G^{**} = PSL(2, q) \times PSL(2, q)$. If $q$ is even then $PGL(2, q) = PSL(2, q)$ and so $G = G^{**} = G^{**}$, implying that $\Gamma(y_1, y_2)$ is connected. If $q$ is odd then $|PGL(2, q) : PSL(2, q)| = 2$ and $G \geq G^\ast > G^{**}$ imply that $|G : G^\ast| \leq 2$ and so $\Gamma(y_1, y_2)$ has at most 2 components. Moreover, the cosets $PSL(2, q)y_1$ and $PSL(2, q)y_2$ are equal if and only if $y_1y_2$ is a square in $GF(q)$. Hence the image of the factor group homomorphism $\phi : G^\ast \to G^\ast/G^{**}$ is a diagonal subgroup of $G/G^{**} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, that is, $G \neq G^\ast$, if and only if $y_1y_2$ is a square. □

**Lemma 3.3.** For all $y_1, y_2 \in GF(q)^*$, the graph $\Gamma(y_1, y_2)$ is near-polygonal.

**Proof.** The group $\bar{H} = G_{\alpha}$ acts sharply 2-transitively on the cosets of $\bar{D} = G_{\alpha\beta}$ so, by Lemma 2.4, the graph $\Gamma = \Gamma(y_1, y_2)$ is sharply $(G, 2)$-dipath transitive. In order to apply
Lemma 1.1, we also need to show that \( \Gamma \) has a \( G \)-dihedral \((2, m)\)-cover for some \( m \). To this end, define
\[
f = f(y_1, y_2) := \bar{p}(1) \cdot (g(y_1), g(y_2)) = \left( \begin{bmatrix} 0 & y_1 \\ -1 & y_1 \end{bmatrix}, \begin{bmatrix} 0 & y_2 \\ -1 & y_2 \end{bmatrix} \right).
\]
We have \( f \in \tilde{H}g \setminus \tilde{D}g \) and so, by Lemma 2.3, the vertex
\[
y := \alpha f^{-1} = \tilde{H} \left( \begin{bmatrix} y_1 \\ 1 \\ -y_1 \\ 0 \end{bmatrix}, \begin{bmatrix} y_2 \\ 1 \\ -y_2 \\ 0 \end{bmatrix} \right)
\]
is different from \( \beta \) and \((y, \alpha, \beta)\) is a 2-dipath. Spinning this 2-dipath by \( f \) as in Construction 2.1 we obtain a cycle \( C = [\delta_0 = \alpha, \delta_1 = \beta, \delta_2, \ldots, \delta_{m-1} = y] \) for some \( m \), and the \((2, m)\)-cover \( \mathcal{C} = \mathcal{C}^G \). We claim that the group element \( z := \left( \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \) is an involution, satisfying \( \delta_i^z = \delta_{m-i} \) for \( 1 \leq i \leq m-1 \).

Next, we describe how to choose the values \( y_1, y_2 \) such that \( \Gamma(y_1, y_2) \) has a \( 1-(2, m) \) cover for some prescribed value \( m \geq 5 \). We define a sequence of polynomials \( u_n(y) \) as \( u_0(y) := 0 \),
\[
u_{2n+1}(y) := \sum_{k=0}^{n} (-1)^k \binom{2n-k}{k} y^{n-k}
\]
and
\[
u_{2n+2}(y) := \sum_{k=0}^{n} (-1)^k \binom{2n+1-k}{k} y^{n-k}
\]
for \( n \geq 0 \).

In particular, \( u_5(y) = y^2 - 3y + 1 \), \( u_6(y) = y^2 - 4y + 3 \), and \( u_7(y) = y^3 - 5y^2 + 6y - 1 \). As far as we know, these polynomials were first studied by Zara [10].

**Lemma 3.4.** Let \( k \geq 1 \) be an integer.

(a) The roots of \( u_k(y) \) in \( \mathbb{C} \) are the numbers \( \xi + \xi^{-1} + 2 \), where \( \xi \) is a \( k \)th root of unity different from \( \pm 1 \).

(b) For all \( k \geq 1 \), we have \( u_{2k}(y) = u_{2k-1}(y) - u_{2k-2}(y) \) and \( u_{2k+1}(y) = yu_{2k}(y) - u_{2k-1}(y) \).

(c) For all \( k \geq 0 \), we have
\[
u_{2k+1}(x + x^{-1} + 2) = \sum_{i=0}^{2k} x^{-k+i}
\]
and
\[
u_{2k+2}(x + x^{-1} + 2) = \sum_{i=0}^{k} x^{-k+2i}.
\]
Lemma 3.5. Let $m \geq 5$ be fixed, and let $q \equiv \pm 1 \pmod{m}$. Let $\xi_1$ and $\xi_2$ be different $m$th roots of unity in $\mathbb{GF}(q^2)$, at least one of them primitive, and suppose that $\xi_1$, $\xi_2$ satisfy $\xi_1^2 \neq 1$. Then $y_i := \xi_i + \xi_i^{-1} + 2 \in \mathbb{GF}(q)$ for $i = 1, 2$, $y_1 \neq y_2$, and the graph $\Gamma(y_1, y_2)$ has a 1-(2, $m$)-cover.

Proof. First we prove that $y_i \in \mathbb{GF}(q)$. Indeed, computing in $\mathbb{GF}(q^2)$ we have $y_i^q = (\xi_i + \xi_i^{-1} + 2)^q = \xi_i^q + \xi_i^{-q} + 2q = \xi_i^q + \xi_i^{-q} + 2$. If $q \equiv 1 \pmod{m}$ then $\xi_i = \rho(q+1)^j$ for a generator $\rho$ of $\mathbb{GF}(q^2)^*$ and for some integer $j$ and so $\xi_i^q = \rho(q^2+q)^j = \rho(1+q^2) = \xi_i$. Similarly, $\xi_i^{-q} = \xi_i^{-1}$. If $q \equiv -1 \pmod{m}$ then $\xi_i = \rho(q-1)^j$ and so $\xi_i^q = \rho(q^2-q)^j = \rho(1-q)j = \xi_i^{-1}$. Similarly, $\xi_i^{-q} = \xi_i$. In any case, $y_i^q = y_i$ implying $y_i \in \mathbb{GF}(q)$. We also have $y_1 \neq y_2$ because $y_1 - y_2 = (\xi_1 - \xi_2)(\xi_1 \xi_2 - 1)/(\xi_1 \xi_2) \neq 0$ by our assumptions on $\xi_1$ and $\xi_2$.

By Lemma 3.4(c), $u_m(y_i) = 0$ and then Lemma 3.4(b) implies $u_{m+1}(y_i) = -u_m(y_i)$ for $i = 1, 2$. Hence, by Lemma 3.4(d), for the spinning element $f = f(y_1, y_2)$ defined in (2) we have $f^m = 1$. Moreover, if $\xi_i$ is a primitive $m$th root of unity then $u_k(y_i) \neq 0$ for $1 \leq k < m$ because for any $k$th root of unity $\chi$, $y_i - (\chi + \chi^{-1} + 2) = (\xi_i - \chi)(\xi_i \chi - 1)/(\xi_i \chi) \neq 0$ and so $y_i$ differs from the roots of $u_k(y)$. Therefore, $f^k \notin \mathbb{H}$. This means that the cycle spinned by $f$ has length $m$ and so $\Gamma(y_1, y_2)$ has a 1-(2, $m$)-cover. □

Our next goal is to estimate the girth of $\Gamma$. By Lemma 3.1,

\[ W^{(1)}(\beta) = \bar{H} g \bar{P} g = \{ \bar{H} \left[ \begin{array}{cc} -1 & y_1 a_1 \\ 0 & -1 \end{array} \right], \left[ \begin{array}{cc} -1 & y_2 a_1 \\ 0 & -1 \end{array} \right] : a_1 \in \mathbb{GF}(q) \} , \]

\[ W^{(2)}(\beta) = \bar{H} g \bar{P} g \bar{P} g \]

\[ = \{ \bar{H} \left[ \begin{array}{cc} -y_1 a_1 & y_2 a_1 a_2 - y_1 \\ 1 & y_1 a_2 \end{array} \right], \left[ \begin{array}{cc} -y_2 a_1 & y_2 a_1 a_2 - y_2 \\ 1 & -y_2 a_2 \end{array} \right] : a_1, a_2 \in \mathbb{GF}(q) \} , \]

\[ W^{(1)}(\gamma) = \bar{H} g \bar{P} f^{-1} \]

\[ = \{ \bar{H} \left[ \begin{array}{cc} -1 - y_1 a_3 & y_1 a_3 \\ 1 & -1 \end{array} \right], \left[ \begin{array}{cc} -1 - y_2 a_3 & y_2 a_3 \\ 1 & -1 \end{array} \right] : a_3 \in \mathbb{GF}(q) \} , \]

and

\[ W^{(2)}(\gamma) = \bar{H} g \bar{P} g \bar{P} f^{-1} \]

\[ = \{ \bar{H} \left[ \begin{array}{cc} y_2 a_3 a_4 - y_1 + y_1 a_3 & y_1 - y_2 a_3 a_4 \\ -1 - y_1 a_4 & y_1 a_4 \end{array} \right], \left[ \begin{array}{cc} y_2 a_3 a_4 - y_2 + y_2 a_3 & y_2 - y_2 a_3 a_4 \\ -1 - y_2 a_4 & y_2 a_4 \end{array} \right] : a_3, a_4 \in \mathbb{GF}(q) \} . \]

Lemma 3.6. If $y_1 \neq y_2$ then the girth of $\Gamma(y_1, y_2)$ is at least 5.
Proof. Suppose first that \( \Gamma = \Gamma(y_1, y_2) \) contains a 3-cycle. Then, since \( \Gamma \) is \((G, 2)\)-dipath transitive, there is a 3-cycle containing the 2-path \([y, \alpha, \beta]\). This implies that \( y \) is a neighbor of \( \beta \) and, using the \((G, 2)\)-dipath transitive property for the 2-paths with middle vertex \( \alpha \), we obtain that every pair of vertices in \( N_{\Gamma}(\alpha) \) is adjacent. Hence the connected component containing \( \alpha \) is a complete graph \( K_{q+1} \), which cannot happen when \( y_1 \neq y_2 \).

Suppose now that \( \Gamma \) contains a 4-cycle. Then, since \( \Gamma \) is \((G, 2)\)-dipath transitive, there is a 4-cycle containing \([y, \alpha, \beta]\) and so \( W^{(1)}(\beta) \cap W^{(1)}(\gamma) \) contains a vertex different from \( \alpha \). However, we shall prove that \( W^{(1)}(\beta) \cap W^{(1)}(\gamma) = \{\alpha\} \). Indeed, by (4) and (6), \( |W^{(1)}(\beta) \cap W^{(1)}(\gamma)| \) is the number of solutions \((a_1, a_3)\) of the equation

\[
\tilde{H} \left( \begin{bmatrix} -1 & y_1 a_1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & y_2 a_1 \end{bmatrix} \right) = \tilde{H} \left( \begin{bmatrix} -1 - y_1 a_3 & y_1 a_3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 - y_2 a_3 & y_2 a_3 \end{bmatrix} \right).
\]

The ratio of the two coset representatives in (8) is

\[
r_4 := \left( \begin{bmatrix} 1 + y_1 a_3 & y_1 a_1 + y_2 a_1 a_3 - y_1 a_3 \\ -1 & -y_1 a_1 + 1 \end{bmatrix}, \begin{bmatrix} 1 + y_2 a_3 & y_2 a_1 + y_2 a_1 a_3 - y_2 a_3 \\ -1 & -y_2 a_1 + 1 \end{bmatrix} \right),
\]

and we must have \( r_4 \in \tilde{H} \). This implies that the two matrices in \( r_4 \) are scalar multiples of each other and, since the \((2, 1)\) entries are identical, the two matrices are actually equal. Using that \( y_1 \neq y_2 \), the comparison of the \((1, 1)\) entries gives \( a_3 = 0 \) and the comparison of the \((2, 2)\) entries gives \( a_1 = 0 \). Hence the only element in \( W^{(1)}(\beta) \cap W^{(1)}(\gamma) \) is \( \alpha \) and so there is no 4-cycle in \( \Gamma \). \( \square \)

Lemma 3.7. Let \( q = p^e \) be a prime power and \( y_1, y_2 \in \text{GF}(q) \). For \( y_1 \neq y_2 \), the number of 5-cycles containing a fixed 2-path in \( \Gamma(y_1, y_2) \) is

\[
\begin{cases} 
2 & \text{if } p = 3 \text{ and } y_1 = -y_2 \text{ and } -1 \text{ is a square in } \text{GF}(q), \\
1 & \text{if } p \neq 3 \text{ and } (y_1 + y_2)^2 = 9y_1 y_2, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. Since \( \Gamma = \Gamma(y_1, y_2) \) is \((G, 2)\)-dipath transitive, it is enough to find the number of 5-cycles containing \([y, \alpha, \beta]\). By Lemma 3.6, \( \Gamma \) contains no 3-cycles. Therefore, the 5-cycles containing \([y, \alpha, \beta]\) are in one-to-one correspondence with the elements of \( W^{(2)}(\beta) \cap W^{(1)}(\gamma) \) and, by (5) and (6), it is enough to find the number of solutions \((a_1, a_2, a_3)\) in \( \text{GF}(q)^3 \) of the equation

\[
\tilde{H} \left( \begin{bmatrix} -y_1 a_1 \\ 1 \end{bmatrix}, \begin{bmatrix} y_1^2 a_1 a_2 - y_1 \\ -y_1 a_2 \end{bmatrix} \right) = \tilde{H} \left( \begin{bmatrix} -1 & y_1 a_3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 - y_2 a_3 & y_2 a_3 \end{bmatrix} \right).
\]

The ratio of the two coset representatives in (10) is

\[
r := \left( \begin{bmatrix} r_{11}^{(1)} & r_{11}^{(2)} \\ r_{12}^{(1)} & r_{12}^{(2)} \end{bmatrix}, \begin{bmatrix} r_{11}^{(2)} & r_{11}^{(2)} \\ r_{12}^{(2)} & r_{12}^{(2)} \end{bmatrix} \right).
\]
with \( r^{(i)}_{11} = y_i a_1 + y_i - y_i^2 a_1 a_2, \) \( r^{(i)}_{12} = - y_i (y_i^2 a_1 a_2 a_3 + y_i (a_1 a_2 - a_3 - a_1 a_3) - 1), \) \( r^{(i)}_{21} = y_i a_2 - 1, \) \( r^{(i)}_{22} = y_i^2 a_2 a_3 - y_i a_3 + y_i a_2, \) for \( i = 1, 2. \) We have \( r \in \mathcal{H} \) if and only if the two matrices in \( r \) are scalar multiples of each other and the (1, 2) entries are 0, that is, the four equations

\[
\begin{align*}
    r^{(1)}_{12} &= 0, & r^{(2)}_{12} &= 0, & r^{(1)}_{11} r^{(2)}_{22} &= r^{(2)}_{11} r^{(1)}_{22}, & r^{(1)}_{11} r^{(2)}_{21} &= r^{(2)}_{11} r^{(1)}_{21}
\end{align*}
\]

are satisfied (note that \( r^{(1)}_{21} r^{(2)}_{22} = r^{(2)}_{21} r^{(1)}_{22} \) is a consequence of these four equations). Using that \( y_1, y_2 \neq 0 \) and \( y_1 \neq y_2, \) \( r^{(1)}_{11} r^{(2)}_{22} = r^{(2)}_{11} r^{(1)}_{22} \) is equivalent to

\[
a_2 (a_1 a_2 + a_3) = 0 \tag{13}
\]

and \( r^{(1)}_{11} r^{(2)}_{21} = r^{(2)}_{11} r^{(1)}_{21} \) is equivalent to

\[
(a_1 + 1) - (y_1 + y_2) a_1 a_2 + y_1 y_2 a_1 a_2^2 = 0. \tag{14}
\]

If \( a_2 = 0 \) then (14) implies \( a_1 = -1. \) Substituting these values into \( r^{(1)}_{12} = 0 \) we obtain the contradiction \( y_1 = 0, \) so \( a_2 \neq 0 \) and (13) is equivalent to \( a_3 = -a_1 a_2. \) Substituting this value into \( r^{(1)}_{12} = 0 \) and \( r^{(2)}_{12} = 0, \) we obtain

\[
y_1^2 a_1^2 a_2^2 - y_1 (2 a_1 a_2 + a_1^2 a_2) + 1 = 0, \tag{15}
\]

\[
y_2^2 a_1^2 a_2^2 - y_2 (2 a_1 a_2 + a_1^2 a_2) + 1 = 0. \tag{16}
\]

Conversely, if \( a_1, a_2 \) satisfy (14)–(16) then \( a_1, a_2 \) and \( a_3 := -a_1 a_2 \) satisfy (12) so it is enough to find the number of solutions of (14)–(16).

Multiplying (15) by \( y_2/(y_1 - y_2) \) and (16) by \(-y_1/(y_1 - y_2), \) and adding them we obtain

\[
y_1 y_2 a_1^2 a_2^2 - 1 = 0. \tag{17}
\]

Multiplying (15) by \( 1/(y_1 - y_2) \) and (16) by \(-1/(y_1 - y_2), \) and adding them we obtain

\[
(y_1 + y_2) a_1^2 a_2^2 - 2 a_1 a_2 - a_1^2 a_2 = 0. \tag{18}
\]

Moreover, since the coefficient matrix

\[
\begin{bmatrix}
    y_2 & 1
    y_1 - y_2 & y_1 - y_2
\end{bmatrix}
\]

of these linear combinations is invertible, the system of equations (14), (17), (18) is equivalent to (14)–(16). From (17) we see that \( a_1 a_2 \neq 0. \) Dividing (18) by \( a_1 a_2 \) and adding the result to (14) yields \( y_1 y_2 a_1 a_2^2 - 1 = 0; \) comparing that with (17), we obtain \( a_1 = 1. \) Hence, using \( a_2 \neq 0, \) (14), (17), (18) simplifies to the two equations \( a_2 (y_1 + y_2) = 3 \) and \( y_1 y_2 a_2^2 = 1. \) If \( q \) is not a power of 3 then these equations have at most one solution \( a_2, \) and a solution exists if and only if \( y_1 \neq -y_2 \) and \( a_2^2 = (3/(y_1 + y_2))^2 = 1/(y_1 y_2), \) that is, \( 9 y_1 y_2 = (y_1 + y_2)^2. \) If \( q \) is a power of 3 then \( a_2 (y_1 + y_2) = 3 \) has solutions if and only if \( y_1 + y_2 = 0, \) (since we already have \( a_2 \neq 0), \) and in this case \( a_2^2 = -1/y_1^2 \) has 0 or 2 solutions \( a_2, \) depending on whether \( -1 \) is a nonsquare or a square in \( \text{GF}(q). \)

\[\square\]

**Lemma 3.8.** Let \( q = p^e \) be a prime power and \( y_1, y_2 \in \text{GF}(q)^*. \) For \( y_1 \neq y_2, \) the number of 6-cycles containing a fixed 2-path in \( \Gamma(y_1, y_2) \) is

\[
\begin{cases}
    2 & \text{if } p \neq 2 \text{ and } 10 y_1 y_2 - 3 (y_1^2 + y_2^2) \text{ is a nonzero square in } \text{GF}(q), \\
    1 & \text{if } p \neq 2 \text{ and } 10 y_1 y_2 - 3 (y_1^2 + y_2^2) = 0, \\
    0 & \text{otherwise.}
\end{cases}
\]
Proof. Since \( \Gamma = \Gamma(y_1, y_2) \) is \((G, 2)\)-dipath transitive, it is enough to find the number of 6-cycles containing \([y_1, y_2]\). By Lemma 3.6, \( \Gamma \) contains no 3-cycles and 4-cycles, so for each \( \delta \in W(2)(\beta) \cap W(2)(\gamma) \) we have that either \( \delta \) is a neighbor of \( \alpha \) or \( \delta \) is the vertex opposite of \( \alpha \) in a 6-cycle containing \([y_1, y_2]\). For \( \delta \in W(2)(\beta) \), \( \delta \) is a neighbor of \( \alpha \) if and only if \( a_1a_2 = 0 \) in the parametrization (5) of \( W(2)(\beta) \). Similarly, for \( \delta \in W(2)(\gamma) \), \( \delta \) is a neighbor of \( \alpha \) if and only if \( a_3a_4 = 0 \) in the parametrization (7) of \( W(2)(\gamma) \). Hence the number of 6-cycles containing \([y_1, y_2]\) is the number of solutions \((a_1, a_2, a_3, a_4) \in (GF(q)^* \setminus \{0\})^4 \) of the equation

\[
\begin{align*}
\bar{H} &\left( \left[ \begin{array}{cc} -y_1a_1 & y_1^2a_1a_2 - y_1 \\ -y_1a_2 & -y_2a_2 \end{array} \right], \left[ \begin{array}{cc} -y_2a_1 & y_2^2a_1a_2 - y_2 \\ -y_2a_2 & -y_2a_2 \end{array} \right] \right) \\
&= \bar{H} \left( \left[ \begin{array}{cc} y_1^2a_3a_4 - y_1 + y_1a_3 & y_1 - y_1^2a_3a_4 \\
-1 - y_1a_4 & y_1a_4 \end{array} \right], \left[ \begin{array}{cc} y_2^3a_3a_4 - y_2 + y_2a_3 & y_2 - y_2^2a_3a_4 \\
-1 - y_2a_4 & y_2a_4 \end{array} \right] \right).
\end{align*}
\]

The ratio \( r \) of the two coset representatives in (19) is

\[
\begin{align*}
r &:= \left( \begin{array}{cc}
r_{11}^{(1)} & r_{12}^{(1)} \\
r_{21}^{(1)} & r_{22}^{(1)}
\end{array} \right) \cdot \left( \begin{array}{cc}
r_{11}^{(2)} & r_{12}^{(2)} \\
r_{21}^{(2)} & r_{22}^{(2)}
\end{array} \right)
\end{align*}
\]

with \( r_{11}^{(i)} = -y_1^3a_2a_3a_4 + y_1^2(a_2 - a_2a_3 + a_3a_4) - y_1, r_{12}^{(i)} / y_i^2 = -y_1^2a_1a_2a_3a_4 + y_i(a_1a_2 - a_3a_4 - a_1a_2a_3 + a_1a_3a_4) + (a_3 - 1 - a_1), r_{21}^{(i)} = y_i(a_2 - a_4) + y_1^2a_2a_4, r_{22}^{(i)} = y_i^3a_1a_2a_4 + y_1^2(a_1a_2 - a_1a_4 - a_4) - y_1, \) for \( i = 1, 2 \). We have \( r \in \bar{H} \) if and only if the two matrices in \( r \) are scalar multiples of each other and the (1, 2) entries are 0, that is, the four equations

\[
\begin{align*}
r_{12}^{(1)} / y_1^2 &= 0, & r_{12}^{(2)} / y_2^2 &= 0, & r_{11}^{(1)} r_{22}^{(2)} &= r_{11}^{(2)} r_{22}^{(1)}, & r_{22}^{(1)} r_{21}^{(2)} &= r_{22}^{(2)} r_{21}^{(1)}
\end{align*}
\]

are satisfied.

The following computations were performed with the help of Mathematica [9]. As in the proof of Lemma 3.7, we take the linear combination of \( r_{12}^{(1)} / y_1^2 = 0 \) and \( r_{12}^{(2)} / y_2^2 = 0 \) with coefficients \( y_2 / (y_1 - y_2) \) and \(-y_1 / (y_1 - y_2)\), and then with coefficients \( 1 / (y_1 - y_2) \) and \(-1 / (y_1 - y_2)\). So we obtain the equations

\[
\begin{align*}
y_1y_2a_1a_2a_3a_4 + (a_3 - 1 - a_1) &= 0, \quad \text{(22)} \\
(a_1a_2 + a_3a_4 - a_1a_2a_3 + a_1a_3a_4) - (y_1 + y_2)a_1a_2a_3a_4 &= 0. \quad \text{(23)}
\end{align*}
\]

We eliminate \( a_1 \) by multiplying (22) by the coefficient of \( a_1 \) in (23), multiplying (23) by the coefficient of \( a_1 \) in (22), and taking the difference, yielding

\[
a_2(-1 + a_3 + a_3a_4y_1)(-1 + a_3 + a_3a_4y_2) = a_2^2a_4. \quad \text{(24)}
\]

At that point, we distinguish two cases: \( a_3 = 1 \) or \( a_3 \neq 1 \).

If \( a_3 = 1 \) then both (22) and (24) simplify to \( y_1y_2a_2a_4 = 1 \) and (23) simplifies to \( a_1(1 - (y_1 + y_2)a_2) = -1 \). We can solve the first of these equations for \( a_2 \) and then the second for \( a_1 \), yielding

\[
a_2 = \frac{1}{y_1y_2a_4} \quad \text{and} \quad a_1 = \frac{y_1y_2a_4}{y_1 + y_2 - y_1y_2a_4}. \quad \text{(25)}
\]

Note that the numerator of the expressions for \( a_1 \) and \( a_2 \) are not 0 and so, if a solution \((a_1, a_2, a_3, a_4) \in (GF(q)^* \setminus \{0\})^4 \) exists, then we can perform the divisions.
Substituting the values for \( a_1, a_2, a_3 \) into \( r_{11}^{(i)}, r_{21}^{(i)}, r_{22}^{(i)} \) in (20), the equation \( r_{11}^{(1)} r_{22}^{(2)} - r_{22}^{(1)} r_{11}^{(2)} = 0 \) simplifies to \( y_1^2 - y_2^2 = 0 \). Since \( y_1 \neq y_2 \), this implies \( y_2 = -y_1 \) and, in particular, \( p \neq 2 \). Also, (25) yields \( a_1 = -1 \). The equation \( r_{21}^{(1)} r_{22}^{(2)} - r_{22}^{(1)} r_{21}^{(2)} = 0 \) yields \( 2(1 + a_4^2 y_1^2)/(a_4^2 y_1) = 0 \) and so \( 1 + a_4^2 y_1^2 = 0 \). From (25) we also have \( y_2 = a_2 + a_4 y_1^2 = 0 \), implying \( a_2 = a_4 \).

So far, we have obtained that if a solution of (21) with \( a_3 = 1 \) exists then necessarily \( y_1 + y_2 = 0 \), \( p \neq 2 \), \( a_1 = -1 \), and \( a_2 = a_4 \) with \( a_4^2 = -1/y_1^2 \). The last of these conditions implies that \(-1\) is a square in \( \text{GF}(q) \) and so \( 10y_1y_2 - 3(y_1^2 + y_2^2) = -16y_1^2 \) is a nonzero square in \( \text{GF}(q) \). Conversely, if \( p \neq 2 \), \( y_1 + y_2 = 0 \), and \( 10y_1y_2 - 3(y_1^2 + y_2^2) = -16y_1^2 \) is a nonzero square then substitution into (20) shows that \( a_1 = -1, a_3 = 1, a_2 = a_4 = \pm \sqrt{-1/y_1} \) are two solutions.

Now we consider solutions \((a_1, a_2, a_3, a_4) \in (\text{GF}(q)^*)^4 \) with \( a_3 \neq 1 \). In this case, (24) yields

\[
a_2 = \frac{a_2^2 a_4}{(-1 + a_3 + a_3 a_4 y_1)(-1 + a_3 + a_3 a_4 y_2)}
\]

and so, from (22),

\[
a_1 = \frac{a_3 - 1}{1 - y_1 y_2 a_2 a_3 a_4} = -\frac{(-1 + a_3 + a_3 a_4 y_1)(-1 + a_3 + a_3 a_4 y_2)}{1 - a_3 - a_3 a_4 y_1 + y_2 + a_3^2 a_4^2 y_1 y_2}.
\]

Substituting these values into \( r_{11}^{(1)} r_{22}^{(2)} = r_{11}^{(2)} r_{22}^{(1)} \), we obtain

\[
0 = \frac{a_3 a_4 y_1 y_2 (y_1 - y_2)(-2 + 2a_3 + a_3 a_4 (y_1 + y_2))}{(-1 + a_3 + a_3 a_4 y_1)(-1 + a_3 + a_3 a_4 y_2)},
\]

implying

\[
a_4 = \frac{2(1 - a_3)}{a_3 (y_1 + y_2)}.
\]

Finally, substituting the values for \( a_4, a_2, a_1 \) from (28), (26), (27) into \( r_{21}^{(1)} r_{22}^{(2)} = r_{21}^{(2)} r_{22}^{(1)} \), we obtain

\[
0 = \frac{4y_1 y_2 (a_3^2 (y_1 - y_2)^2 + a_3 (-y_1^2 + 6y_1 y_2 - y_2^2) + (y_1 - y_2)^2)}{(1 - a_3) a_3 (y_1 - y_2)((y_1 - y_2)^2 + 4a_3 y_1 y_2)},
\]

that is, \( a_3 \) is a solution of the quadratic equation

\[
0 = a_3^2 (y_1 - y_2)^2 + a_3 (-y_1^2 + 6y_1 y_2 - y_2^2) + (y_1 - y_2)^2.
\]

So far, we have obtained that if a solution \((a_1, a_2, a_3, a_4) \in (\text{GF}(q)^*)^4 \) of (21) with \( a_3 \neq 1 \) exists then \( q \) is not a power of \( 2 \) (otherwise \( a_4 = 0 \) from (28), \( y_1 + y_2 \neq 0 \) (otherwise the denominator of \( a_4 \) is 0), and \( a_3 \) is the solution of the quadratic equation (29), implying that the discriminant \((-y_1^2 + 6y_1 y_2 - y_2^2)^2 - 4(y_1 - y_2)^4 = (y_1 + y_2)^2(10y_1 y_2 - 3(y_1^2 + y_2^2)) \)) is a square in \( \text{GF}(q) \). Conversely, suppose that \( p \neq 2 \), \( y_1 + y_2 \neq 0 \), and \( 10y_1 y_2 - 3(y_1^2 + y_2^2) \) is a square in \( \text{GF}(q) \). Then Eq. (29) has one or two solutions, depending on whether \( 10y_1 y_2 - 3(y_1^2 + y_2^2) = 0 \). Moreover, none of the solutions are equal to 1, because substituting \( a_3 = 1 \) into (29) yields \( 0 = (y_1 + y_2)^2 \), contradicting our assumptions. Fixing a solution \( a_3 \) of (29) defines a value \( a_4 \neq 0 \) from (28). Using this value for \( a_4 \) in (26) gives

\[
a_2 = \frac{2a_3 (y_1 + y_2)}{(a_3 - 1)(y_1 - y_2)^2} \neq 0.
\]
Our final substitution is into (27), yielding

\[ a_1 = \frac{(a_3 - 1)(y_1 - y_2)^2}{(y_1 - y_2)^2 + 4a_3y_1y_2}. \]  

(31)

The denominator in (31) is not 0, because subtracting the equation \(0 = (y_1 - y_2)^2 + 4a_3y_1y_2\) from (29) yields \(0 = (a_3^2 - a_3)(y_1 - y_2)^2\), contradicting \(a_3 \in \{0, 1\}\). Substituting the values for \(a_1, a_2, a_4\) obtained from (31), (30), (28), respectively, into (20), using (29) we obtain that the equations in (21) are satisfied. □

We need one more result about solutions of equations in \(\mathbb{GF}(q)\).

**Lemma 3.9.** Let \(q = p^e\) be a prime power. Then there exist distinct \(y_1, y_2 \in \mathbb{GF}(q)\) that are solutions of the system of equations

\[
\begin{align*}
y_1^3 - 5y_1^2 + 6y_1 - 1 &= 0, \\
y_2^3 - 5y_2^2 + 6y_2 - 1 &= 0, \\
(y_1 + y_2)^2 &= 9y_1y_2 \\
\end{align*}
\]

(32) (33) (34)

if and only if \(p = 41\), and in this case the only solution is \(\{y_1, y_2\} = \{32, 39\}\).

**Proof.** Suppose that \(y_1, y_2 \in \mathbb{GF}(q)\) are solutions. Then, subtracting (33) from (32) and dividing by \(y_1 - y_2 \neq 0\), we obtain

\[ y_1^2 + y_1y_2 + y_2^2 - 5(y_1 + y_2) + 6 = 0 \]

and so, using (34) to substitute for \(y_1^2 + y_2^2\),

\[ 8y_1y_2 + 6 = 5(y_1 + y_2). \]  

(35)

Adding (32) and (33) yields

\[ (y_1 + y_2)(y_1^2 - y_1y_2 + y_2^2) - 5(y_1^2 + y_2^2) + 6(y_1 + y_2) - 2 = 0 \]

and using (34) again we obtain

\[ (y_1 + y_2)(6y_1y_2 + 6) - 35y_1y_2 - 2 = 0. \]  

(36)

On one hand, multiplying (36) by 5 and using (35) to eliminate \(5(y_1 + y_2)\) yields the second order equation

\[ 48(y_1y_2)^2 - 91y_1y_2 + 26 = 0 \]  

(37)

for \(y_1y_2\). On the other hand, squaring (35) and using (34) to eliminate \((y_1 + y_2)^2\) yields the second order equation

\[ 64(y_1y_2)^2 - 129y_1y_2 + 36 = 0 \]  

(38)

for \(y_1y_2\). Four times (37) minus three times (38) gives

\[ 23y_1y_2 - 4 = 0. \]  

(39)

Thus (37) simplifies to \(48(y_1y_2)^2 - 91y_1y_2 + 26 = 2(y_1y_2)^2 + (2y_1y_2)(23y_1y_2) - 4(23y_1y_2) + y_1y_2 + 26 = 2(y_1y_2)^2 + 9y_1y_2 + 10 = 0\) and (38) simplifies to \(64(y_1y_2)^2 - 129y_1y_2 + 36 = 0\).
(3y_1y_2)(23y_1y_2) - 5(y_1y_2)^2 - 5(23y_1y_2) - 14y_1y_2 + 36 = -5(y_1y_2)^2 - 2y_1y_2 + 16 = 0. Multiplying these simplified equations by 5 and 2, respectively, and adding them, we obtain

\[ 41(y_1y_2 + 2) = 0. \] (40)

If \( p \neq 41 \) then \( y_1y_2 = -2 \) and from (39) we obtain \(-50 = 0\), that is, \( p = 2 \) or \( p = 5 \). If \( p = 2 \) then \( y_1y_2 = 0 \) and so at least one of \( y_1, y_2 \) is 0. However, 0 is not a solution of (32) and (33), so this case is impossible. If \( p = 5 \) then (34) implies \((y_1 - y_2)^2 = 5y_1y_2 = 0\), contradicting our assumption that \( y_1 \neq y_2 \). Hence \( p \) cannot be 5. Finally, if \( p = 41 \) then (39) implies \( y_1y_2 = 18 \), and (35) gives \( y_1 + y_2 = 30 \). Hence \( y_1, y_2 \) must be the solutions of the quadratic equation \( y^2 - 30y + 18 = 0 \), which are 32 and 39. Conversely, if \( p = 41 \) then substitution into (32)–(34) shows that \( \{y_1, y_2\} = \{32, 39\} \) is indeed a solution. \( \square \)

Now we are in position to prove of our main results. Theorem 1.3 is proved by Lemma 3.5.

**Proof of Theorem 1.4.** Let \( q = p^e \equiv \pm 1 \pmod{5} \) be a prime power, and let \( y_1, y_2 \) be defined as in Lemma 3.5. Then \( \Gamma(y_1, y_2) \) has a 1-(2, 5)-cover. By Lemma 3.6, \( \Gamma(y_1, y_2) \) contains no 3-cycles and 4-cycles, so its girth is 5 and \( \Gamma(y_1, y_2) \) is a polygonal graph. Since \( y_1 + y_2 = 3 \) and \( y_1y_2 = 1 \), we have \((y_1 + y_2)^2 = 9y_1y_2 \). Therefore, by Lemma 3.7, if \( p \neq 3 \) then each 2-path of \( \Gamma(y_1, y_2) \) contains an exactly one 5-cycle and so \( \Gamma(y_1, y_2) \) is strict polygonal. If \( p = 3 \) then \( y_1, y_2 \) are the solutions of \( y^2 + 1 = 0 \), implying that \( y_1 = -y_2 \) and \(-1 \) is a square in \( \text{GF}(q) \). Hence, by Lemma 3.7, \( \Gamma(y_1, y_2) \) is not strict polygonal and the 5-cycles of \( \Gamma(y_1, y_2) \) constitute a 2-(2, 5)-cover. \( \square \)

**Proof of Theorem 1.5.** Let \( q = p^e \equiv \pm 1 \pmod{6} \) be a prime power. Then \( y_1 = 1, y_2 = 3 \) satisfy the conditions for \( y_1, y_2 \) described in Lemma 3.5 and so \( \Gamma(1, 3) \) has a 1-(2, 6)-cover. By Lemma 3.6, \( \Gamma(1, 3) \) contains no 3-cycles and 4-cycles. If \( p \neq 11 \) then \((y_1 + y_2)^2 = 16 \) is not equal to \( 9y_1y_2 = 27 \) so, by Lemma 3.7, \( \Gamma(1, 3) \) contains no 5-cycles as well. Hence the girth of \( \Gamma(1, 3) \) is 6 and \( \Gamma(1, 3) \) is a polygonal graph. Moreover, we have \( 10y_1y_2 - 3(y_1^2 + y_2^2) = 0 \) and so, by Lemma 3.8, \( \Gamma(1, 3) \) is strict polygonal. If \( p = 11 \) then Lemma 3.8 still implies that the 6-cycles in \( \Gamma(1, 3) \) cover all 2-paths exactly once; however, by Lemma 3.7, so do the 5-cycles. In fact, if \( p = 11 \) then the solutions of \( u_5(y) = y^2 - 3y + 1 = 0 \) in \( \text{GF}(q) \) are 9 and 5, and

\[
\tilde{H}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}) = \tilde{H}(\begin{bmatrix} 0 & 9 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ -1 & 0 \end{bmatrix}).
\]

Hence the graph \( \Gamma(1, 3) \) is the same as the strict polygonal graph of girth 5 constructed in the proof of Theorem 1.4. \( \square \)

**Proof of Theorem 1.6.** Let \( q = p^e \equiv \pm 1 \pmod{7} \) be a prime power, and let \( y_1, y_2 \) be defined as in Lemma 3.5. Then \( \Gamma(y_1, y_2) \) has a 1-(2, 7)-cover. By Lemma 3.6, the girth of \( \Gamma(y_1, y_2) \) is at least 5.

If \( p = 2 \) then \( \Gamma(y_1, y_2) \) contains no 5-cycles by Lemmas 3.7 and 3.9, and contains no 6-cycles by Lemma 3.8. Hence \( \Gamma(y_1, y_2) \) is a polygonal graph of girth 7. Suppose that \( p \) is odd and \( p \neq 41 \). Then Lemmas 3.7 and 3.9 imply that \( \Gamma(y_1, y_2) \) contains no 5-cycles (note that in the case \( p = 3 \), (34) is equivalent to \( y_1 = -y_2 \) and so Lemmas 3.7 and 3.9 can be applied in this case as well). Moreover, if \( y_1, y_2 \) satisfy the extra condition that \( 10y_1y_2 - 3(y_1^2 + y_2^2) \) is a nonsquare in \( \text{GF}(q) \) then Lemma 3.8 implies that \( \Gamma(y_1, y_2) \) contains no 6-cycles and so \( \Gamma(y_1, y_2) \) is a polygonal graph of girth 7.
Finally, if $p = 41$ then 16, 32, and 39 are solutions of $y^3 - 5y^2 + 6y - 1 = 0$ in $\text{GF}(41)$, and therefore in $\text{GF}(41^e)$ for any $e$. If $e$ is even then for any two solutions $y_1, y_2, 10y_1y_2 - 3(y_1^2 + y_2^2) \in \text{GF}(41)$ is a square in $\text{GF}(41^e)$. Hence if there exist two solutions $y_1, y_2$ satisfying (1) then $e$ is odd, and in this case computations in $\text{GF}(41)$ give that $\{y_1, y_2\} = \{16, 39\}$ or $\{y_1, y_2\} = \{32, 39\}$. Now Lemmas 3.7–3.9 imply that $\Gamma(16, 39)$ is a polygonal graph of girth 7. This finishes the proof of Theorem 1.6. \[\square\]

4. Locally $s$-path transitive graphs

In this section we prove a slight generalization of Theorem 1.7. Let $\Gamma$ be a graph and $G \leq \text{Aut}(\Gamma)$. If for a positive integer $s$ and for all vertices $\alpha \in V$, $G\alpha$ is transitive on the $s$-arcs or $s$-dipaths of $\Gamma$ starting at the vertex $\alpha$, then $\Gamma$ is said to be locally $(G, s)$-arc transitive or locally $(G, s)$-dipath transitive, respectively.

We need the following simple lemma, which was proved in [7]. Since the proof is very short, we present it for the sake of completeness.

Lemma 4.1. [7, 7.61] Let $\Gamma$ be a locally $s$-arc transitive graph of girth $g$ and minimal valency at least 3. Then $g \geq 2s - 2$ or, equivalently, $s \leq (g + 2)/2$.

**Proof.** By the definition of girth, $\Gamma$ contains a cycle $[\alpha_0, \alpha_1, \ldots, \alpha_{g-1}]$ of length $g$. Let $\gamma$ be a neighbor of $\alpha_{g-1}$, different from $\alpha_{g-2}$ and $\alpha_0$. Then the $g$-arc $(\alpha_0, \alpha_1, \ldots, \alpha_{g-1}, \alpha_0)$ cannot be mapped to the $g$-arc $(\alpha_0, \alpha_1, \ldots, \alpha_{g-1}, \gamma')$ by a graph automorphism, and so $s < g$. Let $\beta$ be a neighbor of $\alpha_{s-1}$, different from $\alpha_{s-2}$ and $\alpha_s$. Since $\Gamma$ is locally $s$-arc transitive, there is an automorphism $\sigma$ of $\Gamma$ mapping the $s$-arc $(\alpha_0, \alpha_1, \ldots, \alpha_{s-1}, \alpha_s)$ to the $s$-arc $(\alpha_0, \alpha_1, \ldots, \alpha_{s-1}, \beta)$. Then $\sigma$ maps the $(g - s + 1)$-arc $(\alpha_{s-1}, \alpha_s, \ldots, \alpha_{g-1}, \alpha_0)$ to another $(g - s + 1)$-arc which starts at $\alpha_{s-1}$ and ends at $\alpha_0$. These two $(g - s + 1)$-arcs form a closed walk, and hence there exists a cycle of length at most $2(g - s + 1)$. Therefore $g \leq 2(g - s + 1)$, and so $g \geq 2s - 2$. \[\square\]

The following result proves Theorem 1.7.

Theorem 4.2. Let $\Gamma$ be a connected graph of minimal valency at least 3 and let $t$ be a positive integer. Then $\Gamma$ is locally $s$-dipath transitive for all $s \leq t$ if and only if one of the following holds:

(i) $\Gamma$ is locally $t$-arc transitive;
(ii) $\Gamma \cong K_n$ for $n \geq 4$, and $3 \leq t \leq n - 1$;
(iii) $\Gamma \cong K_{m,n}$ for $m \geq n \geq 3$, and $4 \leq t \leq 2n - 1$.

**Proof.** Suppose that $\Gamma$ is locally $l$-dipath transitive for all $l \leq t$, but not locally $t$-arc transitive. Let $s$ be the smallest integer such that $\Gamma$ is locally $s$-dipath transitive but not locally $s$-arc transitive. Let $g$ be the girth of $\Gamma$. We note that an $l$-arc is an $l$-dipath if and only if it consists of distinct vertices. Thus, if $l < g$, then all $l$-arcs are $l$-dipaths; in particular, 1-arcs are exactly the 1-dipaths, and 2-arcs are exactly the 2-dipaths. Therefore, local 2-dipath transitivity is equivalent to local 2-arc transitivity, and hence $s \geq 3$. By the definition of $s$, $\Gamma$ contains some $s$-arcs that are not $s$-dipaths, and it follows that $g \leq s$.

By the definition of $s$, $\Gamma$ is locally $(s - 1)$-arc transitive; in particular, $\Gamma$ is locally 2-arc transitive. Hence, by Lemma 4.1, we have $g \geq 2(s - 1) - 2 = 2s - 4$, and so

$$2s - 4 \leq g \leq s.$$  

(41)

Thus $3 \leq s \leq 4$ and, in particular, we have that $g = 3$ or 4.
Assume that $g = 3$. Then (41) implies $2s - 4 \leq 3$ and so $s = 3$. Let $[\alpha, \beta, \gamma]$ be a 3-cycle in $\Gamma$. Now, since $\Gamma$ is locally 2-arc transitive, any 2-arc $(\alpha, \delta_1, \delta_2)$ can be mapped to $(\alpha, \beta, \gamma)$ by a graph automorphism and so $\delta_2$ is adjacent to $\alpha$. Hence $W^{(2)}(\alpha) = W^{(1)}(\alpha)$ and, since $\Gamma$ is connected, it follows that

$$\Gamma \cong K_n, \quad \text{where } n \geq 4.$$ 

Assume now that $g = 4$. Then (41) implies $4 \leq s$ so $s = 4$. Thus $\Gamma$ is locally 3-arc transitive. Let $[\alpha, \beta, \gamma, \delta]$ be a 4-cycle in $\Gamma$, and write $W^{(1)}(\alpha) = \{\beta_0 = \beta, \beta_1 = \delta, \ldots, \beta_{k-1}\}$. Let $G = \text{Aut}(\Gamma)$, and let $G_{\alpha \beta \gamma} = G_\alpha \cap G_\beta \cap G_\gamma$. Since $\Gamma$ is a locally 3-arc transitive graph, considering the 3-arcs $(\gamma, \beta, \alpha, \beta_i)$ we conclude that $G_{\alpha \beta \gamma}$ is transitive on $W^{(1)}(\alpha) \setminus \{\beta\} = \{\beta_1, \beta_2, \ldots, \beta_{k-1}\}$ and all $\beta_j$, $0 \leq j \leq k - 1$, are adjacent to $\gamma$. Thus $W^{(1)}(\alpha) \subseteq W^{(1)}(\gamma)$. Reversing the role of $\alpha$ and $\gamma$ in this argument, we obtain that $W^{(1)}(\alpha) = W^{(1)}(\gamma)$. Let $\Delta_1 = [\alpha] \cup W^{(2)}(\alpha)$, and let $\Delta_2 = W^{(1)}(\alpha)$. Then each vertex in $\Delta_1$ is adjacent to all vertices in $\Delta_2$. Similarly, each vertex in $\Delta_2$ is adjacent to all vertices in $\Delta_1$. Since $\Gamma$ is connected, $\Gamma$ is a complete bipartite graph with parts $\Delta_1$ and $\Delta_2$, and since the minimal valency is at least 3, we have

$$\Gamma \cong K_{m,n}, \quad \text{where } 3 \leq n \leq m \text{ and } 4 \leq t \leq 2n - 1.$$ 

This proves the theorem. \qed

It was proved by Weiss [8] that if a regular graph of valency at least 3 is $s$-arc transitive but not $(s + 1)$-arc transitive then either $s \leq 5$ or $s = 7$. Thus an immediate consequence is

**Corollary 4.3.** Let $\Gamma$ be a regular graph of valency at least 3. Assume that $\Gamma$ is $l$-dipath transitive for all $l \leq s$ but not $(s + 1)$-dipath transitive. Then one of the following holds:

1. $\Gamma$ is $s$-arc transitive but not $(s + 1)$-arc transitive, and either $s \leq 5$ or $s = 7$;
2. $\Gamma \cong K_{s+1}$;
3. $s$ is odd, and $\Gamma \cong K_{\frac{s+1}{2}, \frac{s+1}{2}}$.

**References**