Functional fractional boundary value problems
with singular $\phi$-Laplacian

Alberto Cabada$^a$, Svatoslav Staněk$^b$

$^a$Department of Mathematical Analysis, Faculty of Mathematics, University of Santiago de Compostela, Santiago de Compostela, Spain
 e-mail: alberto.cabada@usc.es

$^b$Department of Mathematical Analysis, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic
 e-mail: svatoslav.stanek@upol.cz

Abstract. This paper discusses the existence of solutions of the fractional differential equations $^cD^\mu(\phi(^cD^\alpha u)) = Fu$, $^cD^\mu(\phi(^cD^\alpha u)) = f(t, u, ^cD^\nu u)$ satisfying the boundary conditions $u(0) = A(u)$, $u(T) = B(u)$. Here $\mu, \alpha \in (0, 1]$, $\nu \in (0, \alpha]$, $^cD$ is the Caputo fractional derivative, $\phi \in C(-a, a)$ ($a > 0$), $F$ is a continuous operator, $A, B$ are bounded and continuous functionals and $f \in C([0,T] \times \mathbb{R}^2)$. The existence results are proved by the Leray-Schauder degree theory.

Key words. Fractional differential equation; functional boundary value problem; singular $\phi$-Laplacian; Caputo derivative; Leray-Schauder degree.

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1 Introduction

C. Bereanu and J. Mawhin [1] studied the functional differential equation

$$(\phi(u'(t)))' = (Hu)(t) \quad (1.1)$$

and the differential equation

$$(\phi(u'(t)))' = h(t, u(t), u'(t)) \quad (1.2)$$

subject to the nonhomogeneous Dirichlet boundary conditions

$$u(0) = A, \quad u(T) = B \quad (A, B \in \mathbb{R}). \quad (1.3)$$

Here $\phi$ satisfies the condition
(φ) $\phi \in C(-a,a)$ ($a \in (0,\infty)$), $\phi$ is increasing and such that $\phi(0) = 0$ and 
$\lim_{x \to \pm a} \phi(x) = \pm \infty$,

(call it singular). By the Leray-Schauder degree method, they proved that
if $H : C^1[0,T] \to C[0,T]$ is continuous and takes bounded sets into bounded
sets and if $|B - A| < aT$, then problem (1.1), (1.3) has a solution. Hence an
immediately consequence is that if $h \in C([0,T] \times \mathbb{R}^2)$ and if $|B - A| < aT$, then
problem (1.2), (1.3) has a solution. The case $A = B = 0$ was discussed in [2]
The purpose of this paper is to show that the procedures given in [1] can be
applied to fractional functional differential equations with singular $\phi$-Laplacian
and functional boundary conditions.

We discuss the fractional functional differential equation

$$^cD^\mu(\phi(D^\alpha u(t))) = (Fu)(t), \quad (1.4)$$

and the fractional differential equation

$$^cD^\mu(\phi(D^\alpha u(t))) = f(t,u(t), \, ^cD^\nu u(t)), \quad (1.5)$$

where $\mu, \alpha \in (0,1], \nu \in (0,\alpha]$ and $^cD^\gamma$ denotes the Caputo fractional derivative
of order $\gamma$.

The Caputo fractional derivative $^cD^\gamma$ of order $\gamma > 0$ of a function $x : [0,T] \to \mathbb{R}$ is defined as [4, 5]

$$^cD^\gamma u(t) = \begin{cases} \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1} \left( x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) ds, & \text{if } \gamma \notin \mathbb{N}, \\ x^{(\gamma)}(t), & \text{if } \gamma \in \mathbb{N}, \end{cases}$$

where $n = [\gamma] + 1$ for $\gamma \notin \mathbb{N}$ and $[\gamma]$ means the integral part of the number $\gamma$. $\Gamma$ is the Euler gamma function.

Fractional differential equations have received a lot of attention recently. In
the literature there is only a few papers dealing with boundary value problems
for fractional differential equations with $\phi$-Laplacian. In [8, 9, 10] fractional
boundary value problems are investigated for equations only with the $p$-Laplacian
operator $\phi_p, \phi_p(s) = |s|^{p-2}s, \ p > 1$. Wang, Xiang and Liu [8] considered the equation

$$D^\gamma(\phi_p(D^\delta u(t))) + f(t,u(t)) = 0, \ 0 < \gamma \leq 1, \ 1 < \delta \leq 2,$$

and in [9] the equation

$$^cD^\gamma(\phi_p(D^\delta u(t))) + f(t,u(t), \, ^cD^\rho u(t)) = 0, \ 0 < \gamma < 1, \ 2 < \delta < 3, \ 0 < \rho \leq 1.$$

Paper [10] deals with the equation

$$D^\gamma(\phi_p(D^\delta u(t))) = f(t,u(t)), \ \gamma, \delta \in (1,2).$$
In these equations $D^\gamma$ denotes the Riemann-Liouville fractional derivative.

Let $\alpha \in (0,1]$ and let $X = \{ u \in C[0,T] : \mathcal{D}^\alpha u \in C[0,T] \}$ be the space equipped with the norm $\| u \|_\infty + \| \mathcal{D}^\alpha u \|_\infty$, where $\| u \|_\infty = \max_{t \in [0,T]} |u(t)|$ is the norm in $C[0,T]$. As we will prove in Lemma 6, $X$ is a Banach space. We denote by $A$ the set of functionals $A : X \to \mathbb{R}$ which are bounded (i.e., $\sup\{|A(u)| : u \in X\} < \infty$) and continuous. If $A \in A$, we write $A_s = \sup\{|A(u)| : u \in X\}$.

We investigate equations (1.4) and (1.5) subject to the functional boundary conditions

$$u(0) = A(u), \quad u(T) = B(u) \quad (A,B \in A). \quad (1.6)$$

Note that if $A,B \in A$ are constant functionals, then (1.6) reduces to (1.3).

We work with the following condition on the operator $F$ in (1.4).

$$(F) \quad F : X \to C[0,T] \text{ is continuous and takes bounded sets into bounded sets.}$$

We say that a function $u : [0,T] \to \mathbb{R}$ is a solution of problem (1.4), (1.6) if $u \in X$, $\mathcal{D}^{\mu}(\phi(\mathcal{D}^\alpha u)) \in C[0,T]$, $u$ satisfies the boundary conditions (1.6) and equality (1.4) holds for $t \in [0,T]$.

A function $u \in X$ is called a solution of problem (1.5), (1.6) if $\mathcal{D}^{\mu}(\phi(\mathcal{D}^\alpha u)) \in C[0,T]$, $u$ satisfies (1.6) and (1.5) holds for $t \in [0,T]$.

Our paper is organized as follows. Section 2 contains the definition of Riemann-Liouville fractional integral and preliminary facts from the fractional calculus, which are used throughout this paper. In Section 3 we introduce an operator $Q_{A,B} : [0,1] \times X \to X$ and give its properties. In particular, it is proved that $u$ is a solution of problem (1.4), (1.6) if and only if it is a fixed point of $Q_{A,B}(1, \cdot)$. In Section 4 our main results are stated and proved. We present an example to illustrate our results.

## 2 Preliminaries

The Riemann-Liouville fractional integral $I^\gamma x$ of order $\gamma > 0$ of a function $x : [0,T] \to \mathbb{R}$ is given by [4, 5, 6]

$$I^\gamma x(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} x(s) \, ds.$$  

**Lemma 1.** ([3], [4, Theorem 2.5], [5, Lemma 2.8]) Let $\gamma \in (0,1]$. Then $I^\gamma : C[0,T] \to C[0,T]$.

**Lemma 2.** ([5, Lemma 2.21]) Let $\gamma \in (0,1]$. Then $\mathcal{D}^{\mu} I^\gamma x(t) = x(t)$ for $t \in [0,T]$ and $x \in C[0,T]$.

**Lemma 3.** ([5, Lemma 2.22]) Let $\gamma \in (0,1]$. Then $I^\gamma \mathcal{D}^{\mu} x(t) = x(t) - x(0)$ for $t \in [0,T]$ and $x \in C[0,T]$. 

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Lemma 4. ([4, Theorem 2.2], [5, Lemma 2.3]) Let \( \beta, \tau \in (0, \infty) \), \( \beta + \tau \geq 1 \) and let \( x \in L^1[0, T] \). Then the equality \( I^\beta I^\tau x(t) = I^{\beta+\tau}x(t) \) is fulfilled for \( t \in [0, T] \).

Remark 1. The equality \( I^\beta I^\tau x(t) = I^{\beta+\tau}x(t) \) in Lemma 4 can be written in the form
\[
\int_0^t (t-s)^{\beta-1} \left( \int_0^s (s-\xi)^{\tau-1} x(\xi) \, d\xi \right) \, ds = \frac{\Gamma(\beta)\Gamma(\tau)}{\Gamma(\beta+\tau)} \int_0^t (t-s)^{\beta+\tau-1} x(s) \, ds. \tag{2.7}
\]

Lemma 5. Let \( x, y \in X, \alpha \in (0, 1] \) and \( \rho \in (0, \alpha) \). Then \( \mathcal{D}^\rho x \in C[0, T] \) and
\[
\|\mathcal{D}^\rho x - \mathcal{D}^\rho y\| \leq \frac{T^{\alpha-\rho}}{\Gamma(1 + \alpha - \rho)} \|\mathcal{D}^\alpha x - \mathcal{D}^\alpha y\|. \tag{2.8}
\]

Proof. Let \( h(t) = I^\alpha x(t) \) for \( t \in [0, T] \). Then \( h \in C[0, T] \) and, by Lemma 3, \( I^\alpha h(t) = x(t) - x(0) \). Hence (cf. (2.7) for \( \beta = 1 - \rho, \gamma = \alpha \))
\[
\mathcal{D}^\rho x(t) = \frac{1}{\Gamma(1 - \rho)} \frac{d}{dt} \int_0^t (t-s)^{-\rho} (x(s) - x(0)) \, ds
= \frac{1}{\Gamma(1 - \rho)\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\rho} \left( \int_0^s (s-\xi)^{\alpha-1} h(\xi) \, d\xi \right) \, ds
= \frac{1}{\Gamma(1 + \alpha - \rho)} \frac{d}{dt} \int_0^t (t-s)^{-\rho} h(s) \, ds
= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-\rho-1} h(s) \, ds, \quad t \in [0, T].
\]

Consequently, \( \mathcal{D}^\rho x = I^{1-\rho} x \in C[0, T] \) by Lemma 1, and the equality
\[
\mathcal{D}^\rho x(t) - \mathcal{D}^\rho y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-\rho-1} \left( \mathcal{D}^\alpha x(s) - \mathcal{D}^\alpha y(s) \right) \, ds
\]
is fulfilled for \( t \in [0, T] \). Therefore
\[
|\mathcal{D}^\rho x(t) - \mathcal{D}^\rho y(t)| \leq \frac{t^{\alpha-\rho}}{\Gamma(1 + \alpha - \rho)} \|\mathcal{D}^\alpha x - \mathcal{D}^\alpha y\|, \quad t \in [0, T],
\]
and (2.8) follows. \( \square \)

Lemma 6. \( X \) is a Banach space.

Proof. If \( \alpha = 1 \), then \( X = C[0, T] \), and so \( X \) is a Banach space. Let \( \alpha \in (0, 1] \). Let \( \{x_n\} \subset X \) be a Cauchy sequence. Then \( \{x_n\} \) and \( \{\mathcal{D}^\alpha x_n\} \) are Cauchy sequences in the space \( C[0, T] \). Hence there exist \( x, y \in C[0, T] \) such that \( \lim_{n \to \infty} \|x_n - x\| = 0, \lim_{n \to \infty} \|\mathcal{D}^\alpha x_n - y\| = 0 \). We need to prove that \( y = \mathcal{D}^\alpha x \). Since
\[
\int_0^t \mathcal{D}^\alpha x_n(s) \, ds = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t-s)^{-\alpha} (x_n(s) - x_n(0)) \, ds, \quad t \in [0, T], \quad n \in \mathbb{N},
\]
letting \( n \to \infty \) gives
\[
\int_0^t y(s) \, ds = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha}(x(s) - x(0)) \, ds, \quad t \in [0, T],
\]
by the Lebesgue dominated convergence theorem. Consequently, \( y = cD^\alpha x \).

3 Operator \( Q_{A,B} \) and its properties

In order to introduce an operator \( Q_{A,B} \), we need the following technical result.

Let
\[
Z = [0, 1] \times X \times \left( -\frac{aT^\alpha}{\alpha}, \frac{aT^\alpha}{\alpha} \right),
\]
where \( a \) is given in (\( \phi \)).

Lemma 7. Let conditions (\( \phi \)) and (\( F \)) be satisfied. Then for each \( (\lambda, x, d) \in Z \) there exists a unique \( m := \Lambda_{\phi}(\lambda, x, d) \) such that
\[
\int_0^T (T - t)^{\alpha - 1} \phi^{-1}[\lambda I^\mu(Fx)(t) - m] \, dt = d.
\]

Moreover, the function \( \Lambda_{\phi} : Z \to \mathbb{R} \) is continuous.

Proof. Let us choose \( (\lambda, x, d) \in Z \). It is not difficult to verify that the function
\[
\rho(c) = \int_0^T (T - t)^{\alpha - 1} \phi^{-1}[\lambda I^\mu(Fx)(t) - c] \, dt
\]
is well defined, continuous and decreasing on \( \mathbb{R} \).

Since
\[
\lim_{c \to \pm\infty} \rho(c) = \mp a \int_0^T (T - t)^{\alpha - 1} \, dt = \mp \frac{aT^\alpha}{\alpha},
\]
there is a unique solution \( m = \Lambda_{\phi}(\lambda, x, d) \) of (3.9).

In order to prove the continuity of \( \Lambda_{\phi} \), let \( (\lambda, x, d) \in Z \) and \( \{(\lambda_n, x_n, d_n)\} \subset Z \) be a convergent sequence such that \( \lim_{n \to \infty} (\lambda_n, x_n, d_n) = (\lambda, x, d) \). Then
\[
\lim_{n \to \infty} \|Fx_n - Fx\|_\infty = 0
\]
and the relation
\[
|\lambda_n I^\mu(Fx_n)(t) - \lambda I^\mu(Fx)(t)| \leq \lambda_n |I^\mu(Fx_n - Fx)(t)| + |\lambda_n - \lambda| \|I^\mu(Fx)(t)\|_\infty
\]
\[
\leq \frac{\|Fx_n - Fx\|_\infty t^\mu}{\Gamma(\mu + 1)} + |\lambda_n - \lambda| \|Fx\|_\infty t^\mu
\]
\[
\leq \frac{T^\mu}{\Gamma(\mu + 1)} (\|Fx_n - Fx\|_\infty + |\lambda_n - \lambda| \|Fx\|_\infty)
\]
holds for \( t \in [0, T] \) and \( n \in \mathbb{N} \). Hence \( \lim_{n \to \infty} \|\lambda_n I^\mu(Fx_n) - \lambda I^\mu(Fx)\|_\infty = 0 \).
Suppose that the sequence \( \{\Lambda_\phi(\lambda_n, x_n, d_n)\} \) is unbounded and let, for example, \( \lim_{n \to \infty} \Lambda_\phi(\lambda_{k_n}, x_{k_n}, d_{k_n}) = \infty \) for a subsequence \( \{\lambda_{k_n}, x_{k_n}, d_{k_n}\} \) of \( \{\lambda_n, x_n, d_n\} \).

Due to the fact that \( \{\lambda_n I^\mu(Fx_n)\} \) is bounded in \( C[0, T] \), letting \( n \to \infty \) in the equality

\[
\int_0^T (T - t)^{\alpha-1} \phi^{-1}[\lambda_{k_n} I^\mu(Fx_{k_n})(s) - \Lambda_\phi(\lambda_{k_n}, x_{k_n}, d_{k_n})] \, dt = \Lambda_{k_n} \tag{3.10}
\]

we get \( -\frac{\alpha T^\alpha}{\alpha} - d \), which is impossible since \( d \in \left(-\frac{\alpha T^\alpha}{\alpha}, \frac{\alpha T^\alpha}{\alpha}\right) \). Consequently, \( \{\Lambda_\phi(\lambda_n, x_n, d_n)\} \) is bounded.

Let \( \{\Lambda_\phi(\lambda_{\ell_n}, x_{\ell_n}, d_{\ell_n})\} \) be a convergent subsequence of \( \{\Lambda_\phi(\lambda_n, x_n, d_n)\} \) and let \( \theta \) be its limit. Passing to the limit as \( n \to \infty \) in (3.10) (with \( \lambda_{k_n} \) replaced by \( \lambda_{\ell_n} \)), by the Lebesgue dominated convergence theorem, we get

\[
\int_0^T (T - t)^{\alpha-1} \phi^{-1}[\lambda I^\mu(Fx)(t) - \theta] \, dt = d,
\]

which say us that \( \theta = \Lambda_\phi(\lambda, x, d) \).

This proves that any convergent subsequence of \( \{\Lambda_\phi(\lambda_n, x_n, d_n)\} \) has the same limit equals to \( \Lambda_\phi(\lambda, x, d) \), and therefore, \( \lim_{n \to \infty} \Lambda_\phi(\lambda_n, x_n, d_n) = \Lambda_\phi(\lambda, x, d) \). As a result \( \Lambda_\phi \) is continuous on \( Z \). \( \square \)

Let \( A, B \in \mathcal{A} \) be such that \( (B - A)_s < \frac{\alpha T^\alpha}{\Gamma(\alpha+1)} \). Note that, by our notation, \( (B - A)_s = \sup \{|B(u) - A(u)| : u \in X\} \). Let \( Q_{A,B} \) be an operator defined on \( [0, 1] \times X \) by the formula

\[
Q_{A,B}(\lambda, x)(t) = \lambda A(x) + \int_0^t (t-s)^{\alpha-1} \phi^{-1}\left[\lambda I^\mu(Fx)(s) - \Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x)))\right] \, ds,
\]

where \( \Lambda_\phi(\lambda, x, d) \) is defined in Lemma 7.

By the Riemann-Liouville fractional integral \( I^\alpha \), the operator \( Q_{A,B} \) can be written in the form

\[
Q_{A,B}(\lambda, x)(t) = \lambda A(x) + I^\alpha \phi^{-1}\left[\lambda I^\mu(Fx)(t) - \Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x)))\right].
\]

The properties of the operator \( Q_{A,B} \) are collected in the following two lemmas.

**Lemma 8.** Let \( (\phi) \) and \( (F) \) be fulfilled. Then, if \( A, B \in \mathcal{A} \) are such that \( (B - A)_s < \frac{\alpha T^\alpha}{\Gamma(\alpha+1)} \), the following properties hold

(a) \( Q_{A,B} : [0, 1] \times X \to X \),

(b) \( Q_{A,B} \) is a completely continuous operator.

**Proof.** (a) Let us choose \( (\lambda, x) \in [0, 1] \times X \). Then, by Lemma 1, the function

\[
p(t) := \phi^{-1}\left[\lambda I^\mu(Fx)(t) - \Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x)))\right]
\]

satisfies

\[
\int_0^1 |p(t)| \, dt < \infty.
\]

By the Lebesgue dominated convergence theorem, we get

\[
\int_0^1 |p(t)| \, dt \to 0 \quad \text{as} \quad n \to \infty,
\]

which is possible only if \( P_{A,B} \) is the zero operator. \( \square \)
is a continuous function on $[0, T]$ and, since the equalities
\[ Q_{A,B}(\lambda, x)(t) = \lambda A(x) + I^\alpha p(t), \]
\[ 'D^\alpha Q_{A,B}(\lambda, x)(t) = 'D^\alpha (\lambda A(x) + I^\alpha p(t)) = 'D^\alpha I^\alpha p(t) = p(t) \]
are fulfilled for $t \in [0, T]$, we have $Q_{A,B}(\lambda, x) \in X$. Consequently, assertion (a)
holds.
(b) We start by proving that $Q_{A,B}$ is continuous. Let $\{(\lambda_n, x_n)\} \subset [0, 1] \times X$
be convergent and let $\lim_{n \to \infty} (\lambda_n, x_n) = (\lambda, x) \in [0, 1] \times X$. Put
\[ q_n(t) = \phi^{-1}[\lambda_n I^\alpha (Fx_n)(t) - \Lambda_\phi(\lambda_n, x_n, \lambda_n \Gamma(\alpha)(B(x_n) - A(x_n)))] ; \]
\[ q(t) = \phi^{-1}[\lambda I^\alpha (Fx)(t) - \Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x)))] \]
for $t \in [0, T]$ and $n \in \mathbb{N}$.
It follows from $\lim_{n \to \infty} (B(x_n) - A(x_n)) = B(x) - A(x)$, from conditions (φ)
and (F) and from Lemma 7, that $\lim_{n \to \infty} \|q_n - q\|_\infty = 0$.
Since
\[ Q_{A,B}(\lambda_n, x_n)(t) = \lambda_n A(x_n) + I^\alpha q_n(t), \quad Q_{A,B}(\lambda, x)(t) = \lambda A(x) + I^\alpha q(t), \]
\[ 'D^\alpha Q_{A,B}(\lambda_n, x_n)(t) = q_n(t), \quad 'D^\alpha Q_{A,B}(\lambda, x)(t) = q(t) \]
for $t \in [0, T]$ and $n \in \mathbb{N}$, we see that $\lim_{n \to \infty} Q_{A,B}(\lambda_n, x_n) = Q_{A,B}(\lambda, x)$ in $X$.
Hence $Q_{A,B}$ is a continuous operator.
We now prove that the set $\{Q_{A,B}(\lambda, x) : (\lambda, x) \in [0, 1] \times \Omega\}$ is relatively compact in $X$
for each bounded $\Omega \subset X$.
First, we note that
\[ |Q_{A,B}(\lambda, x)(t)| \leq A_s + \frac{a}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} ds \leq A_s + \frac{a T^\alpha}{\Gamma(\alpha + 1)}, \]
\[ |'D^\alpha Q_{A,B}(\lambda, x)(t)| = \phi^{-1}[\lambda I^\alpha (Fx)(t) - \Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x)))] < a \]
for $t \in [0, T]$ and $(\lambda, x) \in [0, 1] \times X$. As consequence the set $\{Q_{A,B}(\lambda, x) : (\lambda, x) \in [0, 1] \times X\}$ is bounded in $X$.
Let $\Omega \subset X$ be a bounded set. Then, by (F), there exists a positive constant $Q$
such that $\|Fx\|_\infty \leq Q$ for $x \in \Omega$. Hence
\[ |\lambda I^\alpha (Fx)(t)| \leq \frac{\|Fx\|_\infty}{\Gamma(\mu + 1)} t^\mu \leq \frac{Q T^\mu}{\Gamma(\mu + 1)} \quad \text{for } t \in [0, T] \text{ and } x \in \Omega. \quad (3.11) \]
Let $0 \leq t_1 < t_2 \leq T$ and put
\[ M(\lambda, x) := \Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x))) \quad \text{for } (\lambda, x) \in [0, 1] \times X. \]
Then the estimates
\[
|Q_{A,B}(\lambda, x)(t_2) - Q_{A,B}(\lambda, x)(t_1)|
\]
\[
= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} \phi^{-1}[\lambda I^\mu(Fx)(s) - M(\lambda, x)] \, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} \phi^{-1}[\lambda I^\mu(Fx)(s) - M(\lambda, x)] \, ds \right|
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}) \phi^{-1}[\lambda I^\mu(Fx)(s) - M(\lambda, x)] \, ds \right|
\]
\[
+ \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \phi^{-1}[\lambda I^\mu(Fx)(s) - M(\lambda, x)] \, ds \right|
\]
\[
\leq \frac{a}{\Gamma(\alpha)} \left( \int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \, ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \, ds \right)
\]
\[
= \frac{a}{\Gamma(\alpha + 1)} (t_1^\alpha + 2(t_2 - t_1)^\alpha - t_2^\alpha)
\]

and
\[
\left| \mathcal{D}^\alpha Q_{A,B}(\lambda, x)(t)|_{t=t_2} - \mathcal{D}^\alpha Q_{A,B}(\lambda, x)(t)|_{t=t_1} \right|
\]
\[
= \left| \phi^{-1}[\lambda I^\mu(Fx)(t)|_{t=t_2} - M(\lambda, x)] - \phi^{-1}[\lambda I^\mu(Fx)(t)|_{t=t_1} - M(\lambda, x)] \right|
\]

are fulfilled for \((\lambda, x) \in [0, 1] \times \Omega\). From (3.12) it follows that the set \(\{Q_{A,B}(\lambda, x) : (\lambda, x) \in [0, 1] \times \Omega\}\) is equicontinuous on \([0, T]\). It remains to prove that the set \(\{\mathcal{D}^\alpha Q_{A,B}(\lambda, x) : (\lambda, x) \in [0, 1] \times \Omega\}\) is equicontinuous on \([0, T]\). First we prove that
\[
L := \sup\{|M(\lambda, x)| : (\lambda, x) \in [0, 1] \times \Omega\} < \infty.
\]

By the definition of \(\Lambda_\phi(\lambda, x, \lambda \Gamma(\alpha)(B(x) - A(x)))\), the equality
\[
\int_0^T (T - t)^{\alpha-1} \phi^{-1}[\lambda I^\mu(Fx)(t) - M(\lambda, x)] \, ds = \lambda \Gamma(\alpha)(B(x) - A(x))
\]
holds for \((\lambda, x) \in [0, 1] \times \Omega\). Hence due to the mean value theorem for integrals there exists \(\xi = \xi(\lambda, x) \in [0, T]\) such that
\[
\phi^{-1}[\lambda I^\mu(Fx)(t)|_{t=\xi} - M(\lambda, x)] \int_0^T (T - t)^{\alpha-1} \, dt = \lambda \Gamma(\alpha)(B(x) - A(x)).
\]

Consequently,
\[
M(\lambda, x) = \lambda I^\mu(Fx)(t)|_{t=\xi} - \phi \left( \frac{\lambda \Gamma(\alpha + 1)(B(x) - A(x))}{T^\alpha} \right).
\]

In view of \(|\lambda I^\mu(Fx)(t)|_{t=\xi} \leq \frac{Q_{T^\mu}}{\Gamma(\alpha + 1)}\) and
\[
\left| \frac{\lambda \Gamma(\alpha + 1)(B(x) - A(x))}{T^\alpha} \right| \leq \frac{\Gamma(\alpha + 1)(B - A)s}{T^\alpha} =: \varepsilon < a
\]
we get

\[ |M(\lambda, x)| \leq \frac{QT^\mu}{\Gamma(\mu + 1)} + \phi(\varepsilon) < \infty \text{ for } (\lambda, x) \in [0, 1] \times \Omega, \]

which proves (3.14). We have

\[
\begin{align*}
&\left|\lambda I^\mu(Fx)(t)|_{t=t_2} - \lambda I^\mu(Fx)(t)|_{t=t_1}\right| \\
&\leq \frac{1}{\Gamma(\mu)} \left| \int_{t_1}^{t_2} ((t_2 - s)^{\mu-1} - (t_1 - s)^{\mu-1})(Fx)(s) \, ds \right| \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{\mu-1}(Fx)(s) \, ds \\
&\leq \frac{Q}{\Gamma(\mu + 1)}(t_1^{\mu} + 2(t_2 - t_1)^{\mu} - t_2^{\mu})
\end{align*}
\]

and, by (3.11) and (3.14),

\[
\left|\lambda I^\mu(Fx)(t)|_{t=t_j} - M(\lambda, x)\right| \leq \frac{QT^\mu}{\Gamma(\mu + 1)} + L, \text{ } (\lambda, x) \in [0, 1] \times \Omega, \ j = 1, 2.
\]

Now, since \(\phi^{-1}\) is uniformly continuous on the interval \(\left[-\frac{QT^\mu}{\Gamma(\mu + 1)} - L, \frac{QT^\mu}{\Gamma(\mu + 1)} + L\right]\), we conclude, from (3.13), that the set \(\{\mathcal{D}_A^\alpha \mathcal{Q}_{A,B}(\lambda, x) : (\lambda, x) \in [0, 1] \times \Omega\}\) is equicontinuous on \([0, T]\). Hence, from the Arzelà-Ascoli theorem, \(\mathcal{Q}_{A,B}(\lambda, x) : (\lambda, x) \in [0, 1] \times \Omega\) is relatively compact in \(X\). So, we have proved that \(\mathcal{Q}_{A,B}\) is a completely continuous operator. \(\square\)

**Lemma 9.** Assume that (\(\phi\)) and (H) are fulfilled. Let \(A, B \in \mathcal{A}\) be such that \((B - A)_s < \frac{\alpha T^\alpha}{\Gamma(\alpha + 1)}\). Then \(u\) is a solution of problem (1.4), (1.6) if and only if it is a fixed point of the operator \(\mathcal{Q}_{A,B}(1, \cdot)\).

**Proof.** Let \(u\) be a solution of problem (1.4), (1.3). Then \(u \in X, u(0) = A(u), u(T) = B(u)\) and \(u\) satisfies the equality (1.4) for \(t \in [0, T]\).

Hence, by Lemma 3,

\[ \mathcal{D}_A^\alpha u(t) = \phi^{-1}[I^\mu(Fu)(t) - c] \text{ for } t \in [0, T], \]

where \(c \in \mathbb{R}\).

Now, again by Lemma 3,

\[ u(t) = A(u) + I^\alpha \phi^{-1}[I^\mu(Fu)(t) - c] \text{ for } t \in [0, T]. \]

The last equality at \(t = T\) together with the condition \(u(T) = B(u)\), implies that

\[ \int_0^T (T - t)^{\alpha-1} \phi^{-1}[I^\mu(Fu)(t) - c] \, dt = \Gamma(\alpha)(B(u) - A(u)). \]

Since \(\Gamma(\alpha)(B - A)_s < \frac{\alpha T^\alpha}{\alpha}\), we conclude, from Lemma 7, that \(c = \Lambda(1, u, \Gamma(\alpha)(B(u) - A(u)))\). Therefore \(u\) is a fixed point of \(\mathcal{Q}_{A,B}(1, \cdot)\).
Assume now that \( u \) is a fixed point of \( Q_{A,B}(1, \cdot) \). Then \( u \in X \) and the equality

\[
\begin{align*}
  u(t) &= A(u) + \int_0^t (t-s)^{\alpha-1} \phi^{-1} \left[ I^\mu(Fu)(s) - \Lambda_\phi(1, u, \Gamma(\alpha)(B(u) - A(u))) \right] \, ds \\
  \text{(3.15)}
\end{align*}
\]

is fulfilled for \( t \in [0, T] \). Hence \( u(0) = A(u) \).

On the other hand, by Lemma 7,

\[
\int_0^T (T-t)^{\alpha-1} \phi^{-1} \left[ I^\mu(Fu)(t) - \Lambda_\phi(1, u, \Gamma(\alpha)(B(u) - A(u))) \right] \, ds = \Gamma(\alpha)(B(u) - A(u)),
\]

we have that \( u(T) = A(u) + (B(u) - A(u)) = B(u) \).

From Lemma 2 and equality (3.15), using the fact that \( \mathcal{D}^\mu c = 0 \) for any constant function \( c \), it follows that

\[
\phi(\mathcal{D}^\alpha u(t)) = I^\mu(Fu)(t) - \Lambda_\phi(1, u, \Gamma(\alpha)(B(u) - A(u))) \text{ for } t \in [0, T].
\]

Consequently, again by Lemma 2, we have \( \mathcal{D}^\mu(\phi(\mathcal{D}^\alpha u(t))) = (Fu)(t) \) for \( t \in [0, T] \). Hence \( u \) is a solution of problem (1.4), (1.6).

The following result is needed for applying the Leray-Schauder degree in the proof of the main result of this paper, Theorem 1.

**Lemma 10.** Let (\( \phi \)) and (H) hold. Let

\[
B = \{x \in X : x = Q_{A,B}(\lambda, x) \text{ for some } \lambda \in [0, 1]\}.
\]

Then there exists a positive constant \( W \) such that the estimate

\[
\|x\|_{\infty} < W, \quad \|\mathcal{D}^\alpha x\|_{\infty} < W \tag{3.16}
\]

holds for all \( x \in B \).

**Proof.** Let \( x \in B \). Then \( x = Q_{A,B}(\lambda, x) \), where \( \lambda \in [0, 1] \). Hence the relations

\[
|x(t)| \leq A_s + \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi^{-1} \left[ \Gamma(\alpha) \right] \, ds \right|
\]

\[
\leq A_s + \frac{a}{\Gamma(\alpha)} \int_0^T (t-s)^{\alpha-1} \, ds
\]

\[
\leq A_s + \frac{aT^\alpha}{\Gamma(\alpha + 1)}
\]

and

\[
\|\mathcal{D}^\alpha x(t)\| \leq \left| \phi^{-1} \left[ \Gamma(\alpha) \right] \right| < a
\]

hold for \( t \in [0, T] \). Therefore, estimate (3.16) holds for all \( x \in B \) with \( W = \max \{ A_s + \frac{aT^\alpha}{\Gamma(\alpha + 1)}, a \} + 1 \).
4 Existence results

**Theorem 1.** Let $(\phi)$ and $(H)$ hold. Let $A, B \in \mathcal{A}$ be such that $(B - A)_s < \frac{aT^\alpha}{\Gamma(\alpha + 1)}$. Then problem (1.4), (1.6) has a solution.

**Proof.** By Lemma 9 we need to prove that $Q_{A,B}(1, \cdot)$ has a fixed point. Let $W$ be a positive constant given in Lemma 10 and let

$$
\Omega = \{x \in X : \|x\|_{\infty} < W, \|D^\alpha x\|_{\infty} < W\}.
$$

To simplify notation, we use the same letter $Q_{A,B}$ for the restriction of $Q_{A,B}$ on $[0, 1] \times \overline{\Omega}$. Then, by Lemmas 8 and 10, $Q_{A,B} : [0, 1] \times \overline{\Omega} \to X$ is a compact operator and $Q_{A,B}(\lambda, x) \neq x$ for $(\lambda, x) \in [0, 1] \times \partial\Omega$. Hence, by the homotopy property (see, e.g., [7])

$$
\text{deg} (I - Q_{A,B}(1, \cdot), \Omega, 0) = \text{deg} (I - Q_{A,B}(0, \cdot), \Omega, 0),
$$

where ”deg” stands for the Leray-Schauder degree and $I$ is the identical operator on $X$.

In view of $\phi^{-1}(0) = 0$ we conclude, from Lemma 7, that $\Lambda(0, x, 0) = 0$ for $x \in X$, and therefore $Q_{A,B}(0, x) = 0$ for $x \in \Omega$. Hence

$$
\text{deg} (I - Q_{A,B}(0, \cdot), \Omega, 0) = \text{deg} (I, \Omega, 0) = 1,
$$

which gives (cf. (4.17)) $\text{deg} (I - Q_{A,B}(1, \cdot), \Omega, 0) = 1$. Consequently, there exist a fixed point of $Q_{A,B}(1, \cdot)$. \hfill $\square$

**Corollary 1.** Let $(\phi)$ be fulfilled and let $f \in C([0, T] \times \mathbb{R}^2)$. Let $A, B \in \mathcal{A}$ satisfying that $(B - A)_s < \frac{aT^\alpha}{\Gamma(\alpha + 1)}$. Then problem (1.5), (1.6) has a solution.

**Proof.** For $x \in X$ define an operator $F$ by the formula

$$(F.x)(t) = f(t, x(t), D^\nu x(t)), \quad \nu \in (0, \alpha].$$

Since, by Lemma 5, $D^\nu x \in C[0, T]$ for each $x \in X$, we have that $F : X \to C[0, T]$.

Let $\{x_n\} \subset X$ be a convergent sequence in $X$ to $x \in X$. Then, Lemma 5 again implies that $\lim_{n \to \infty} \|D^\nu x_n - D^\nu x\|_{\infty} = 0$, and so $\lim_{n \to \infty} \|F x_n - F x\|_{\infty} = 0$. Hence $F$ is continuous.

Let $\Omega \subset X$ be bounded. Then the continuity of $f$ and Lemma 5 guarantee that

$$
\sup\{\|f(t, x(t), D^\nu x(t))\| : t \in [0, T], x \in \Omega\} < \infty.
$$

Consequently, $F$ takes bounded sets into bounded sets.

By Theorem 1, there exists a solution $u$ of problem (1.4), (1.6). Since $(Fu)(t) = f(t, u(t), D^\nu u(t))$ for $t \in [0, T]$, $u$ is a solution of problem (1.5), (1.6). \hfill $\square$

**Remark 2.** If $\alpha = \mu = \nu = 1$ and if $A, B \in \mathcal{A}$ are constant functionals, then $\mathcal{D}^\mu \phi(\mathcal{D}^\alpha x) = (\phi(x^\prime))$, $f(t, x, D^\nu x) = f(t, x, x^\prime)$ and $|B - A| < \frac{aT^\alpha}{\Gamma(\alpha + 1)} = aT$. Hence the existence results for problems (1.1), (1.3) and (1.2), (1.3) given in [1] follows from Theorem 1 and Corollary 1.
Example 1. Let $F: X \to C[0, T]$ be continuous such that $\|F(x)\|_\infty \leq p(\|x\|_\infty, \|D^\alpha x\|_\infty)$ for some $p(u, v) \in C(\mathbb{R}^2)$, and let $f \in C([0, T] \times \mathbb{R}^2)$ and $\phi(u) = \frac{u}{\sqrt{1-u^2}}$. Then $F$ satisfies condition $(F)$ and $\phi$ condition $(\phi)$ with $a = 1$.

Let $A(x) = \frac{\sin(D^\alpha x(\tau))}{1+\|x\|_\infty} \int_0^T x(t) \, dt$ and $B(x) = A(x) + \mu \arctan(x(\xi)) + \|D^\alpha x\|_\infty$ for $x \in X$, where $\mu \in \mathbb{R}$, $\tau, \xi \in [0, T]$. Then $A, B \in \mathcal{A}$ and $(B-A)_s = \frac{\mu \pi}{2}$.

By Theorem 1 and Corollary 1, problems (1.4), (1.6) and (1.5), (1.6) are solvable for each $\mu$ satisfying $|\mu| < \frac{2T^\alpha}{\pi(\alpha+1)}$.

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