Expression of the Lebesgue $\Delta$–integral on time scales as a usual Lebesgue integral. Application to the calculus of $\Delta$ – antiderivatives.

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1 Introduction

The theory of dynamic equations appears in the literature in 1988 in the Ph. D. of S. Hilger [18]. The aim of this theory consists in to study differential and difference equations under the same formulation. Thus, the concepts of $\Delta$–derivative and $\Delta$–integral have been defined. These two concepts cover both the classical derivative and the Riemann integral together with the forward and sum operators. After this, a wide number of very important tools have been developed with the intention of solving different kinds of dynamic equations. Functions as the exponential $e_p(t, s)$, the trigonometric $\cos_p(t, s)$ and $\sin_p(t, s)$ or the hyperbolic $\cosh_p(t, s)$ and $\sinh_p(t, s)$, together with operators $\oplus$ and $\ominus$ have been introduced and represent fundamental tools for the calculus on time scales, see [1, 3, 8, 9, 10] and references therein. Using these operators, some topics on dynamic equations as, among others, the general expression of the solutions of first and second order linear equations [8], the expression of the Green's functions of some linear boundary value problems [4, 5, 8, 11, 12, 13] or oscillation properties of first and second order nonlinear equations on time scales [6, 14, 15, 19], have recently been considered in the literature.

An exhaustive study of the Riemann $\Delta$ – integral has been done by Guseinov in [16], Guseinov and Kaymakçalân in [17] and Bohner and Guseinov in [7]. Recently, the Lebesgue $\Delta$ – integral has been introduced by Bohner and Guseinov in [9, Chapter 5]. On those papers the fundamental theory of such integrals

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is developed, moreover the relationship between Riemann and Lebesgue $\Delta$ –
integrals is also given.

Despite this very important efforts, to obtain the exact value of a $\Delta$–integral of a rd – continuous function in an arbitrary time scale remains as an open problem, in fact, the majority of the $\Delta$ – antiderivatives of the elemental continuous functions is unknown for arbitrary time scales. We note that the $\Delta$ – primitives of functions as $t^n$, $1/t^n$, $\cos t$, $\sin t$ or $e^t$ have been obtained only for particular cases of time scales.

The aim of this paper is to connect the classical Lebesgue integral on real measurable sets with the Lebesgue $\Delta$–integral presented in [9] for functions defined on an arbitrary bounded time scale $T$ such that $\min T = a$ and $\max T = b$.

In order to do this, we set out several basic notions from abstract measure and integration theory, [20, 21, 22], adapted to the measurable space $(T, \mathcal{M}(m_1^t))$ equipped by the Lebesgue $\Delta$–measure $\mu_\Delta$, which was introduced in [9] and we remind it in section 2.

In section 3, after proving that, for any arbitrary time scale, the set of right scattered points is at most countable, we show the relation between the Lebesgue $\Delta$–measure $\mu_\Delta$ and the Lebesgue measure $\mu_L$ and we establish a criterion for $\Delta$–measurability of sets, from which, in section 4, we connect Lebesgue $\Delta$ – measurable functions to Lebesgue measurable functions.

The most important part of this paper is section 5. There, we compare Lebesgue $\Delta$–integrability with Lebesgue integrability and give a formula to calculate the Lebesgue $\Delta$–integral as a Lebesgue integral on $T$ plus a sum of real numbers, in which only the right scattered points appear. This formula point out the division of the time scales calculus between the “continuous” and the “discrete” parts.

In section 6, we use the mentioned formula to deduce the antiderivatives of some important functions as, among others, the potential, the exponential or the trigonometric ones. To obtain such expressions, we must describe carefully the set of right scattered points of the considered time scale in each case. To illustrate the obtained results, we calculate some integrals on the ternary Cantor set.

2 Preliminaries

To begin, we set out the method used in [9, Chapter 5] by Bohner and Guseinov to define the Lebesgue $\Delta$–measure on $T$.

First, denoting for every $x, y \in \mathbb{R}$, as $[x, y) = \{t \in \mathbb{R}, x \leq t < y\}$, they define a countably additive measure $m_1$ on the set

$$\mathcal{F}_1 = \{[\tilde{a}, \tilde{b}) \cap T : \tilde{a}, \tilde{b} \in T, \tilde{a} \leq \tilde{b}\},$$

that assigns to each interval $[\tilde{a}, \tilde{b}) \cap T \in \mathcal{F}_1$ its length, that is,

$$m_1([\tilde{a}, \tilde{b})) = \tilde{b} - \tilde{a}.$$
The interval $[\hat{a}, \hat{a})$ is understood as the empty set. Using $m_1$, they generate the outer measure $m_1^*$ on $P(T)$, defined for each $E \in P(T)$ as

$$m_1^*(E) = \begin{cases} \inf_{R} \sum_{i \in I_R} (b_i - \hat{a}_i) \in \mathbb{R}^+, & \text{if } b \notin E, \\ +\infty, & \text{if } b \in E, \end{cases}$$

with

$$I_R = \left\{ [\hat{a}_i, b_i) \cap T \in \mathcal{F}_1 \right\} \quad : \quad I_R \subset \mathbb{N}, \quad E \subset \bigcup_{i \in I_R} \left( (\hat{a}_i, b_i) \cap T \right).$$

A set $A \subset T$ is said to be $\Delta$–measurable if the following equality holds for all subset $E$ of $T$.

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (T \setminus A)),$$

Now, defining the family

$$\mathcal{M}(m_1^*) = \{ A \subset T : A \text{ is } \Delta \text{–measurable} \},$$

the Lebesgue $\Delta$–measure, denoted by $\mu_\Delta$, is the restriction of $m_1^*$ to $\mathcal{M}(m_1^*)$.

Now, we introduce several concepts from general measure and integration, see [20, 21, 22], applied to the measurable space $(T, \mathcal{M}(m_1^*))$ with the Lebesgue $\Delta$–measure $\mu_\Delta$.

**Definition 2.1** We say that $f : T \to \mathbb{R} \equiv [-\infty, +\infty]$ is $\Delta$–measurable, if for every $\alpha \in \mathbb{R}$, the set

$$f^{-1}([-\infty, \alpha)) = \{ t \in T : f(t) < \alpha \}$$

is $\Delta$–measurable.

**Definition 2.2** We say that $S : T \to \mathbb{R}$ is simple if it only takes a finite number of values $\alpha_1, \ldots, \alpha_n$, all of them different.

If $A_j = \{ t \in T : S(t) = \alpha_j \}$, then

$$S = \sum_{j=1}^{n} \alpha_j \chi_{A_j},$$

with $\chi_{A_j} : T \to \mathbb{R}$ the characteristic function of $A_j$, i. e.

$$\chi_{A_j}(t) = \begin{cases} 1, & \text{if } t \in A_j, \\ 0, & \text{if } t \in T \setminus A_j. \end{cases}$$

**Remark 2.1** It is not difficult to verify that if $S : T \to \mathbb{R}$ is simple with $S = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$, then $S$ is $\Delta$–measurable if and only if $A_j$ is $\Delta$–measurable for all $j \in \{1, \ldots, n\}$. \hfill 3
Definition 2.3 Let $E \subset \mathbb{T}$ be a $\Delta$–measurable set and let $S : \mathbb{T} \to [0, +\infty)$ be a simple and $\Delta$–measurable function with

$$S = \sum_{j=1}^{n} \alpha_j \chi_{A_j}.$$ 

The Lebesgue $\Delta$–integral of $S$ on $E$ it is defined as

$$\int_{E} S(s) \, \Delta s = \sum_{j=1}^{n} \alpha_j \mu_{\Delta}(A_j \cap E).$$

The agreement $0 \cdot \infty = 0$ is used.

Definition 2.4 Let $E \subset \mathbb{T}$ be a $\Delta$–measurable set and let $f : \mathbb{T} \to [0, +\infty]$ be a $\Delta$–measurable function. The Lebesgue $\Delta$–integral of $f$ on $E$ it is defined as

$$\int_{E} f(s) \, \Delta s = \sup \int_{E} S(s) \, \Delta s,$$

where the supremum is taken on all simple $\Delta$–measurable functions $S$ such that $0 \leq S \leq f$ in $\mathbb{T}$.

Remark 2.2 Note that if $f$ is a simple function, Definitions 2.3 and 2.4 are equivalent.

Definition 2.5 Let $E \subset \mathbb{T}$ be a $\Delta$–measurable set and let $f : \mathbb{T} \to \bar{\mathbb{R}}$ be a $\Delta$–measurable function. We say that $f$ is Lebesgue $\Delta$–integrable on $E$ if at least one of the elements

$$\int_{E} f^+(s) \, \Delta s \quad \text{or} \quad \int_{E} f^-(s) \, \Delta s,$$

is finite, where the positive and negative parts of $f$, $f^+$ and $f^-$ respectively, are defined as

$$f^+ := \max\{f, 0\} \quad \text{and} \quad f^- := \max\{-f, 0\}. \quad (2.1)$$

In which case, we define the Lebesgue $\Delta$–integral of $f$ on $E$ as

$$\int_{E} f(s) \, \Delta s = \int_{E} f^+(s) \, \Delta s - \int_{E} f^-(s) \, \Delta s.$$

Moreover, we define the Lebesgue $\Delta$–integral of $|f|$ on $E$ as

$$\int_{E} |f(s)| \, \Delta s = \int_{E} f^+(s) \, \Delta s + \int_{E} f^-(s) \, \Delta s.$$

Definition 2.6 Let $E \subset \mathbb{T}$ be a $\Delta$–measurable set and let $f : \mathbb{T} \to \bar{\mathbb{R}}$ be a $\Delta$–measurable function. We say that $f$ belongs to $L^1(E)$ provided that

$$\int_{[a,b] \cap E} |f(s)| \, \Delta s < \infty.$$
Now we enunciate the monotone convergence theorem adapted to our measurable space. We refer to [21] for a proof on an arbitrary measurable space equipped by a measure.

**Theorem 2.1** Let $E \subset \mathbb{T}$ be a $\Delta$–measurable set and let $\{f_m\}_{m \in \mathbb{N}}$ be a sequence of $\Delta$–measurable functions such that for every $t \in \mathbb{T}$ the following conditions are satisfied

a) $0 \leq f_m(t) \leq f_{m+1}(t) \leq \infty$ for all $m \in \mathbb{N}$.

b) $\lim_{m \to \infty} f_m(t) = f(t)$.

Then, $f$ is $\Delta$–measurable and

$$ \lim_{m \to \infty} \int_E f_m(s) \, \Delta s = \int_E f(s) \, \Delta s. $$

### 3 Measurable and $\Delta$–measurable Sets

In this section, we present a criterion for $\Delta$–measurability of sets and we show the relation between the Lebesgue $\Delta$–measure $\mu_\Delta$ and the Lebesgue measure $\mu_L$.

First, we prove an important property in the development of this paper.

**Lemma 3.1** The set of all right–scattered points of $\mathbb{T}$ is at most countable, that is, there are $I \subset \mathbb{N}$ and $\{t_i\}_{i \in I} \subset \mathbb{T}$ such that

$$ R := \{t \in \mathbb{T} : t < \sigma(t)\} = \{t_i\}_{i \in I}. \tag{3.1} $$

**Proof:** Let $g : [a, b] \to \mathbb{R}$ be defined as

$$ g(t) = \begin{cases} 
  t, & \text{if } t \in \mathbb{T}, \\
  \sigma(s), & \text{if } t \in (s, \sigma(s)), s \in \mathbb{T}.
\end{cases} $$

It is obvious that function $g$ is monotonous on $[a, b]$ and continuous on the set

$$ [a, b] \setminus \{t \in \mathbb{T} : t < \sigma(t)\}. $$

Since the set of points where a monotonous function has discontinuities is at most countable, [20], we arrive at the desired result. \qed

**Remark 3.1** Note that, since $\mathbb{T} = \bigcup_{n \in \mathbb{N}} (\mathbb{T} \cap (-n, n))$, the previous result is valid even in the case of $\mathbb{T}$ unbounded.

In all the paper, given $E \subset \mathbb{T}$, we denote by

$$ I_E = \{i \in I : t_i \in E \cap R\}, \tag{3.2} $$

with $I \subset \mathbb{N}$ and $R = \{t_i\}_{i \in I}$ given in (3.1).

The following result links the outer measure $m^*_1$ defined above to the Lebesgue measure $\mu^*$. 5
Lemma 3.2 If $E \subset T$, then the following properties are satisfied:

1. $\mu^*(E) \leq m_1^*(E)$.

2. If $b \notin E$ and $E$ have not any right–scattered points, then
   $$\mu^*(E) = m_1^*(E).$$

3. The sets $R$, defined in (3.1), and $T \setminus R$ are Lebesgue measurables and
   $$\mu_L(R) = 0 \quad \text{and} \quad \mu_L(T \setminus R) = b - a = \mu_{\Delta}([a,b] \cap T).$$

4. $$\mu_{\Delta}(E \cap R) = \sum_{i \in I_E} (\sigma(t_i) - t_i) \leq (b - a) = \mu_{\Delta}([a,b] \cap T).$$

5. If $b \notin E$, then
   $$m_1^*(E) = \sum_{i \in I_E} (\sigma(t_i) - t_i) + \mu^*(E).$$

6. $m_1^*(E) = \mu^*(E)$ if and only if $b \notin E$ and $E$ have not any right–scattered points.

Proof: Assertions 1, 2, 3 and 4 are readily verified from the definitions of $m_1^*$ and $\mu^*$. We only prove 5 because 6 is an obvious consequence from 5.

Suppose that $b \notin E$. By 3 we know that $R$ and $T \setminus R$ are Lebesgue measurable sets and $\mu_L(R) = 0$, so that, from the properties of the Lebesgue outer measure $\mu^*$ we deduce that

$$\mu^*(E) = \mu^*(E \cap (R \cup (T \setminus R))) = \mu^*(E \cap R) + \mu^*(E \cap (T \setminus R)) = \mu^*(E \cap T \setminus R),$$

hence, since $b \notin E \cap (T \setminus R)$ and $E \cap (T \setminus R)$ have not any right-scattered points, from 2 we arrive at

$$\mu^*(E) = \mu^*(E \cap (T \setminus R)) = m_1^*(E \cap (T \setminus R)),$$

whence it follows, as $R$ is a $\Delta$–measurable set, that

$$m_1^*(E) = m_1^*(E \cap R) + m_1^*(E \cap (T \setminus R))$$

$$= \sum_{i \in I_E} (\sigma(t_i) - t_i) + \mu^*(E),$$

and we get the assertion 5. \hfill \Box

Now we can prove the criterion for $\Delta$–measurability of sets.

Proposition 3.1 Let $A \subset T$. Then $A$ is $\Delta$–measurable if and only if $A$ is Lebesgue measurable.

In such a case, the following properties hold for every $\Delta$–measurable set $A$:
1. If $b \notin A$, then
   \[ \mu_\Delta(A) = \sum_{i \in I_A} (\sigma(t_i) - t_i) + \mu_L(A). \] (3.3)

2. $\mu_\Delta(A) = \mu_L(A)$ if and only if $b \notin A$ and $A$ have not any right-scattered points.

**Proof:** Let $A$ be a $\Delta$–measurable set; let us see that $A$ is Lebesgue measurable.

Suppose first that $b \notin A$. Let be $E \subset [a, b]$.

i) If $b \notin E$, then, from the equality $[a, b] \setminus A = (T \setminus A) \cup ([a, b] \setminus T)$ and property 5 in Lemma 3.2, using that $A$ is $\Delta$–measurable and $T$ is Lebesgue measurable, we obtain

\[
\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap ([a, b] \setminus A)) \\
\leq \mu^*(E \cap A) + \mu^*(E \cap (T \setminus A)) + \mu^*(E \cap ([a, b] \setminus T)) \\
= m_1^*(E \cap A) + m_1^*(E \cap (T \setminus A)) - \sum_{i \in I_{E \cap T}} (\sigma(t_i) - t_i) \\
+ \mu^*(E \cap ([a, b] \setminus T)) \\
= m_1^*(E \cap T) - \sum_{i \in I_{E \cap T}} (\sigma(t_i) - t_i) + \mu^*(E \cap ([a, b] \setminus T)) \\
= \mu^*(E \cap T) + \mu^*(E \cap ([a, b] \setminus T)) = \mu^*(E),
\]

so that,

\[ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap ([a, b] \setminus T)). \]

ii) If $b \in E$, then, as $\mu^*\{b\} = 0$, it follows from i) that

\[
\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap ([a, b] \setminus A)) \\
\leq \mu^*((E \cap [a, b]) \cap A) + \mu^*((E \cap [a, b]) \cap ([a, b] \setminus A)) \\
= \mu^*(E \cap [a, b]) \leq \mu^*(E),
\]

hence,

\[ \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap ([a, b] \setminus A)). \]

This proves that $A$ is Lebesgue measurable.

Now suppose that $b \in A$.

$A \setminus \{b\}$ is $\Delta$–measurable as the difference of two $\Delta$–measurable sets, from the previous argument we know that $A \setminus \{b\}$ is Lebesgue measurable.

Since $\{b\}$ is Lebesgue measurable it turns out that $A$ is Lebesgue measurable.

The fact that if $A$ is Lebesgue measurable then $A$ is $\Delta$–measurable, follows similarly.

The last part is an immediate consequence from properties 5 and 6 in Lemma 3.2. \[ \square \]
From this Proposition we have the following Corollary.

**Corollary 3.1** If $A \subset [a, b]$ is a Lebesgue measurable set, then the set $A \cap \mathbb{T}$ is $\Delta$–measurable.

## 4 Lebesgue and $\Delta$–measurable Functions

In this section, bearing in mind the criterion for $\Delta$–measurability of sets, proved in the previous section, we compare Lebesgue $\Delta$-measurable functions with Lebesgue measurable functions.

In order to do this, given a function $f : \mathbb{T} \to \bar{\mathbb{R}}$, we need an auxiliary function which extends $f$ to the interval $[a, b]$, $\tilde{f} : [a, b] \to \bar{\mathbb{R}}$ defined as

$$
\tilde{f}(t) := \begin{cases} 
  f(t), & \text{if } t \in \mathbb{T}, \\
  f(t_i), & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I,
\end{cases}
$$

(4.1)

with $I \subset \mathbb{N}$ and $\{t_i\}_{i \in I}$ defined in (3.1).

The obtained result is the following.

**Proposition 4.1** Assume that $f : \mathbb{T} \to \bar{\mathbb{R}}$ and $\tilde{f} : [a, b] \to \bar{\mathbb{R}}$ is the extension of $f$ to $[a, b]$, defined in (4.1). Then, $f$ is $\Delta$–measurable if and only if $\tilde{f}$ is Lebesgue measurable.

**Proof:** Suppose first that $f$ is $\Delta$–measurable.

Let $\alpha \in \mathbb{R}$. By denoting as $A_\alpha = f^{-1}([\infty, \alpha)) \subset \mathbb{T}$ we have that

$$
\tilde{f}^{-1}([\infty, \alpha)) = \{ t \in \mathbb{T} : \tilde{f}(t) < \alpha \} \cup \{ t \in \bigcup_{i \in I} (t_i, \sigma(t_i)) : \tilde{f}(t) < \alpha \}
$$

$$
= \{ t \in \mathbb{T} : f(t) < \alpha \} \cup \left( \bigcup_{i \in I \setminus A_\alpha} (t_i, \sigma(t_i)) \right)
$$

$$
= A_\alpha \cup \left( \bigcup_{i \in I \setminus A_\alpha} (t_i, \sigma(t_i)) \right).
$$

Since $f$ is $\Delta$–measurable we have that the set $A_\alpha$ is $\Delta$–measurable and, as a consequence of Proposition 3.1, we have that $A_\alpha$ is Lebesgue measurable.

Hence, $\tilde{f}^{-1}([\infty, \alpha))$ is Lebesgue measurable as countable union of Lebesgue measurable sets.

As $\alpha \in \mathbb{R}$ is arbitrary, it turns out that $\tilde{f}$ is Lebesgue measurable.

Now suppose that $\tilde{f}$ is Lebesgue measurable.

Given $\alpha \in \mathbb{R}$, $f^{-1}([\infty, \alpha))$ is Lebesgue measurable, hence, equality

$$
f^{-1}([\infty, \alpha)) = \tilde{f}^{-1}([\infty, \alpha)) \cap \mathbb{T},
$$

and Corollary 3.1 ensure that $f^{-1}([\infty, \alpha))$ is $\Delta$–measurable.

As a consequence, $f$ is $\Delta$–measurable. $\square$
5 Lebesgue and Lebesgue $\Delta$–Integrability

In this section, we establish an equivalence between Lebesgue $\Delta$–integrable and Lebesgue integrable functions. Likewise we get a formula to calculate the Lebesgue $\Delta$–integral as a sum of suitable Lebesgue integrals.

The following results give us this formula for simple and nonnegative functions respectively.

Lemma 5.1 Let $E \subset \mathbb{T}$ be a $\Delta$–measurable set such that $b \notin E$. If $S : \mathbb{T} \to [0, +\infty)$ is a simple and $\Delta$–measurable function with $S = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$ and $\tilde{S} : [a, b] \to [0, +\infty)$ is the extension of $S$ to $[a, b]$, defined in (4.1). Then $\tilde{S}$ is Lebesgue measurable and

$$\int_E S(s) \Delta s = \int_{\tilde{E}} \tilde{S}(s) \, ds,$$

with

$$\tilde{E} := E \cup \bigcup_{i \in I_E} (t_i, \sigma(t_i)).$$

(5.1)

Proof: From the definition of $\tilde{S}$ we deduce that $\tilde{S}$ is simple and

$$\tilde{S} = \sum_{j=1}^{n} \alpha_j \chi_{\tilde{A}_j},$$

with $\tilde{A}_j$ defined in (5.1) (with obvious notation).

Let $j \in \{1, \ldots, n\}$; since $S$ is $\Delta$-measurable, $A_j$ is $\Delta$-measurable and by Proposition 3.1 we know that $A_j$ is Lebesgue measurable, so that $\tilde{A}_j$ is Lebesgue measurable as countable union of Lebesgue measurable sets.

Whence, $\tilde{S}$ is Lebesgue measurable.

On the other hand, for every $j \in \{1, \ldots, n\}$, it follows from the countably additivity of $\mu_L$ that

$$\mu_L(\tilde{A}_j \cap \tilde{E}) = \mu_L(A_j \cap \tilde{E}) + \mu_L \left( \left( \bigcup_{i \in I_{A_j}} (t_i, \sigma(t_i)) \right) \cap \tilde{E} \right)$$

$$= \mu_L(A_j \cap E) + \sum_{i \in I_{A_j \cap E}} (\sigma(t_i) - t_i),$$

and since $b \not\in A_j \cap E$, by Proposition 3.1 we have that

$$\mu_L(A_j \cap E) + \sum_{i \in I_{A_j \cap E}} (\sigma(t_i) - t_i) = \mu_\Delta(A_j \cap E).$$
As a consequence we arrive at

\[\int_{\tilde{E}} \tilde{S}(s) \, ds = \sum_{j=1}^{n} \alpha_j \mu_L(\tilde{A}_j \cap \tilde{E}) = \sum_{j=1}^{n} \alpha_j \mu(\tilde{A}_j \cap E) = \int_{E} S(s) \, \Delta s,\]

and the proof is complete. \(\square\)

**Lemma 5.2** Assume that \(E \subset \mathbb{T}\) is a \(\Delta\)-measurable set such that \(b \notin E\), \(f : \mathbb{T} \to [0, +\infty]\) is a \(\Delta\)-measurable function and \(\tilde{f} : [a, b] \to [0, +\infty]\) the extension of \(f\) to \([a, b]\), defined in (4.1). Then,

\[\int_{E} f(s) \, \Delta s = \int_{E} \tilde{f}(s) \, ds,\]

where \(\tilde{E}\) is defined in (5.1).

**Proof:** Since \(f\) is \(\Delta\)-measurable, by Proposition 4.1 we know that \(\tilde{f}\) is Lebesgue measurable and hence (see [21]), there exists a sequence \(\{S_m\}_{m \in \mathbb{N}}\) of simple and Lebesgue measurable functions such that for every \(t \in [a, b]\) the following assertions carry over

a) \(0 \leq S_m(t) \leq S_{m+1}(t) \leq \tilde{f}(t)\) for all \(m \in \mathbb{N}\).

b) \(\lim_{m \to \infty} S_m(t) = \tilde{f}(t)\).

For every \(m \in \mathbb{N}\), we define \(h_m : [a, b] \to \mathbb{R}\) as

\[h_m(t) := \begin{cases} S_m(t) & \text{if } t \in \mathbb{T}, \\ S_m(t_i) & \text{if } t \in (t_i, \sigma(t_i)), \text{ for some } i \in I, \end{cases}\]

with \(I \subset \mathbb{N}\) and \(\{t_i\}_{i \in I}\) given in (3.1).

In view of the definition, the hypotheses of monotone convergence theorem hold for \(\{h_m\}_{m \in \mathbb{N}}\) and \(\tilde{f}\), so that

\[\int_{\tilde{E}} \tilde{f}(s) \, ds = \lim_{m \to \infty} \int_{E} h_m(s) \, ds. \tag{5.2}\]

Now, for every \(m \in \mathbb{N}\), we define \(f_m := h_m|_{\mathbb{T}}\). Clearly, function \(f_m\) is simple and \(\tilde{f}_m = h_m\), from Proposition 4.1 and Lemma 5.1 we deduce that \(f_m\) is \(\Delta\)-measurable and

\[\int_{E} f_m(s) \, \Delta s = \int_{\tilde{E}} \tilde{f}_m(s) \, ds = \int_{\tilde{E}} h_m(s) \, ds. \tag{5.3}\]

By construction, \(\{f_m\}_{m \in \mathbb{N}}\) verifies conditions a) and b) of Theorem 2.1 and so,

\[\int_{E} f(s) \, \Delta s = \lim_{m \to \infty} \int_{E} f_m(s) \, \Delta s.\]
Hence, as a consequence of (5.2) and (5.3) we obtain that

$$\int_E f(s) \Delta s = \lim_{m \to \infty} \int_E h_m(s) \, ds = \int_E \tilde{f}(s) \, ds,$$

which leads the desired result. \(\square\)

The following Theorem gives us the characterization of the \(\Delta\) – integrable functions.

**Theorem 5.1** Let \(E \subset T\) be a \(\Delta\)–measurable set such that \(b \notin E\), let \(\tilde{E}\) be the set defined in (5.1), let \(f : T \to \mathbb{R}\) be a \(\Delta\)–measurable function and let \(\tilde{f} : [a, b] \to \mathbb{R}\) be the extension of \(f\) to \([a, b]\), defined in (4.1).

Then, \(f\) is Lebesgue \(\Delta\)–integrable on \(E\) if and only if \(\tilde{f}\) is Lebesgue integrable on \(\tilde{E}\).

In which case, the following equality hold:

$$\int_E f(s) \Delta s = \int_{\tilde{E}} \tilde{f}(s) \, ds.$$

**Proof:** It is easy to prove that

$$\tilde{f}^+ = (\tilde{f})^+ \quad \text{and} \quad \tilde{f}^- = (\tilde{f})^-.$$

Since \(f^+\) and \(f^-\) are nonnegative functions, by Lemma 5.2 we arrive at

$$\int_E f^\pm(s) \Delta s = \int_{\tilde{E}} \tilde{f}^\pm(s) \, ds = \int_{\tilde{E}} (\tilde{f})^\pm(s) \, ds.$$

Hence, our claim follows from the concepts of \(\Delta\)-integrability and Lebesgue integrability. \(\square\)

As a direct consequence of this result, we have the following characterization of the space \(L^1(E)\), with \(E \subset T\).

**Corollary 5.1** Let \(E \subset T\) be a \(\Delta\) – measurable set, \(f : T \to \mathbb{R}\) and let \(\tilde{f} : [a, b] \to \mathbb{R}\) be the extension of \(f\) to \([a, b]\), defined in (4.1).

Then, \(f \in L^1(E)\) if and only if \(\tilde{f} \in L^1(\tilde{E})\).

From the previous Theorem we obtain the main result of this paper, which gives a formula to calculate the Lebesgue \(\Delta\)–integral.

**Theorem 5.2** Let \(E \subset T\) be a \(\Delta\) – measurable set, with \(b \notin E\). If \(f : T \to \mathbb{R}\) is \(\Delta\)–integrable on \(E\), then

$$\int_E f(s) \Delta s = \int_E f(s) \, ds + \sum_{i \in I_E} f(t_i)(\sigma(t_i) - t_i), \quad (5.4)$$

with \(I_E\) defined in (3.2).
Proof: Since \( f : \mathbb{T} \to \mathbb{R} \) is \( \Delta \)-integrable on \( E \), by Theorem 5.1 we know that \( \tilde{f} \) is Lebesgue integrable on \( \tilde{E} \) and

\[
\int_E f(s) \, \Delta s = \int_{\tilde{E}} \tilde{f}(s) \, ds. \tag{5.5}
\]

From (5.1) and [20] we have that

\[
\int_{\tilde{E}} \tilde{f}(s) \, ds = \int_{\tilde{E} \cap [a,b] \setminus \mathbb{T}} \tilde{f}(s) \, ds
\]

\[
= \int_E f(s) \, ds + \sum_{i \in I_E} \int_{(t_i, \sigma(t_i))]} f(t_i) \, ds
\]

\[
= \int_E f(s) \, ds + \sum_{i \in I_E} f(t_i)(\sigma(t_i) - t_i).
\]

The equality (5.5) ends the proof. \( \square \)

Remark 5.1 Note that expression (5.4) allow us to apply the theory of Lebesgue integrals coupled with the theory of convergence of series. As a consequence we can calculate, in some situations, the value of the integral or, if it is not possible, give some estimations of such value.

6 Applications to the Calculus of Antiderivatives.

In this section, we will use expression (5.4) to calculate the Antiderivative of several elemental functions defined on an arbitrary bounded time scale.

As we have seen, (5.4) gives us a method to calculate the exact value of the Lebesgue \( \Delta \)-integral of a Lebesgue \( \Delta \)-integrable function \( f \) on a \( \Delta \)-measurable set \( E \subset \mathbb{T} \). However, since \( E \) is an arbitrary subset of \( \mathbb{T} \), not necessarily a real interval, this formulation does not allow us to use the calculus of Antiderivatives on the real case. In order to do this, we rewrite equation (5.4) according to the values of Lebesgue integrals on adequate intervals. The result is the following.

Theorem 6.1 Let \( f : [a,b] \to \mathbb{R} \) be a Lebesgue integrable function on \( [a,b] \). Then for all \( r, t \in \mathbb{T} \), with \( r \leq t \), the following expression holds:

\[
\int_{[r,t] \cap \mathbb{T}} f(s) \, \Delta s = \int_{[r,t]} f(s) \, ds + \sum_{i \in I_{r,t}} \int_{(t_i, \sigma(t_i))} (f(t_i) - f(s)) \, ds, \tag{6.1}
\]

with \( I_{r,t} \equiv I_{[r,t] \cap \mathbb{T}} \) defined in (3.2).
Proof: Since \([r, t) = ([r, t) \cap \mathbb{T}) \cup \bigcup_{i \in I_{r,t}} (t_i, \sigma(t_i))]\). We have that
\[
\int_{[r, t)} f(s) \, ds = \int_{[r, t) \cap \mathbb{T}} f(s) \, ds + \sum_{i \in I_{r,t}} \int_{(t_i, \sigma(t_i))} f(s) \, ds.
\]

The result is a direct consequence of equation (5.4) and the fact that
\[
\int_{(t_i, \sigma(t_i))} f(s) \, ds = f(t_i) (\sigma(t_i) - t_i).
\]
\[\square\]

It is well known that if the bounded function \(f : [a, b] \rightarrow \mathbb{R}\) is Riemann integrable from \(r\) to \(t\), then it is Lebesgue integrable on \([r, t)\) and both values are equal, see [22, Theorem 6.29]. The same property holds in the case of \(\Delta\)-Riemann and \(\Delta\)-Lebesgue integrable functions, as Bohner and Guseinov prove in [9, Theorem 5.81].

On the other hand, the classical Lebesgue criterion for Riemann integrability [22, Theorem 6.29] establishes that the bounded function \(f : [a, b] \rightarrow \mathbb{R}\) is Riemann integrable from \(r\) to \(t\) if and only if the set of points where \(f\) is discontinuous has Lebesgue measure zero. Thus it is that, by equation (3.3), the set of all right dense points of \([r, t) \cap \mathbb{T}\) at which \(f\vert_{\mathbb{T}}\) is discontinuous is a set of \(\Delta\)-measure zero. It follows from [9, Theorem 5.82] that \(f\vert_{\mathbb{T}}\) is Riemann \(\Delta\)-integrable from \(r\) to \(t\).

Consequently, if the bounded function \(f : [a, b] \rightarrow \mathbb{R}\) is Riemann integrable from \(r\) to \(t\), from equality (6.1) we arrive at
\[
\int_{r}^{t} f(s) \, \Delta s = \int_{r}^{t} f(s) \, ds + \sum_{i \in I_{r,t}} \int_{t_i}^{\sigma(t_i)} (f(t_i) - f(s)) \, ds, \tag{6.2}
\]
where the integral on the left side of the expression denotes the Riemann \(\Delta\)-integral from \(r\) to \(t\) and the ones on the right denote the Riemann integral from \(r\) to \(t\) and from \(t_i\) to \(\sigma(t_i)\) respectively.

In the sequel we obtain the expressions of the integrals of some elemental Riemann \(\Delta\)-integrable functions. The expressions are valid on any time scale and depend strongly on the set of right – scattered points.

**Proposition 6.1** The following equality holds for all \(n \in \mathbb{N}\) and \(r, t \in \mathbb{T}\) such that \(r \leq t\):
\[
\int_{r}^{t} s^n \, \Delta s = \frac{t^{n+1} - r^{n+1}}{n+1} + \sum_{i \in I_{r,t}} \left( \frac{n}{n+1} t_i^n - \sum_{j=0}^{n-1} \frac{(\sigma(t_i))^{n-j} t_i^j}{n+1} \right) \mu(t_i),
\]
with \(I_{r,t}\) as in Theorem 6.1.

**Proof:** It follows immediately from expression (6.2), by defining \(f(s) = s^n\) for all \(s \in [a, b]\).
\[\square\]
Analogously, it is easy to prove the following equalities:

\[ \int_r^t \frac{1}{s} \, \Delta s = \log \frac{t}{r} + \sum_{i \in I_{t, r}} \left( \frac{\mu(t_i)}{t_i} - \log \frac{\sigma(t_i)}{t_i} \right), \]

\[ \int_r^t \frac{1}{s^n} \, \Delta s = \frac{t^{1-n} - r^{1-n}}{1-n} + \sum_{i \in I_{t, r}} \left( \mu(t_i) t_i^{-n} + \frac{(\sigma(t_i))^{1-n} - t_i^{1-n}}{1-n} \right), \]

whenever \( n \in \mathbb{N} \), \( n \neq 1 \) and \( 0 \not\in [r, t] \).

\[ \int_r^t \alpha s \, \Delta s = \frac{1}{\log \alpha} \left[ a^\alpha - a^\beta + \sum_{i \in I_{t, r}} a^{t_i} \left( \mu(t_i) \log a + 1 - a^{\mu(t_i)} \right) \right], \]

whenever \( a > 0 \).

\[ \int_r^t \cos \alpha s \, \Delta s = \frac{1}{\alpha} \left[ \sin \alpha t - \sin \alpha r + \sum_{i \in I_{t, r}} \alpha \mu(t_i) \cos \alpha t_i \right. \]

\[ \left. -2 \sum_{i \in I_{t, r}} \cos \left( \alpha \left( t_i + \frac{\mu(t_i)}{2} \right) \right) \sin \frac{\alpha \mu(t_i)}{2} \right], \]

\[ \int_r^t \sin \alpha s \, \Delta s = \frac{1}{\alpha} \left[ \cos \alpha r - \cos \alpha t + \sum_{i \in I_{t, r}} \alpha \mu(t_i) \sin \alpha t_i \right. \]

\[ \left. -2 \sum_{i \in I_{t, r}} \sin \left( \alpha \left( t_i + \frac{\mu(t_i)}{2} \right) \right) \sin \frac{\alpha \mu(t_i)}{2} \right], \]

whenever \( \alpha \neq 0 \).

By using the integration by parts formulas in the real Riemann integral, we obtain the following identity:

\[ \int_r^t \frac{e^s}{s} \Delta s = (t - 1) e^t + (1 - r) e^r \]

\[ + \sum_{i \in I_{t, r}} e^{t_i} \left[ t_i \mu(t_i) + 1 \right) - (\sigma(t_i) - 1) e^{\mu(t_i)} - 1 \].

The Change of Variable Theorem on the real Riemann integral allows us to deduce the following formula:

\[ \int_r^t \frac{s}{1 + s^2} \, \Delta s = \log \sqrt{\frac{1 + t^2}{1 + r^2}} + \sum_{i \in I_{t, r}} \left( \frac{t_i \mu(t_i)}{1 + t_i^2} - \log \sqrt{\frac{1 + (\sigma(t_i))^2}{1 + t_i^2}} \right). \]
If we are interested in to solve the $\Delta$–Riemann integral of a function $f$ that depends on $\rho(t)$ or $\sigma(t)$, we can obtain such integrals by redefining the function $f$ in the whole interval $[a, b]$ with as it is defined in $T$, but identifying $\rho(t) = \sigma(t) = t$ in $[a, b] \setminus T$. We illustrate this fact in the following examples.

\[ \int_r^t \frac{1}{\rho(s)^j s^k \sigma(s)^l} \Delta s = \frac{\frac{t^{1-n} - r^{1-n}}{1-n}}{\frac{t^{1-n} - r^{1-n}}{1-n}}\]

for every $r, t \in T$ such that $r \leq t$ and $j, k, l \in \mathbb{Z}$ such that $j + k + l = n > 0$

\[ \int_r^t \frac{1}{\rho(s)^j s^k \sigma(s)^l} \Delta s = \frac{\frac{t^{1-n} - r^{1-n}}{1-n}}{\frac{t^{1-n} - r^{1-n}}{1-n}}\]

whenever $j, k, l \in \mathbb{Z}$ and $j + k + l = n > 1$.

\[ \int_r^t \frac{1}{\rho(s)^j s^k \sigma(s)^l} \Delta s = \frac{\frac{t^{1-n} - r^{1-n}}{1-n}}{\frac{t^{1-n} - r^{1-n}}{1-n}}\]

whenever $j, k, l \in \mathbb{Z}$ and $j + k + l = 1$.

In all the cases, it is assumed that if some exponent is negative then $0 \not\in [r, t) \cap T$.

Moreover, we obtain the following expressions for the solutions that appear in the resolution of first and second order linear dynamic equations, see [8] for details.

\[ e_p(t, r) = \exp \left( \int_r^t p(s) \, ds \right) \prod_{i \in I_{r,t}} \frac{(1 + \mu(t_i)p(t_i))}{\exp \left( \int_{t_i}^{\sigma(t_i)} p(s) \, ds \right)} \]

and

\[ \int_r^t \frac{1}{1 + p(s)\mu(s)} \Delta s = t - r + \sum_{i \in I_{r,t}} (\oplus p)(t_i)(\mu^2(t_i)) \]

whenever $p : [a, b] \to \mathbb{R}$ is Riemann integrable with $p|_{T}$ regressive, i.e. $1 + \mu(s)p(s) \not\equiv 0$ for all $s \in T$, and $r, t \in T$ are such that $r < t$.

\[ \sin_p(t, r) = \left( \prod_{i \in I_{r,t}} \sqrt{1 + p^2(t_i)\mu^2(t_i)} \right) \sin \left( \sum_{i \in I_{r,t}} \arctan \left( p(t_i)\mu(t_i) \right) \right) \]
and

$$\cos_p(t, s) = \left( \prod_{i \in I_{s,t}} \sqrt{1 + p^2(t_i)\mu^2(t_i)} \right) \cos \left( \sum_{i \in I_{s,t}} \arctan(p(t_i)\mu(t_i)) \right)$$

### 6.1 Calculus on the ternary Cantor set.

Now we consider the classical ternary Cantor set. A description of this set and the values of $\mu(t)$ for all $t \in C$ can be founded in [8].

To calculate some integrals on subsets of $C$, it is necessary, as we have been in this section, to describe the set of right scattered points in $C$. One can verify that the set of such points is given by the following recursively equation:

$$R = \{ t \in T : t < \sigma(t) \} = \bigcup_{m=0}^{\infty} \left( \bigcup_{l=1}^{2^m} \{ t_{m,l} \} \right),$$

where $t_{0,1} = \frac{1}{3}$ and, for all $m \geq 1$,

$$t_{m,l} = \begin{cases} \frac{1}{3^{m+1}}, & \text{if } l = 1, \\ t_{m,k} + \frac{2}{3^{m-j}}, & \text{if } l = 2^j + k; j = 0, \ldots, m - 1, k = 1, \ldots, 2^j. \end{cases}$$

To give the value of the integral of a rd – continuous function $f$ in $[0, t) \subset C$, we must look for the right scattered points $s \in C$ such that $s < t$.

Thus, let $t \in R$ a right scattered point in $C$, then $t = t_{m,l}$ for some $m \in \mathbb{N}$ and $l \in \{1, \ldots, 2^m\}$. By construction, we have that

$$R \cap [0, t) = \bigcup_{k=0}^{\infty} \left( \bigcup_{j=1}^{i_k(t_{m,l})} \{ t_{k,j} \} \right),$$

where, for every $k \in \mathbb{N}$, the value of $i_k$ is given by

$$i_k(t_{m,l}) = \begin{cases} (2l - 1) \cdot 2^{k-(m+1)}, & \text{if } m \leq k - 1, \\ l - 1, & \text{if } m = k, \\ j - 1 + \frac{(-1)^l + 1}{2}, & \text{if } m = k + 1, \\ i_k(t_{m-1,l}), & \text{if } m \geq k + 2, \end{cases}$$

with $j = \frac{1}{2} \left[ l + \frac{1 - (-1)^l}{2} \right]$. 

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As a consequence, the Lebesgue $\Delta$–integral on $[0,t)$ of a rd–function $f$ is given by

$$\int_{[0,t) \cap T} f(s) \Delta s = \int_{[0,t)} f(s) \, ds + \sum_{k=0}^{\infty} \sum_{j=1}^{i_k(t_m,l)} \left( f(t_{k,j}) - f(s) \right) ds.$$  

In particular, when $f(s) = s$ for all $s \in T$, we have, using that $\mu(t_{k,j}) = 3^{-(k+1)}$ for all $k \in \mathbb{N}$ and $j \in \{1, \ldots, 2^m\}$, the following expression

$$\int_{[0,t) \cap T} s \, \Delta s = \frac{t^2}{2} - \frac{1}{2} \sum_{k=0}^{\infty} \sum_{j=1}^{i_k(t_m,l)} \mu^2(t_{k,j})$$  

$$= \frac{t^2}{2} - \frac{1}{2} S(t_{m,l}),$$

where

$$S(t_{m,l}) = \sum_{k=0}^{\infty} \frac{i_k(t_m,l)}{3^{2(k+1)}}$$

$$= \sum_{k=0}^{m-2} \frac{i_k(t_{m-1,l})}{3^{2(k+1)}} + \left( j - 1 + \frac{(-1)^l + 1}{2} \right) \frac{1}{3^{2m}}$$  

$$+ \frac{l - 1}{3^{2(m+1)}} + (2l - 1) \sum_{k=m+1}^{\infty} \frac{2k-(m+1)}{3^{2(k+1)}}$$

$$= \sum_{k=0}^{m-2} \frac{i_k(t_{m-1,l})}{3^{2(k+1)}} + \left( j - 1 + \frac{(-1)^l + 1}{2} \right) \frac{1}{3^{2m}}$$  

$$+ \frac{9l - 2}{7 \cdot 3^{2(m+1)}}.$$

References


References:


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