The Embedding of Partial Triple Systems
When 4 Divides $\lambda$

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We show that if 4 divides $\lambda$, then any partial triple system of order $r$ and index $\lambda$ can be embedded in a proper triple system of index $\lambda$ and order $n$ whenever $n$ is $\lambda$-admissible and $n \geq 2r + 1$. Moreover we find a set of necessary conditions for the embedding of a partial triple system of index $\lambda$ when $\lambda$ is even and show that when 4 divides $\lambda$, then a very closely related set of conditions is sufficient.

1. INTRODUCTION

A partial triple system of order $r$ and index $\lambda$, a $PTS(r, \lambda)$ for short, is a collection of triples of elements of an $r$-set such that each pair of elements is in at most $\lambda$ of the triples. Such a $PTS(r, \lambda)$ is maximal if no further triples can be added to the collection without contravening one of the rules. If each pair of elements is in exactly $\lambda$ triples, then we have a triple system of order $r$ and index $\lambda$, a $TS(r, \lambda)$ for short. It is well known that these exist if and only if $r \geq 3$ and the following two conditions are satisfied (see [16], for example):

$$\lambda r(r - 1) \equiv 0 \pmod{3}$$

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and

$$\lambda(r - 1) \equiv 0 \pmod{2}.$$  

When $r$ and $\lambda$ obey these conditions, we call $r$ $\lambda$-admissible.

A well known conjecture of Lindner [14, 15] is that any $PTS(r, 1)$ can be extended to a $TS(n, 1)$ by the introduction of further elements and triples if $n$ is 1-admissible and $n \geq 2r + 1$. The number $2r + 1$ here is best possible. It is natural to extend Lindner's conjecture to all values of $\lambda$. Thus we have the following more general conjecture.

**Conjecture 1.** Any $PTS(r, \lambda)$ can be extended to a $TS(n, \lambda)$ whenever $n$ is $\lambda$-admissible and $n \geq 2r + 1$.

Often the $PTS(r, \lambda)$ is said to be embedded in the $TS(n, \lambda)$.

Early results on Lindner's conjecture were given by Treash [22] and Lindner [14]. Conjecture 1 was proved for $\lambda = 1$ and $n \geq 4r + 1$ by Andersen et al. [5] and more recently for any value of $\lambda$ and $n \geq 4r$ (except possibly if $r \leq 14$ when $\lambda$ is even) by Rodger and Stubbs [19] and Stubbs [21].

Our main achievement in this paper is to prove Conjecture 1 in the case when $4 | \lambda$.

**Theorem 1.1.** Let $4 | \lambda$. Then any $PTS(r, \lambda)$ can be extended to a $TS(n, \lambda)$ whenever $n$ is $\lambda$-admissible and $n \geq 2r + 1$.

Our method of proof is quite different from that used by Andersen et al. [5], Rodger and Stubbs [19], or Stubbs [21]. It bears a close affinity to work on latin squares, timetables, and Hamiltonian decompositions of complete graphs by Andersen, Hilton, Nash-Williams, and Rodger [1–4, 10–12, 17, 18]. It brings out the very close relationship between these embedding problems and some problems on edge-colouring graphs, and it leads to results which are really much more detailed and illuminating than Theorem 1.

Unfortunately there is one conjecture on edge-colouring which we are unable to prove (Conjecture 2); if proved it would enable us to prove an extremely strong conjecture on embedding $PTS(r, \lambda)$ when $\lambda$ is even (Conjecture 3), and would incidentally solve Conjecture 1 whenever $\lambda$ is even. The connection between Conjectures 2 and 3 is explained in detail elsewhere [13].

2. TWO CONJECTURES

In this section we explain briefly two conjectures which we believe to incorporate the real truth about embedding maximal $PTS(r, \lambda)$'s when $\lambda$ is
even. There is quite a lot of evidence in the rest of this paper to support these conjectures. It is explained elsewhere [13] why a slight extension of Conjecture 2 on edge-colourings implies Conjecture 3 on embedding $PTS(r, \lambda)$'s when $\lambda$ is even.

First we need to explain a number of graph-theoretical concepts, some of which are non-standard. For terminology not defined here, see [8].

A graph consists of a set $V(G)$ of vertices, a set $L^{1/2}(G)$ of half-loops, and a set $E(G)$ of edges. Each edge is incident with two distinct vertices and contributes one to the degree of each of the vertices with which it is incident. Each half-loop is incident with one vertex and contributes one to the degree of that vertex. A loop at a vertex is the union of two half-loops at that vertex and so contributes two to the degree of the vertex. The set of all loops of $G$ is denoted by $L(G)$. The set of all half-loops of $G$ which are not part of a loop of $G$ are denoted by $H(G)$. Thus $L^{1/2}(G) = (\bigcup_{L \in L(G)} I) \cup H(G)$. A graph $G$ in which all half-loops occur in pairs, each pair forming a loop (so $H(G) = \emptyset$), is a normal graph. A graph is regular of degree $d$ if the sum of the degrees due to the edges and the half-loops incident with each vertex is $d$. Clearly each normal graph $G$ with maximum degree $\Delta = \Delta(G)$ can be turned into a regular graph of degree $\Delta$ by the introduction of the appropriate number of half-loops at each vertex.

An edge colouring of a graph $G$ is a function $\alpha : E(G) \cup L(G) \cup H(G) \to C$, where $C$ is a set of colours. Note that this implies that each loop receives one colour, and so the two corresponding half-loops receive the same colour.

We now introduce a variant of the idea of an edge-colouring. A split-loop colouring is a function $\alpha : E(G) \cup L^{1/2}(G) \to C$. Thus a split-loop colouring may be thought of as a kind of edge colouring in which each loop receives two colours (possibly the same colour twice). In fact, if we “split” each loop of $G$ by inserting a vertex in each loop, forming a loopless graph $G^*$, then a split-loop colouring of $G$ naturally corresponds to an edge-colouring of $G^*$.

A $\lambda$-half-loop factor of a graph $H$ is a subgraph which is regular of degree $\lambda$. A $\lambda$-factor of a graph $H$ is a subgraph which is normal and is regular of degree $\lambda$. Thus a $\lambda$-factor has its usual meaning, and any loop contributes two to the vertex it is on; but a $\lambda$-half-loop factor may have some half-loops. A $\lambda$-half-loop factorization of a regular graph $H$ is a partition of $E(H) \cup L^{1/2}(H)$ into $\lambda$-half-loop factors. Thus a $\lambda$-half-loop factorization of $H$ is a split-loop colouring of $H$ in which each colour class is a regular half-loop factor.

A split-loop colouring of a graph $H$ with colours $c_1, \ldots, c_k$ is said to be equalized if, for each $i, j \in \{1, \ldots, k\}, i \neq j$,

$$||C_i| - |C_j|| \leq 1,$$
where, for $1 \leq i \leq k$, $C_i$ is the set of edges of $H$ of colour $c_i$ (so half-loops are not included in $C_i$).

If $v$ is a vertex in $H$ then a split-loop colouring of $H$ is skew-free on $v$ if not more than half the half-loops at $v$ have the same colour. The colouring is skew-free if, for each $v \in V(H)$, it is skew-free on $v$.

We are now in a position to state Conjecture 2.

Conjecture 2. Let $\lambda$ be even and let $x \geq 2$. Let $H$ be a normal regular connected graph of degree $x\lambda$. Then $H$ has an equalized skew-free $\lambda$-half-loop factorization if and only if the following two conditions are satisfied:

(i) when $x > 2$, $H$ does not have exactly one loop,
(ii) when $x = 2$, the number of loops of $H$ is even.

Two simple cases where we cannot prove Conjecture 2 are when $\lambda = 2$, $x = 3$ and $H$ is a normal graph with two or three loops. Theorems 5.3 and 5.4 show that, when $x$ and $\lambda$ are even, if the number of loops is even, or if there is a vertex on which the number of loops is at least 2, then $H$ does have an equalized skew-free $\lambda$-half-loop factorization. It is convenient now to show that conditions (i) and (ii) are necessary for the existence of such a factorization.

Proof of the Necessity in Conjecture 2. Suppose first that $H$ has exactly two half-loops, both on the same vertex, say $v_0$ (so $H$ has one loop). If $H$ has a skew-free half-loop factorization, then $H$ would have a $\lambda$-factor $F$ with exactly one half-loop on $v_0$. Remove the half-loop from $F$. Then $F$ would be a graph without loops or half-loops with exactly one vertex ($v_0$) of odd degree, the remaining vertices having even degree ($\lambda$). This is impossible. Therefore $H$ cannot have exactly one loop. This proves (i).

Now suppose that $x = 2$. Let $e(H)$ and $l(H)$ be the number of (proper) edges and half-loops, respectively, of $H$. Then $2e(H) + l(H) = (x\lambda)|V(H)|$. Suppose that $H$ had an equalized skew-free $\lambda$-half-loop factorization, and let $F_1$ and $F_2$ be the $\lambda$-half-loop factors. The number of half loops in $F_1$ must equal the number of half-loops in $F_2$ since this factorization is skew-free. Then $F_1$ and $F_2$ contain the same number of edges, so $e(H)$ is even. Therefore 4 divides $l(H)$, so the number of loops is even since $H$ is normal.

We now turn to the main conjecture, Conjecture 3, on embedding a $PTS(r, \lambda)$ when $\lambda$ is even. Conjecture 3 takes its inspiration from Ryser's theorem [20] on embedding latin rectangles, or more particularly from a development of Ryser's theorem due to Cruse [9].

Given a $PTS(r, \lambda)$ $T$, let $G$ be the normal loopless multigraph in which the vertex set is the set of elements of $T$, and two vertices of $G$ are joined by $x$ edges if they are in $\lambda - x$ triples of $T$. We call $G$ the missing-edge graph of $T$. If we are trying to embed $T$ in a $TS(n, \lambda)$, then the integer $n$ will be
known to us; assuming that \( \Delta(G) \leq \lambda(n-r) \) and that \( \lambda(n-r) - d_G(v) \) is even for each \( v \in V(T) \), we define a normal regular graph \( G^\circ \) by adjoining the requisite number of loops at each vertex of \( G \) to make \( G^\circ \) regular of degree \( \lambda(n-r) \). For \( v \in V(T) \), let \( N(v) \) be the number of triples which contain the vertex \( v \).

We conjecture now state Conjecture 3.

**Conjecture 3.** Let \( \lambda \) be even. A PTS(\( r, \lambda \)) \( T \) can be embedded in a TS(\( n, \lambda \)) without inserting any further triples on the elements of \( T \) if and only if the following four conditions are satisfied:

(i) \( n \) is \( \lambda \)-admissible,

(ii) \( N(v) \geq \lambda(2r-n-1)/2 \) (for all \( v \in V(T) \)),

(iii) \( \sum_{v \in V(T)} N(v) \leq \lambda((n-r)^r) + \binom{n}{2} - r(n-r)/2 \), with equality if \( n-r=2 \),

(iv) \( G^\circ \) contains no component with exactly one loop, and if \( n-r=2 \), \( G^\circ \) contains no component with an odd number of loops.

In this conjecture, condition (ii) ensures that \( G^\circ \) can be formed (see Lemma 2.1). The difficulty in proving Conjecture 3 really lies in trying to cope with condition (iv). It is convenient to postpone the proof of the necessity in Conjecture 3 until Section 4 (see Theorem 4.3).

We now show that condition (ii) in Conjecture 3 implies that the definition of \( G^\circ \) is valid.

**Lemma 2.1.** Let \( \lambda \) be even. Let \( T \) be a PTS(\( r, \lambda \)). If condition (ii) of Conjecture 3 is satisfied then \( \Delta(G) \leq \lambda(n-r) \) and \( \lambda(n-r) - d_G(v) \) is even (for all \( v \in V(G) \)) (and so the definition of \( G^\circ \) is valid).

**Proof.** Since \( T \) is a PTS(\( r, \lambda \)), there can be at most \( \lambda(r-1) \) triples incident with each \( v \in V(T) \). From the definition of \( G \) as the missing-edge graph, we have

\[
d_G(v) + 2N(v) = \lambda(r-1) \quad \text{(for all } v \in V(T))
\]  

From (ii) it follows that

\[
d_G(v) \leq \lambda(r-1) - \lambda(2r-n-1) - \lambda(n-r),
\]

and so \( \Delta(G) \leq \lambda(n-r) \). Since \( \lambda \) is even, it follows from (1) that

\[
\lambda(n-r) - d_G(v) \equiv d_G(v) \pmod{2}
\]

\[
= \lambda(r-1) - 2N(v)
\]

\[\equiv 0 \pmod{2}.
\]
3. **Amalgamated Triple Systems**

Roughly speaking, an amalgamated TS is what you get if you take a TS and amalgamate several of the vertices. Let $\lambda K_n$ be the graph on $n$ vertices in which each pair of vertices is joined by $\lambda$ edges. Here, and in the rest of this paper, we think of a TS($n, \lambda$) as $\lambda K_n$, in which the edges are coloured with $\lambda n(n - 1)/6$ colours, each colour class forming a $K_3$.

We now give a formal definition of an amalgamation of a graph. Given a normal graph $G$, let $U \subseteq V(G)$ and let $Q$ be an element, $Q \notin V(G)$. Then $H$ is an amalgamation of $G$ formed by amalgamating the vertices in $U$ if $H$ is a normal graph with vertex set $\{Q\} \cup (V(G) \setminus U)$ and if there is a bijection

$$\phi: E(G) \cup L(G) \rightarrow E(H) \cup L(H)$$

such that

(Ai) $\phi(e)$ joins two vertices $x$ and $y$ in $V(H)$ if $e$ joins $x$ and $y$ and $x, y \in V(G) \setminus U$,

(Aii) $\phi(e)$ joins two vertices $x$ and $Q$ in $V(H)$ if $e$ joins $x$ and $y$ and $x \in V(G) \setminus U$, $y \in U$,

(Aiii) $\phi(e)$ is a loop on $x$ if $e$ is a loop on $x$ and $x \in V(G) \setminus U$, and

(Aiv) $\phi(e)$ is a loop on $Q$ if either $e$ joins two vertices $x, y \in U$ or $e$ is a loop on a vertex $x$ and $y \in U$.

The vertex $Q$ is called the amalgamated vertex (or the source vertex). It is understood that if $G$ has an edge colouring $\alpha$, then this will be transferred to $H$ by the bijection $\phi$. Notice that $G$ and $H$ are normal graphs.

With a TS($n, \lambda$) as an edge-coloured $\lambda K_n$ in mind, if $U \subseteq V(\lambda K_n)$ then the amalgamation $W$ of $\lambda K_n$ formed by amalgamating $U$ is called an amalgamated TS($n, \lambda$) (or an amalgamated triple system of order $n$ and index $\lambda$). Thus, for example if $U = V(\lambda K_n)$ then $W$ would be a graph consisting of one vertex ($Q$) and $\lambda(n - 1)/2$ loops; the loops would be coloured with $\lambda n(n - 1)/6$ colours and there would be three loops of each colour.

The method of proof of our main theorem is essentially to reverse the process of amalgamation of TS($n, \lambda$)'s. To this end we need to study amalgamation itself in some detail. So we start by giving a number of properties satisfied by amalgamated TS($n, \lambda$)'s.

**Lemma 3.1.** An amalgamated TS($n, \lambda$) $S$ with $n - r$ vertices amalgamated satisfies the following properties.

(Bi) Each vertex has degree $\lambda(n - 1)$ except for the source vertex $Q$
which has degree $\lambda(n-r)(n-1)$. (If $r = n - 1$, then any vertex can be designated as the source vertex.)

(Bii) The source vertex is incident with $\lambda(n-r)(n-r-1)/2$ loops; no other vertex has any loops on it.

(Biii) The number of edges between two vertices $x$ and $y$, where $x \neq y$, is $d(x)d(y)/\lambda(n-1)^2$.

(Note that $d(x)$ denotes the degree of the vertex $x$.)

Proof. These properties follow directly from the definitions.

Recall that a $TS(n, \lambda)$ is a $\lambda K_n$ whose edges are coloured with $\lambda n(n-1)/6$ colours so that each colour class forms a $K_3$.

Lemma 3.2. An amalgamated $TS(n, \lambda)$ $S$ with $n-r$ vertices amalgamated has the following property:

(Biv) The three edges of any given colour induce a subgraph of one of the following four types:

Note. We call these triangles 3-triangles, 2-triangles, 1-triangles, and 0-triangles, respectively, and sometimes denote them by $\{Q, Q, Q\}$, $\{Q, Q, x\}$, $\{Q, x, y\}$, and $\{x, y, z\}$, respectively.

Proof. When $S$ is formed by amalgamating $n-r$ vertices of $\lambda K_n$, the edge colouring of $\lambda K_n$ is carried over to $S$.

Associated with an amalgamated $TS(n, \lambda)$ $S$ with $n-r$ vertices amalgamated, we have a $PTS(r, \lambda)$ $T$ consisting of all the 0-triangles of $S$. Based on $T$, we can construct the missing-edge graph $G$ and the associated normal regular graph $G^\circ$ as described before Conjecture 3. The missing-edge graph $G$ is then the set of the edges which are in 1-triangles of $S$ and which are not incident with $Q$. Each loop of $G^\circ$ corresponds to a 2-triangle of $S$. 
The amalgamated $TS(n, \lambda) S$ induces a skew-free split-loop colouring of the graph $G^\circ$ in a fairly obvious way. Suppose that vertices $v_1, ..., v_{n-r}$ of $\lambda K_n$ are amalgamated to form the vertex $Q$. For each $i \in \{1, ..., n-r\}$, let those edges of $\lambda K_n - \{v_1, ..., v_{n-r}\}$ which are in triples (coloured triangles) of $\lambda K_n$ with one vertex $v_i \in \{v_1, ..., v_{n-r}\}$, but with neither of the other two vertices in $\{v_1, ..., v_{n-r}\}$, be coloured $c_i$; let those edges of $\lambda K_n$ which are in triples of $\lambda K_n$ with two vertices $v_i$ and $v_j$ both in the set $\{v_1, ..., v_{n-r}\}$, but with the third vertex $v \notin \{v_1, ..., v_{n-r}\}$, and which join $v$ to $v_i$ be coloured $c_i$ also. Transferring this partial edge colouring of $\lambda K_n$ to $S$, we have that the edges and the half-loops of $G^\circ$ are coloured with $c_1, ..., c_{n-r}$. Each edge and each half-loop receives one colour. Also each loop of $G^\circ$ receives two distinct colours since each loop of $G^\circ$ corresponds to a triangle $\{v, v_i, v_j\}$ of $\lambda K_n$. Thus $G^\circ$ has a skew-free split-loop colouring with $c_1, ..., c_{n-r}$.

**Lemma 3.3.** In an amalgamated $TS(n, \lambda) S$, the induced skew-free split-loop colouring of $G^\circ$ is a $\lambda$-half-loop factorization of $G^\circ$.

**Proof.** Suppose that vertices $v_1, ..., v_{n-r}$ of the edge-coloured $\lambda K_n$ are amalgamated to form $S$. Each $v_i \in \{v_1, ..., v_{n-r}\}$ and each $v \in V(\lambda K_n) \setminus \{v_1, ..., v_{n-r}\}$ are joined in $\lambda K_n$ by $\lambda$ edges, each of which is, in $S$, either in a 2-triangle or in a 1-triangle. Each such edge gives rise in $G^\circ$ to a colour $i$ on a half-loop of $G^\circ$ or on an edge of $G^\circ$, respectively. Therefore each vertex in $G^\circ$ is incident with $\lambda$ edges or half-loops coloured $i$. Therefore the split-loop colouring of $G^\circ$ is a $\lambda$-half-loop factorization of $G^\circ$ (there are $n-r$ $\lambda$-half-loop factors).

**Lemma 3.4.** Let $\lambda$ be even. An amalgamated $TS(n, \lambda)$, with $n-r$ vertices amalgamated, has the following property:

\((Bv)\) If $n-r = 2$ then each component of $G^\circ$ has an even number of loops.

**Proof.** In this case $G^\circ$ is a regular graph of degree $2\lambda$. It has a skew-free split-loop colouring with two colours such that each of the two colour classes is, by Lemma 3.3, a $\lambda$-half-loop factor. If $G^\circ$ had a component with an odd number of loops, then, since the colouring is skew-free, each of the two $\lambda$-half-loop factors would have a component with an odd number of half-loops. Let $F$ be a component of one of the half-loop factors that has an odd number of half-loops, and let $F'$ denote $F$ with the half-loops removed. Then, since $\lambda$ is even, $F'$ has an odd number of vertices of odd degree, which is impossible. Therefore each connected component of $G^\circ$ has an even number of loops.
LEMMA 3.5. Let $\lambda$ be even. An amalgamated $TS(n, \lambda)$, with $n - r$ vertices amalgamated, has the property:

(Bvi) If $n - r > 2$ then no component of $G^o$ has exactly one loop.

Proof. In this case, $G^o$ is a regular graph of degree $(n - r)\lambda$. It has a skew-free split-loop colouring with $n - r$ colours such that each colour class is, by Lemma 3.3, a $\lambda$-half-loop factor. If $G^o$ had a component with exactly one loop, then, since the colouring is skew-free, two of the $\lambda$-half-loop factors would have a component with exactly one half-loop. Let $F$ be a component of one of the $\lambda$-half-loop factors that has exactly one half-loop, and let $F'$ denote $F$ with the half-loop removed. Then, since $\lambda$ is even, $F'$ has exactly one vertex of odd degree, which is impossible. Therefore no connected component of $G^o$ has exactly one loop.

Let $\mu(n, \lambda)$ denote the largest number of triples in a $PTS(n, \lambda)$. The value of $\mu(n, \lambda)$ was established by Bermond [6] and is given for even $\lambda$ in the following lemma.

LEMMA 3.6. For $n \geq 3$ and $\lambda$ even,

$$\mu(n, \lambda) = \begin{cases} \lambda \binom{n}{2}/3 & \text{if either } n \equiv 0 \text{ or } 1 \pmod{3} \text{ or } \lambda \equiv 0 \pmod{6}, \\ \left(\lambda \binom{n}{2} - 2\right)/3 & \text{if } n \equiv 2 \pmod{3} \text{ and } \lambda \equiv 2 \pmod{6}, \\ \left(\lambda \binom{n}{2} - 4\right)/3 & \text{if } n \equiv 2 \pmod{3} \text{ and } \lambda \equiv 4 \pmod{6}. \end{cases}$$

LEMMA 3.7. An amalgamated $TS(n, \lambda)$ with $n - r$ vertices amalgamated, has the following property:

(Bvii) The number of 3-triangles is at most $\mu(n - r, \lambda)$.

Proof. Suppose that vertices $v_1, ..., v_{n-r}$ were amalgamated to form $S$. Before amalgamation, by the definition of $\mu(n - r, \lambda)$, the vertices $v_1, ..., v_{n-r}$ could have had no more than $\mu(n - r, \lambda)$ triangles on them. Each coloured triangle on $v_1, ..., v_{n-r}$ before amalgamation becomes a 3-triangle after amalgamation. The lemma now follows.

4. QUASI TRIPLE SYSTEMS

In this section we define quasi $TS(n, \lambda)$'s when $\lambda$ is even. From the definition it is clear that an amalgamated $TS(n, \lambda)$ is a quasi $TS(n, \lambda)$. We show
that Conjecture 3 can be reformulated to state that, when \( \lambda \) is even, a quasi \( TS(n, \lambda) \) is an amalgamated \( TS(n, \lambda) \); this reformulation seems to be a more illuminating version of Conjecture 3.

Before defining a quasi \( TS(n, \lambda) \), we need the following lemma. (Recall that \( A(G) \) is the maximum degree of \( G \).)

**Lemma 4.1.** Suppose that \( H \) is a normal graph with \( \lambda n(n - 1)/2 \) edges (counting parallel edges) and loops. Suppose that \( H \) has an edge colouring with \( \lambda n(n - 1)/6 \) colours with three edges of each colour. Suppose \( H \) has a special vertex \( Q \), the source vertex, and that \( H \) satisfies conditions (Bi)-(Biv). Let \( G \) be the graph whose vertex set is \( V(H) \setminus \{Q\} \) and whose edge set is the set of those edges of 1-triangles of \( H \) which are not incident with \( Q \). Then the following two conditions are satisfied:

- (Bviii) \( \lambda(n-r) - d_G(v) \) is even, for all \( v \in V(G) \).
- (Bix) \( A(G) \leq \lambda(n-r) \).

*Proof.* From (Bi) and (Biii), the number of edges between \( v \in V(G) \) and \( Q \) is

\[
\frac{d_H(v) d_H(Q)}{\lambda(n-1)^2} = \frac{\lambda(n-1) \lambda(n-r)(n-1)}{\lambda(n-1)^2} = \lambda(n-r).
\]

Each such edge is in a coloured triangle so the number of 1-triangles incident with \( v \) is at most \( \lambda(n-r) \). Therefore \( A(G) \leq \lambda(n-r) \). Also, if \( y \) is the number of 2-triangles incident with \( v \), then \( d_G(v) = \lambda(n-r) - 2y \), so \( \lambda(n-r) - d_G(v) \) is even. This proves Lemma 4.1.

In view of Lemma 4.1, the definition of \( G^0 \) is valid (it is the regular graph of degree \( \lambda(n-r) \) formed from \( G \) by adjoining the appropriate number of loops to each vertex).

We now define a quasi \( TS. \) For \( \lambda \) even, a quasi \( TS(n, \lambda) \) is a normal graph \( H \) with \( \lambda n(n-1)/2 \) edges (counting parallel edges) and loops; it has an edge colouring with \( \lambda n(n-1)/6 \) colours with three edges of each colour; it contains a special vertex \( Q \); and for some positive integer \( r \) it satisfies (Bi)-(Biv).

For convenience, we restate the conditions (Bi)-(Bix) here:

- (Bi) Each vertex has degree \( \lambda(n-1) \) except for the source vertex which has degree \( \lambda(n-1)(n-r) \).
- (Bii) The source vertex is incident with \( \lambda(n-r)(n-r-1)/2 \) loops; no other vertex is incident with any loops.
- (Biii) The number of edges between \( x \) and \( y \), \( x \neq y \) is \( d(x) d(y)/\lambda(n-1)^2 \).
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(Biv) The three edges of each colour induce a 3-, 2-, 1-, or 0-triangle.

(Bv) If \( n - r = 2 \) then each component of \( G^\circ \) has an even number of loops.

(Bvi) If \( n - r > 2 \) then no component of \( G^\circ \) has exactly one loop.

(Bvii) The number of 3-triangles is at most \( \mu(n - r, \lambda) \).

(Bviii) \( \lambda(n - r) - d_G(v) \) is even for all \( v \in V(G) \).

(Bix) \( \Delta(G) \leq \lambda(n - r) \).

Since we have shown that if \( \lambda \) is even then an amalgamated \( TS(n, \lambda) \) satisfies (Bi)–(Bvi), it follows that when \( \lambda \) is even, an amalgamated \( TS(n, \lambda) \) is a quasi \( TS(n, \lambda) \).

We now make the following conjecture.

**Conjecture 4.** Let \( \lambda \) be even and \( n \) be \( \lambda \)-admissible. Then a quasi \( TS(n, \lambda) \) is an amalgamated \( TS(n, \lambda) \).

If Conjecture 4 is true then every quasi \( TS(n, \lambda) \) can be "undone" to produce a proper \( TS(n, \lambda) \). We show below that Conjectures 3 and 4 are equivalent.

Before doing this, we note the following further properties of a quasi \( TS(n, \lambda) \), \( S \). These properties are not of great importance, but may help the reader in understanding quasi \( TS(n, \lambda) \)'s. Let \( p \) and \( q \) be the number of 2- and 3-triangles of \( S \), respectively. Let \( s(G) = |E(G)| \).

**Lemma 4.2.** In a quasi \( TS(n, \lambda) \) \( S \) in which \( Q \) has degree \( \lambda(n - 1)(n - r) \), we have the following further properties:

(Bx) The number of vertices distinct from \( Q \) is \( r \).

(Bxi) The number of edges joining \( Q \) to other vertices is \( \lambda r(n - r) \).

(Bxii) The number of edges which are in 1-triangles and which join \( Q \) to other vertices is \( \lambda r(n - r) - 2p \).

(Bxiii) \( s(G) = \lambda r(n - r)/2 - p \).

(Bxiv) The average number of edges (excluding loops or half-loops) in a \( \lambda \)-half-loop factor of \( G^\circ \) is \( \lambda r/2 - p/(n - r) \).

(Bxv) \( p \leq \lambda(n - r)(n - r - 1)/2 \).

(Bxvi) \( q = (\lambda(n - r) - p)/3 \).

**Proof.** These results follow easily from the definitions and conditions (Bi)–(Bix).

We next use all this information on quasi \( TS \)'s to show that the conditions (i)–(iv) in Conjecture 3 are necessary for a \( PTS(r, \lambda) \) \( T \) to be
embeddable in a $TS(n, \lambda)$ without inserting any further triples on the elements of $T$.

**Theorem 4.3.** The conditions (i)–(iv) of Conjecture 3 are a necessary set of conditions for Conjecture 3.

**Proof.** Suppose that $T$ can be embedded in a $TS(n, \lambda)$ without inserting any further triples on the elements of $T$. Then clearly condition (i) is satisfied. From this $TS(n, \lambda)$, construct an amalgamated $TS(n, \lambda) S$ by amalgamating the $n - r$ vertices not in $T$. Then $S$ satisfies the conditions (Bi)–(Bxvi). From (Bv) and (Bvi) it follows that condition (iv) is satisfied.

For $v \in V(T)$, the number $N(v)$ is the number of 0-triangles on the vertex $v$. Let $R(v)$ be the number of 1-triangles on $v$. Then

$$R(v) + 2N(v) = \lambda(r - 1).$$

By (Bix) it follows that $R(v) \leq \Delta(G) \leq \lambda(n - r)$. Therefore

$$N(v) \geq (\lambda(r - 1) - \lambda(n - r))/2 = \lambda(2r - n - 1)/2.$$

Therefore condition (ii) is satisfied.

The number $\sum_{v \in V(T)} N(v)$ is the total number of edges which are in 0-triangles of $T$. Therefore

$$\lambda \binom{r}{2} - \Delta(G) = \sum_{v \in V(T)} N(v).$$

Therefore by (Bxiii),

$$\lambda \binom{r}{2} - \lambda r(n - r)/2 + p = \sum_{v \in V(T)} N(v). \quad (2)$$

By (Bxv) therefore,

$$\sum_{v \in V(T)} N(v) \leq \lambda \left( \binom{r}{2} + \binom{n - r}{2} - r(n - r)/2 \right).$$

If $n - r = 2$ then $p = \lambda$ and so by (2), equality holds. Therefore condition (iii) is satisfied.

We now go on to show that Conjectures 3 and 4 are equivalent.

**Theorem 4.4.** Conjecture 4 is equivalent to the sufficiency in Conjecture 3.

**Proof.** First suppose that the sufficiency in Conjecture 3 is correct. We need to show that any quasi $TS(n, \lambda)$ with $\lambda$ even an $n \lambda$-admissible is an
amalgamated $TS(n, \lambda)$. So let $S$ be a quasi $TS(n, \lambda)$ in which $Q$ has degree $\lambda(n-1)(n-r)$, and let $T$ be the associated $PTS(r, \lambda)$ obtained from $S$ by deleting $Q$ and the edges of all 1-, 2-, and 3-triangles. The graphs $G$ and $G^\circ$ may now be defined (we saw in Lemma 4.1 that this is possible), and we have that

$$d_G(v) + 2N(v) = \lambda(r-1).$$

From (Bix) it follows that $d_G(v) \leq A(G) \leq \lambda(n-r)$. Therefore

$$N(v) \geq (\lambda(r-1) - \lambda(n-r))/2 = \frac{\lambda(2r-n-1)}{2},$$

so condition (ii) is satisfied. Of course, since $n$ is $\lambda$-admissible, condition (i) is satisfied. By (Bv) and (Bvi), condition (iv) is satisfied. Condition (iii) follows as in the proof of Theorem 4.3.

By the sufficiency of Conjecture 3, it follows that $T$ can be completed to a $TS(n, \lambda)$. Amalgamating the $(n-r)$ new vertices, we obtain $S$ again. Thus $S$ is an amalgamated $TS(n, \lambda)$ as required. Therefore, assuming the sufficiency of Conjecture 3, it follows that, when $n$ is $\lambda$-admissible and $\lambda$ is even, then any quasi $TS(n, \lambda)$ is an amalgamated $TS(n, \lambda)$.

Now suppose that $n$ is $\lambda$-admissible, $\lambda$ is even, and any quasi $TS(n, \lambda)$ is an amalgamated $TS(n, \lambda)$. Let $T$ be a $PTS(r, \lambda)$ and suppose that conditions (i) to (iv) of Conjecture 3 are satisfied. It is shown in Lemma 2.1 that the definition of $G^\circ$ is valid. We wish to show that $T$ can be embedded in a $TS(n, \lambda)$. All we need to do is to complete $T$ to a quasi $TS(n, \lambda)$ with one further vertex $Q$, with the appropriate number of edges on $Q$, the edges being coloured so that each colour class is a 0-, 1-, 2-, or 3-triangle.

Let the triples of $T$ be the 0-triangles. Join each of the $r$ vertices of $T$ to a further vertex $Q$ by $\lambda(n-r)$ edges, and introduce $\lambda(n-r)(n-r-1)/2$ loops on $Q$. Corresponding to each edge $uv$ of $G$, select two edges, one joining $u$ to $Q$, the other joining $v$ to $Q$, and let these three edges be a 1-triangle; do this in such a way that the 1-triangles are edge-disjoint. For this to be possible, we need to know that the number of edges joining $v$ to $Q$ is at least $d_G(v)$; that is, $d_G(v) \leq \lambda(n-r)$. But this is a consequence of Lemma 2.1.

Also by Lemma 2.1, we have that $\lambda(n-r) - d_G(v)$ is even (for all $v \in V(G)$). On each $v \in V(G)$ we now insert $(\lambda(n-r) - d_G(v))/2$ 2-triangles. To be sure that this can be done, all we still need to know is that there are a sufficient number of loops on $Q$; that is,

$$\sum_{v \in V(G)} (\lambda(n-r) - d_G(v))/2 \leq \lambda \binom{n-r}{2}.$$

For $n-r \geq 3$, this is equivalent to the condition

$$\lambda r(n-r)/2 - \left(\lambda \binom{r}{2} - \sum_{v \in V(G)} N(v)\right) \leq \lambda \binom{n-r}{2}.$$
But this follows from condition (iii). Therefore the 2-triangles can be inserted, as described. For \( n - r = 2 \), of course we must use up all \( \lambda \) loops on \( Q \), so that \( \sum_{v \in V(G)} (\lambda(n - r) - d_G(v))/2 = \lambda(n^2 - r^2) \). This is equivalent to (iii) in this case.

The remaining loops on \( Q \) must now be partitioned into 3-triangles. From the fact that \( n \) is \( \lambda \)-admissible, it follows that the number of loops left is divisible by three, and so this partitioning can be performed.

We must now check that conditions (Bi)-(Bvi) are satisfied. In fact (Bi)-(Biv) are clear, and (Bv) and (Bvi) follow from condition (iv).

5. Evidence for Conjecture 2

In this section we prove Conjecture 2 in two important cases. The remaining case in which we have been unable to find a proof is when the half-loops of \( H \) are paired off into loops, the number of loops being odd, there being no two loops on the same vertex.

Before our first main result we need a few preliminary results. An edge colouring of a normal graph \( G \) with colours \( c_1, \ldots, c_k \) is equitable if, for each \( i, j \in \{1, \ldots, k\}, i \neq j, \)

\[ ||C_i(v) - |C_j(v)|| \leq 1 \quad \text{for all } v \in V(G), \]

where \( C_i(v) \) denotes the number of edges or loops of colour \( c_i \) incident with \( v \) (each loop is counted twice). It is balanced if, in addition, for each \( i, j \in \{1, \ldots, k\}, i \neq j, \)

\[ ||C_i(u, v) - |C_j(u, v)|| \leq 1 \quad \text{for all } u, v \in V(G), u \neq v, \]

where \( C_i(u, v) \) is the number of edges joining \( u \) and \( v \) of colour \( c_i \).

Recall that an edge colouring is equalized if, for each \( i, j \in \{1, \ldots, k\}, i \neq j, \)

\[ ||C_i - |C_j|| \leq 1, \]

where \( C_i \) denotes the number of edges (but not half-loops) coloured \( c_i \).

**Lemma 5.1.** Let \( k \geq 1 \) be given, and let \( G \) be a normal bipartite (multi)-graph. Then \( G \) has an equalized balanced edge colouring with \( k \) colours.

de Werra [23] proved this result without the qualification that the edge-colouring must be equalized. An elementary proof of his result may be found in [7] or [2]. The fact that the balanced edge colouring can be equalized is given as an exercise in [7]. Since it is crucial to our argument, we give an elementary proof here.
**Proof.** Let $G$ have a balanced edge colouring with colours $c_1, \ldots, c_k$, but suppose the edge colouring is not equalized. Then, for some $i, j \in \{1, \ldots, k\}$, $i \neq j$, we have

$$|C_i| \leq |C_j| - 2.$$ 

Consider the subgraph $H_0$ of edges coloured $c_i$ and $c_j$. From each multiple edge, remove pairs of edges, one of each colour, until there is either 0 or 1 edge left. The subgraph $H_1$ remaining after this is simple and has an equitable edge colouring with $c_i$ and $c_j$. $H_1$ consists of paths and even circuits. Since $|C_i| \leq |C_j| - 2$, there must be at least one alternately coloured path with the edge of each end coloured $c_j$. Interchange the colours on this path. Repeat this until the edge colouring of $H_1$ is equalized. Now restore the pairs of parallel edges. Then in $G$ we now have $||C_i| - |C_j|| \leq 1$, and the edge colouring is still balanced. Repeat this with different pairs of colours as necessary until the balanced edge colouring is equalized.

The next lemma demonstrates that when the requirement that the split-loop colouring must be skew-free is omitted, then one can find an equalized $\lambda$-half-loop factorization easily enough.

**Lemma 5.2.** Let $\lambda$ be even and $x \geq 1$. Let $H$ be a regular graph of degree $x \lambda$. Then $H$ has an equalized $\lambda$-half-loop factorization.

**Proof.** We let the loopless graph $H^*$ denote the graph $H$ with all its half-loops removed (since each loop is a pair of half-loops, all loops are removed). First note that the number of half-loops of $H$ is $\lambda x |V(H)| - 2 |E(H^*)|$, and so is even (since $\lambda$ is even). Pair off the half-loops and form a graph $H'$ by replacing each pair of half-loops with either a whole loop (if the pair of half-loops are on the same vertex) or an edge joining the two corresponding vertices (if the pair of half-loops are not on the same vertex). Then $H'$ is a regular normal graph. Since $H'$ has even degree, each component of $H'$ has an Eulerian cycle. Orient each such an Eulerian cycle, and let $D'$ be the directed graph thus formed. Then each vertex $D'$ has indegree $\lambda x / 2$ and outdegree $\lambda x / 2$. Let the vertices of $D'$ be $v_1, \ldots, v_n$. Form a bipartite graph $B'$ as follows. Let the vertex sets be $\{v_1', \ldots, v_n'\}$ and $\{v_1'', \ldots, v_n''\}$. For each directed edge from $v_i$ to $v_j$ in $D'$, place an edge between the vertices $v_i'$ and $v_j''$ in $B'$. Finally form a graph $B^*$ by omitting all edges of $B'$ corresponding to edges or loops of $H'$ inserted when the half-loops of $H$ were replaced. Then each edge of $B^*$ corresponds to an edge of $H^*$.

By Lemma 5.1, $B$ has an equalized equitable edge colouring with $x$ colours $c_1, \ldots, c_x$. We may form an edge colouring of $H^*$ by colouring an edge joining $v_i$ and $v_j$ with colour $c_i$ whenever the corresponding edge
joining $v_i'$ to $v_j''$ in $B$ is coloured $c_j$. Since the edge colouring of $B$ is equalized, the edge colouring of $H^*$ will also be equalized. Since the edge colouring of $B$ is equitable and the maximum degree of $B$ is not more than $\lambda x/2$, the number of edges of any given colour at any vertex of $B$ is not more than $\lambda/2$, and so it follows that the number of edges of any given colour at any vertex of $H'$ is not more than $\lambda$. We may now add back the half-loops of $H$ and colour them so as to produce a $\lambda$-half-loop factorization. Then this $\lambda$-half-loop factorization of $H$ is equalized, as required.

We remark that in this proof we only use the fact that $B$ has an equitable equalized edge colouring (rather than a balanced equalized edge colouring).

We now give the first important case in which we can prove Conjecture 2.

**Theorem 5.3.** Let $\lambda$ and $x$ both be even. Let $H$ be a normal regular connected graph of degree $x\lambda$. Suppose that the number of loops of $H$ is even. Then $H$ has an equalized skew-free $\lambda$-half-loop factorization.

**Proof.** For each vertex $v$ and for each loop on $v$, introduce a new vertex $v^*$, and replace the loop by two edges between $v$ and $v^*$. Let the graph formed be $H^*$. Let $l(H)$ be the number of loops of $H$ (so $2l(H)$). Then

$$x\lambda |V(H)|/2 = e(H) + l(H) = e(H^*) - l(H).$$

Since $x\lambda/2$ and $l(H)$ are both even, it follows that $e(H^*)$ is even. So $H^*$ has an Eulerian cycle of even length. Traverse an Eulerian cycle of $H^*$, colouring the edges alternately $\alpha$ and $\beta$. This yields a skew-free equalized $(\lambda x/2)$-half-loop factorization of $H$. Let $H_\alpha$ and $H_\beta$ be the $(\lambda x/2)$-half-loop factors coloured $\alpha$ and $\beta$, respectively. Since $x$ is even, by Lemma 5.2, $H_\alpha$ has an equalized $\lambda$-half-loop factorization (into $x/2$ $\lambda$-half-loop factors), and similarly so does $H_\beta$. Combining these, we obtain an equalized skew-free $\lambda$-half-loop factorization of $H$, as required.

We now give a second important case in which we can prove Conjecture 2.

**Theorem 5.4.** Let $\lambda$ and $x$ both be even and let $x \geq 4$. Let $H$ be a normal regular connected graph of degree $x\lambda$. Let the number of loops of $H$ be odd. If there is at least one vertex that is incident with at least two loops then $H$ has an equalized skew-free $\lambda$-half-loop factorization.

**Proof.** Insert a vertex in each loop of $H$, forming a graph $H^*$. In this case, $e(H^*)$ (and $e(H)$) is odd. Let $v_0$ be a vertex that is incident with at least two loops, and let $v^*$ be the vertex inserted in one of these loops.
Traverse an Eulerian cycle of $H^*$, colouring the edges alternately $\alpha$ and $\beta$, starting at $v^*$ with an edge coloured $\alpha$. Then both edges incident with $v^*$ are coloured $\alpha$. This edge colouring corresponds to a half-loop factorization of $H$ into two $(x\lambda/2)$-half-loop factors, say $H_\alpha$ and $H_\beta$. Each original loop, except for one on $v_0$, splits into two half-loops, one in $H_\alpha$ and one in $H_\beta$. The one exceptional loop on $v_0$ is coloured $\alpha$, and so both its constituent half-loops are in $H_\alpha$.

From Lemma 5.2, each of $H_\alpha$ and $H_\beta$ has a $\lambda$-half-loop factorization which is equalized on the edges. Together, these factorizations give a split-loop colouring which is skew-free on all vertices except possibly $v_0$ (since $H_\alpha$ contains more than half of the half-loops on $v_0$). We now ensure that the skew-free property can be obtained on $v_0$ as well.

**Remark.** If $4|\lambda$ then $H_\alpha$ and $H_\beta$ can still be formed as described here for any value of $x$. This is used in Lemma 6.2.

Consider the component $J$ of $H_\alpha$ which contains $v_0$. Pair off all the half-loops in $J$ and replace each pair by an edge or a loop, forming a regular normal graph $J^*$ of degree $x\lambda/2$. Let $D$ be a directed Eulerian cycle of $J^*$, and let $B'$ be the corresponding bipartite graph (as described in the proof of Lemma 5.2). Then $B'$ is regular of degree $x\lambda/4$. From $B'$ remove all edges corresponding to half-loops (or loops) of $J$. Let the bipartite graph thus formed be denoted by $B$. Then either $d_B(v_0^*) \leq (x\lambda/4) - 2$ or $d_B(v_0^*') \leq (x\lambda/4) - 2$ (possibly both of these are true); we may suppose that, in fact, $d_B(v_0^*) \leq (x\lambda/4) - 2$.

By Lemma 5.1, $B$ has an equitable equalized edge colouring with $x/2 \geq 2$ colours. In this edge colouring, at least two colours occur at $v_0^*$ less than $\lambda/2$ times. Therefore, in the corresponding split-loop colouring of $J$, there are two half-loops at $v_0^*$, each receiving one of these two colours. Since no colour occurs at a vertex of $B$ on more than $\lambda/2$ edges, the corresponding split-loop colouring of $J$ partitions the edges and half-loops of $J$ into $\lambda$-half-loop factors. Since the edge-colouring of $B$ is equalized, the split-loop colouring of $J$ is equalized. We can combine this $\lambda$-half-loop factorization of $J$ with $\lambda$-half-loop factorizations of the other components of $H_\alpha$, to obtain an equalized $\lambda$-half-loop factorization of $H_\alpha$. Combining this with the $\lambda$-half-loop factorization of $H_\beta$ gives an equalized skew-free $\lambda$-half-loop factorization of $H$.

6. Some Preliminary Results in the Case When 4 Divides $\lambda$

Up until now, all that has been written applies whenever $\lambda$ is even. We now give some more specialized results for when $4|\lambda$; our point in giving
them is solely to use them in the proof of the PTS embedding problem when $4 | \lambda$. Recall that we define $\varepsilon(H) = |E(H)|$.

Given a normal regular connected graph $H$ of degree $x \lambda$ with $x > 1$, let $y$ satisfy

$$y \in \{ \lfloor \varepsilon(H)/x \rfloor, \lceil \varepsilon(H)/x \rceil \}$$ if $x$ does not divide $\varepsilon(H)$ and $x > 2$, 
$$y \in \{ -1 + \varepsilon(H)/x, \varepsilon(H)/x, 1 + \varepsilon(H)/x \}$$ if $x | \varepsilon(H)$ and $x > 2$, 
and

$$y = \varepsilon(H)/x$$ if $x = 2$.

As described in the proof of Theorem 5.3, if $4 | \lambda$, $x > 2$, and $H$ is a normal regular graph of degree $x \lambda$ with an even number of loops, then $H$ has an equalized skew-free $(x \lambda/2)$-half-loop factorization into two $(x \lambda/2)$-half-loop factors $H_\alpha, H_\beta$. In this case we write $H = (H_\alpha, H_\beta)$.

**Lemma 6.1.** Let $4 | \lambda$ and $x \geq 2$. Let $H$ be a normal regular connected graph of degree $x \lambda$. Let the number of loops of $H$ be even. Let $H = (H_\alpha, H_\beta)$ (so that exactly half of the half-loops on each vertex are in $H_\alpha$). Then $H_\alpha$ has a $\lambda$-half-loop factor $F$ with

$$\varepsilon(F) = \{ y \}$$ if $2x$ does not divide $\varepsilon(H)$,

$$\varepsilon(F) = \{ \varepsilon(H)/x \}$$ if $2x | \varepsilon(H)$.

**Proof.** Clearly $\varepsilon(H_\alpha) = \varepsilon(H_\beta) = \varepsilon(H)/2$. If $x = 2$ this proves the lemma, so now assume that $x > 2$. By Lemma 5.2, each of these has an equalized $(\lambda/2)$-half-loop factorization into $x$ $(\lambda/2)$-half-loop factors. If $x | \varepsilon(H)/2$ then each of these has exactly $\varepsilon(H)/2x$ edges; combining two of the $(\lambda/2)$-half-loop factors in $H_\alpha$ yields the desired $\lambda$-half-loop factor $F$. So from now on suppose that $x > 2$ and that $2x$ does not divide $\varepsilon(H)$.

Since $\varepsilon(H)$ is even, we may write $\varepsilon(H) = x(2p) + (2q)$, where $0 < q < x$. Then $\varepsilon(H_\alpha) = xp + q$. The $(\lambda/2)$-half-loop factorization of $H_\alpha$ has $q$ $(\lambda/2)$-half-loop factors with $\lceil \varepsilon(H)/2x \rceil$ edges and $x - q$ with $\lfloor \varepsilon(H)/2x \rfloor$ edges.

If $q = x/2$, then $x \geq 4$ and $x | \varepsilon(H)$. Then $H_\alpha$ has at least two $(\lambda/2)$-half-loop factors with $(-1 + \varepsilon(H)/x)/2$ edges, and at least two with $(1 + \varepsilon(H)/x)/2$ edges. Combining these appropriately yields our $\lambda$-half-loop factor $F$.

If $q < x/2$ then $\varepsilon(H) = x(2p) + r$, where $0 < r = 2q < x$; also $H_\alpha$ has at least one $(\lambda/2)$-half-loop factor with $p + 1$ edges and at least two with $p$ edges. Therefore these can be combined to give $\lambda$-half-loop factors with either $2p$ or $2p + 1$ edges. But $2p = \lfloor \varepsilon(H)/x \rfloor$ and $2p + 1 = \lceil \varepsilon(H)/x \rceil$ in this case. The lemma follows therefore if $q < x/2$. 
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If $q > x/2$ then $s(H) = x(2p + 1) + s$, where $0 < s < x$; also $H_x$ has at least two $(\lambda/2)$-half-loop factors with $p + 1$ edges and at least one with $p$ edges. These can be combined to give $\lambda$-half-loop factors with either $2p + 1$ or $2p + 2$ edges. But now $2p + 1 = \lceil \varepsilon(H)/x \rceil$ and $2p + 2 = \lceil \varepsilon(H)/x \rceil$. The lemma therefore follows in this final case also.

We next give the analogous result in the case when the number of loops of $H$ is odd and two of the loops are on the same vertex. First we introduce some notation. From the remark in the proof of Theorem 5.4, if $4 \mid \lambda$, $x \geq 3$, and $H$ is a normal regular graph of degree $x\lambda$ with an odd number of loops, two of which occur at the same vertex $v_0$, then $H$ has a $(x\lambda/2)$-half-loop factorization into two $(x\lambda/2)$-half-loop factors $H_\alpha$ and $H_\beta$ with $s(H_\alpha) + 1 = s(H_\beta)$ and with two more half-loops in $H_\alpha$ at $v_0$ than in $H_\beta$ at $v_0$; also, at each vertex other than $v_0$, there are the same number of half-loops in $H_\alpha$ as in $H_\beta$. To describe this briefly, we write $H = (H_\alpha, H_\beta; v_0)$.

We now give the analogous result in the case when the number of loops of $H$ is odd and two of the loops are on the same vertex.

**Lemma 6.2.** Let $4 \mid \lambda$ and $x \geq 3$. Let $H$ be a normal regular connected graph of degree $x\lambda$. Let the number of loops of $H$ be odd and at least three. Let $H$ contain a vertex $v_0$ that is incident with at least two loops. Then either $H_\alpha$ or $H_\beta$ has a half-loop factor $F$ with the following properties:

(a) $F$ has at least one half-loop incident with $v_0$, and

(b) for each $v \in V(H)$, the number of half-loops in $F$ incident with $v$ is not more than the number of half-loops in $H - F$ incident with $v$.

Moreover, either we can choose $\varepsilon(F) = z$ for at least two consecutive values of $z$ satisfying

$$\lceil \varepsilon(H)/x \rceil - 1 \leq z \leq \lceil \varepsilon(H)/x \rceil + 1,$$

or $2x \mid (\varepsilon(H) + 3)$ and $H$ contains no vertex that is incident with more than three loops, in which case we can take $\varepsilon(F) = -1 + (\varepsilon(H) + 3)/x$.

**Proof.** Let $H = (H_\alpha, H_\beta; v_0)$. As described in the proof of Theorem 5.4, we can give $H_\alpha$ (or $H_\beta$) an equalized $(\lambda/2)$-half-loop factorization into $x$ $(\lambda/2)$-half-loop factors. Moreover, if the procedure in the proof of Theorem 5.4 is followed, then no colour appears on more than two more half-loops at $v_0$ than any other colour. We shall call such $(\lambda/2)$-half-loop factorizations standard.

Let $\varepsilon(H_\alpha) = px + q$, where $0 \leq q < x$. Then the standard $(\lambda/2)$-half-loop factorization has $x - q$ $(\lambda/2)$-half-loop factors with $\lceil \varepsilon(H_\alpha)/x \rceil$ edges and $q$ $(\lambda/2)$-half-loop factors with $\lceil \varepsilon(H_\alpha)/x \rceil + 1$ edges. Call $(\lambda/2)$-half-loop factors of $H_\alpha$ with no half-loops on $v_0$, $(x, 0)$-$(\lambda/2)$-half-loop factors, and
those with at least one half-loop on \( v_0 \). \((\alpha, 1)-(\lambda/2)\)-half-loop factors. Make similar definitions for the half-loop factors of \( H_\beta \). If there is an \((\alpha, 1)-(\lambda/2)\)-half-loop factor \( F_1 \) and also two \((\alpha, 0)-(\lambda/2)\)-half-loop factors \( F_2 \) and \( F_3 \), with different numbers of edges in \( F_2 \) and \( F_3 \), then the lemma follows by taking \( F_1 \cup F_2 \) and \( F_1 \cup F_3 \). If all \((\alpha, 0)-(\lambda/2)\)-half-loop factors have the same number of edges, but there are two \((\alpha, 1)-(\lambda/2)\)-half-loop factors with differing numbers of edges, then the lemma follows similarly. So suppose that in \( H_\alpha \) all \((\alpha, 1)-(\lambda/2)\)-half-loop factors have the same number \( m(\alpha, 1) \) of edges, and similarly, if there are any, then all \((\alpha, 0)-(\lambda/2)\)-half-loop factors have the same number \( m(\alpha, 0) \) of edges.

The lemma follows in the corresponding cases when we consider a standard \((\lambda/2)\)-half-loop factorization of \( H_\beta \); we may suppose therefore that in \( H_\beta \) all \((\beta, 1)-(\lambda/2)\)-half-loop factors have the same number \( m(\beta, 1) \) of edges and that if there are any, then all \((\beta, 0)-(\lambda/2)\)-half-loop factors have the same number \( m(\beta, 0) \) of edges.

If \( m(\alpha, 1) = m(\alpha, 0) \) (so \( m(\alpha, 1) = \varepsilon(H_\alpha)/x \)) and \( m(\beta, 1) \) and \( m(\beta, 0) \) are defined, then since \( \varepsilon(H_\beta) = \varepsilon(H_\alpha) + 1 \), all the \((\lambda/2)\)-half-loop factors of \( H_\beta \) have \( \varepsilon(H_\alpha)/x \) edges except for one, which has \( 1 + \varepsilon(H_\alpha)/x \) edges. So either \( m(\beta, 1) = m(\alpha, 1) + 1 \) and \( m(\beta, 0) = m(\alpha, 0) \) or \( m(\beta, 0) = m(\alpha, 0) + 1 \) and \( m(\beta, 1) = m(\alpha, 1) \). In either case we can get suitable \( \lambda \)-half-loop factors of \( H \) with different numbers of edges by taking the union of two suitable \((\lambda/2)\)-half-loop factors from \( H_\alpha \) and the union of two suitable \((\lambda/2)\)-half-loop factors from \( H_\beta \). A similar argument applies if \( m(\beta, 1) = m(\beta, 0) \) and \( m(\alpha, 1) \) and \( m(\alpha, 0) \) are defined, and also if there are no \((\alpha, 0)-(\lambda/2)\)-half-loop factors or no \((\beta, 0)-(\lambda/2)\)-half-loop factors.

We may therefore suppose now that \( m(\alpha, 0) \), \( m(\alpha, 1) \), \( m(\beta, 0) \), and \( m(\beta, 1) \) are defined and that \( m(\alpha, 0) \neq m(\alpha, 1) \) and \( m(\beta, 0) \neq m(\beta, 1) \). Recall that the number of half-loops on \( v_0 \) in \( H_\alpha \) is two more than the number of half-loops on \( v_0 \) in \( H_\beta \). Recall also that the \((\lambda/2)\)-half-loop factorizations of \( H_\alpha \) and \( H_\beta \) are standard. Bearing this in mind, we consider three cases.

Let \( s(\beta, 1) \) be the number of \((\beta, 1)-(\lambda/2)\)-half-loop factors in the \((\lambda/2)\)-half-loop factorization of \( H_\beta \); define \( s(\beta, 0), s(\alpha, 1), \) and \( s(\alpha, 0) \) similarly.

If \( v_0 \) has at least four loops on it in \( H \), then \( s(\beta, 1) \geq 2 \) and so combining two \((\beta, 1)-(\lambda/2)\)-half-loop factors together, and one such factor with one \((\beta, 0)-(\lambda/2)\)-half-loop factor yields the desired result.

If \( v_0 \) has three loops it in \( H \) and \( s(\beta, 1) = 2 \), then the argument above applies again. So assume that \( v_0 \) has three loops on it in \( H \) and \( s(\beta, 1) = 1 \). Since \( m(\alpha, 1) \neq m(\alpha, 0) \) and \( s(\beta, 1) = 1 \), it must follow that the number of edges in the \((\beta, 1)-(\lambda/2)\)-half-loop factor is less than the number of edges in a \((\beta, 0)-(\lambda/2)\)-half-loop factor. Therefore \( x | (\varepsilon(H_\beta) + 1) = \varepsilon(H_\alpha) + 2 = (\varepsilon(H) + 3)/2 \), so \( 2x | (\varepsilon(H) + 3) \) and there is a \( \lambda \)-half-loop factor \( F \) satisfying (a) and (b) such that \( \varepsilon(F) = -1 + (\varepsilon(H) + 3)/x \).

We may now suppose that there are exactly two loops on \( v_0 \) in \( H \), so
that \( s(\beta, 1) = 1 \). Again, since \( m(\alpha, 1) \neq m(\alpha, 0) \) and \( s(\beta, 1) = 1 \), it follows that the \((\beta, 1)-(\lambda/2)\)-half-loop factor has one fewer edge than the \((\beta, 0)-(\lambda/2)\)-half-loop factors, so that \( x | (e(H_\beta) + 1) = e(H_\alpha) + 2 \). Furthermore \( s(\alpha, 1) = 2 \) and each of the \((\alpha, 1)-(\lambda/2)\)-half-loop factors has one fewer edge than each of the \((\alpha, 0)-(\lambda/2)\)-half-loop factors. Since \( e(H_\beta) + 1 = e(H_\alpha) + 2 = (e(H) + 3)/2 \) it follows again that \( 2x | (e(H) + 3) \) and there is a \( \lambda \)-half-loop factor \( F \) satisfying (a) and (b) such that \( e(F) = -1 + (e(H) + 3)/x \).

We now give a crucial structural lemma, around which our proof of the main result when \( 4 | \lambda \) pivots.

**Lemma 6.3.** Let \( 4 | \lambda \) and \( x \geq 3 \). Let \( H \) be a normal regular connected graph of degree \( x\lambda \). Either let \( H \) contain an even number of loops or let \( H \) contain an odd number \(( \geq 3 ) \) of loops and a vertex \( v_0 \) that is incident with at least two loops. Let \( F \) be a \( \lambda \)-half-loop factor of \( H_\alpha \) or of \( H_\beta \), as described in Lemma 6.1 or 6.2. Then each component of \( H - F \) which does not contain a vertex that is incident with at least one half-loop of \( F \) contains an even number of loops.

**Proof.** Suppose that \( F \) is in \( H_\alpha \) and consider a component \( C \) of \( H - F \). Suppose that no vertex of \( V(C) \) is incident with a half-loop of \( F \). Then \( C \) is the union of a number of components of \( H_\alpha - F \) and a number of components of \( H_\beta - F \). To each half-loop of \( C \) in \( H_\alpha - F \) there corresponds a half-loop of \( C \) in \( H_\beta \), and vice versa. The number of half-loops of \( C \) in \( H_\beta \) must be even (since the number of vertices of odd degree in a normal graph is even). Therefore the number of loops in \( C \) is even. If \( F \) is in \( H_\beta \) then the argument is the same.

7. A Sufficient Condition for Embedding a \( PTS(r, \lambda) \) When \( 4 \) Divides \( \lambda \)

In Theorem 7.1 we give a sufficient set of conditions for the embedding of a \( PTS(r, \lambda) \) in a \( TS(n, \lambda) \). These conditions are very like those of Conjecture 3, the only differences being that in condition (iv) we require more about the components of \( G^o \) and the bound in condition (iii) is a little tighter. First we need to define some numbers.

For a given \( PTS(r, \lambda) \), let \( y_0 \) denote the number of components \( C \) of \( G^o \) such that

(a) \( 2(n - r) | (e(C) + 3) \), and

(b) \( C \) contains an odd number of loops and does not contain any vertex incident with more than three loops.
Then define

$$k_0(T, n) = \begin{cases} (n - r - 3) y_0 & \text{if } n \geq r + 3, \\ 0 & \text{if } n \leq r + 2. \end{cases}$$

Let $y_2$ denote the number of components $C$ of $G^\circ$ with an odd number of loops, and let $y_1 = y_2 - y_0$. Then define

$$k_1(T, n) = \begin{cases} (n - r) y_1 & \text{if } n \geq r + 3, \\ 0 & \text{if } n \leq r + 2 \end{cases}$$

(notice that $y_1 = 0$ if $n - r = 3$). Finally let

$$k^*_1(T, n) = \begin{cases} k_1 & \text{if } n \geq r + 6 \\ 3k_1/2 & \text{if } n = r + 5 \\ 3k_1 & \text{if } n = r + 4 \\ 0 & \text{if } n \leq r + 3. \end{cases}$$

Recall that $N(v)$ is the number of triples containing $v$.

**Theorem 7.1.** Let $4 | \lambda$. A $PTS(r, \lambda)$ $T$ can be embedded in a $TS(n, \lambda)$ without inserting any further triples on the elements of $T$ if the following four conditions are satisfied:

(i) $n$ is $\lambda$-admissible,

(ii) $N(v) \geq \lambda(2r - n - 1)/2$ (for all $v \in V(T)$),

(iii) $\sum_{v \in V(T)} N(v) \leq \lambda(("^2\lambda" + (5) - r(n - r)/3) - k_0(T, n) - k^*_1(T, n)$,

(iv) If $C$ is a component of $G^\circ$ with an odd number of loops then $n - r \neq 2$ or 3 and $C$ contains a vertex that is incident with at least 2 loops.

**Proof.** Recall that, as shown in Lemma 2.1, the fact that $G^\circ$ is properly defined follows from condition (ii). In Theorem 4.4 we showed that Conjecture 4 was equivalent to the sufficiency of Conjecture 3. When $4 | \lambda$ let us define a pseudo $TS(n, \lambda)$ to be a quasi $TS(n, \lambda)$ which satisfies, in addition, conditions (iii) and (iv) of this theorem. The argument of Theorem 4.4 goes through in an unfettered way to show that Theorem 7.1 is equivalent to the statement that, if $4 | \lambda$ and $n$ is $\lambda$-admissible, then any pseudo $TS(n, \lambda)$ is an amalgamated $TS(n, \lambda)$. Our $PTS(r, \lambda)$ $T$ corresponds to a pseudo $TS(n, \lambda)$ in which the vertex $Q$ has degree $\lambda(n - 1)(n - r)$. So we prove Theorem 7.1 by assuming that $n$ is $\lambda$-admissible and that $S$ is a pseudo $TS(n, \lambda)$ and by proving that $S$ is an amalgamated $TS(n, \lambda)$.

This process itself is done in stages. We take a pseudo $TS(n, \lambda)$ $S$ with a vertex $Q$ with degree $\lambda(n - 1)(n - r)$, and from it we produce a pseudo $TS(n, \lambda)$ $S'$ with a vertex $Q'$ with degree $\lambda(n - 1)(n - r - 1)$ such that $S$ is an amalgamation of $S'$. We repeat this until we have a pseudo $TS(n, \lambda)$ $S^*$ with all vertices having degree $\lambda(n - 1)$; but any such pseudo $TS(n, \lambda)$ is
actually a $TS(n, \lambda)$. Furthermore $S^*$ contains $n-r$ vertices whose amalgamation produces $S$. Thus our main task will be to produce $S'$ from $S$.

A nice feature of our argument is that $S'$ will have no components with an odd number of loops, so that the numbers $k_0$ and $k_1$ for $S'$ are both 0. Thus for $S'$ the bounds in (iii) of this theorem are the same as those in (iii) of Conjecture 3. In view of the argument in Theorem 4.4, this means that for $S'$ we do not have to verify the bounds in (iii) separately, as they follow from the other numerical conditions satisfied by quasi $TS(n, \lambda)'s$, as we saw in Lemma 4.2 and Theorem 4.3.

Let $G$ be the missing edge graph of $S$ and let $G^0$ be the associated normal regular graph of degree $(n-r)\lambda$ formed by adjoining $((n-r)\lambda-d_G(v))/2$ loops to each vertex $v \in V(G)$. Lemma 6.1 describes a type of $\lambda$-half-loop factor contained in each component of $G^0$ with an even number of loops, and Lemma 6.2 describes a type of $\lambda$-half-loop factor contained in each component of $G^0$ with an odd number of loops, when at least one vertex in each such component is incident with at least two loops. By condition (iv), every component of $G^0$ is one of these two types. Except in a special circumstance detailed below, we may combine the $\lambda$-half-loop factors in the various components together to form a $\lambda$-half-loop factor $F$ such that $|F|$ takes either of two consecutive values of $z$ in the range

$$\left(\varepsilon(G) + 3y_0\right)/(n-r) - y_0 - y_1 \leq z \leq \left(\varepsilon(G) + 3y_0\right)/(n-r) - y_0 + y_1.$$ (3)

To see this, consider the following points. Each factor of a component $C$ that contributes to $y_0$ has $-1 + (\varepsilon(C) + 3)/(n-r)$ edges. We can choose each factor of each of the other $y_1$ components $C$ with an odd number of loops to contain as few as $\lfloor \varepsilon(C)/(n-r) \rfloor$ or as many as $\lceil \varepsilon(C)/(n-r) \rceil$ edges except possibly for one such factor which may have to have as few as $-1 + \lfloor \varepsilon(C)/(n-r) \rfloor$ or as many as $1 + \lceil \varepsilon(C)/(n-r) \rceil$ edges. Then (3) follows by using similar observations for the components with an even number of loops and by using the fact that

$$\lfloor \varepsilon(C_1)/(n-r) \rfloor + \cdots + \lfloor \varepsilon(C_x)/(n-r) \rfloor \geq \lfloor \varepsilon(C_1) + \cdots + \varepsilon(C_x)/(n-r) \rfloor - (x-1).$$

The special circumstance may arise when each component $C_e$ of $G^0$ with an even number of loops satisfies $2(n-r)\lfloor \varepsilon(C_e) \rfloor$ and each component $C_0$ of $G^0$ with an odd number of loops satisfies $2(n-r)\lceil \varepsilon(C_0) + 3 \rceil$ and contains no vertex with more than three loops. In this case

$$z = -y_0 + (\varepsilon(G) + 3y_0)/(n-r).$$ (4)

We now go on to describe the procedure we adopt if $n-r \geq 3$ in forming the pseudo $TS(n, \lambda) S'$. First we describe this procedure and then
afterwards we justify it, showing that the various numerical conditions make sense. If \( n - r = 2 \), the procedure is altogether simpler and is described later.

We "split off" a vertex \( u \) from \( Q \). That is, we introduce a further vertex \( u \), place some triangles on it, remove some triangles from \( Q \), and alter other triangles, in a way we now describe. For each edge \( v_1v_2 \) of \( F \) we form a new 0-triangle \( \{v_1, v_2, u\} \). For each edge \( v_1v_2 \) of \( E(G) \setminus F \) we retain the 1-triangle \( \{v_1, v_2, Q\} \). For each half-loop of \( F \) on a vertex \( v \) we form a new 1-triangle \( \{v, u, Q\} \). If \( h(F, v) \) denotes the number of half-loops of \( F \) on the vertex \( v \) and \( h(G^0, v) \) denotes the number of half-loops of \( G^0 \) on the vertex \( v \), then we retain \( (h(G^0, v) - 2h(F, v))/2 \) 2-triangles incident with \( v \). If \( t(u) \) denotes the number of new 1-triangles incident with \( u \), then we form \( (\lambda(n - r - 1) - t(u))/2 \) new 2-triangles on \( u \). Finally we remove \( (\lambda(n - r - 1) - t(u))/2 \) 3-triangles from \( Q \).

The number of half-loops of \( F \) equals the number of 1-triangles on \( u \). Thus if one further edge were added to \( F \), this would result in the number of 1-triangles on \( u \) being reduced by two and the number of new 2-triangles on \( u \) being increased by one. (Recall that, except in one case, we do have this freedom in choosing the number of edges in \( F \).) Each 2-triangle in \( S' \) corresponds to a loop when \( (G')^0 \) is formed at the next stage, where \( G' \) denotes the missing-edge graph of \( S' \). The fact that we usually have a choice of two consecutive values of \( z \) means that we can usually choose \( z \) so that the number of loops in the component of \( (G')^0 \) containing \( u \) is even. Lemma 6.3 tells us that, in any case, the number of loops in every other component of \( (G')^0 \) is even (since a vertex incident with a half-loop in \( F \) is adjacent to \( u \) in \( (G')^0 \)). Thus in any case there will be at most one component, the one containing \( u \), of \( (G')^0 \) with an odd number of loops. If \( z \) is not too high, note that there will be more than one loop in \( (G')^0 \) on \( u \).

We discuss in more detail below the question of the choice of \( z \). Before doing that, we justify the procedure for forming \( S' \) explained above.

We start by justifying the various numerical assumptions that were made in our description of the procedure. First note that from Lemmas 6.1 and 6.2, it follows that for each \( v \in V(G) \), \( h(F, v) \leq h(G^0 - F, v) \), and so \( h(G^0, v) \geq 2h(F, v) \). Since \( S \) is a quasi \( TS(n, \lambda) \), it follows from (Bviii) that \( h(G^0, v) \) is even. Therefore \( (h(G^0, v) - 2h(F, v))/2 \) is a non-negative integer.

Let \( h(F) \) denote the number of half-loops of \( F \). Then \( h(F) = \lambda r - 2\varepsilon(F) \). Thus \( t(u) \), the number of 1-triangles in \( S' \) on \( u \), satisfies \( t(u) = \lambda r - 2\varepsilon(F) \). We need to show that \( \lambda(n - r - 1) - t(u) = \lambda(n - r - 1) - \lambda r + 2\varepsilon(F) = \lambda(n - 2r - 1) + 2\varepsilon(F) \) is even and nonnegative. The number \( \lambda(n - r - 1) - t(u) \) is clearly even. Thus if \( z \) is chosen satisfying (3), then we need to show that

\[
\lambda(n - 2r - 1)/2 + \left( (\varepsilon(G) + 3y_0)/(n - r) \right) - y_0 - y_1 \geq 0.
\]
Since \((\sum_{v \in \nu(T)} N(v)) + \varepsilon(G) = \lambda(t_u')\), condition (iii) is equivalent to 
\[
\lambda(n - 2r - 1)/2 + (\varepsilon(G) - k_0 - k_1)/2 + \gamma_n(n - r) \geq 0,
\]
which implies the inequality above.

Finally we need to know that the number of 2-triangles we place on \(u\), 
namely 
\[
\lambda(n - r - 1)/2 = \lambda(n - 2r - 1)/2 + \gamma_n(n - r)/(n - r) - y_0 + y_1,
\]
is not more than the original number of 3-triangles. But the 
original number of 3-triangles was 
\[
\lambda(n - 1)/6 - \lambda r(n - r)/2 - \left(\sum_{v \in \nu(T)} N(v)\right)/3
\]
\[
= \lambda(n - 1)/6 - \lambda r(n - r)/2 - \lambda \binom{r}{2}/3 + \varepsilon(G)/3
\]
\[
= \lambda \binom{n - r}{2}/3 - \lambda r(n - r)/6 + \varepsilon(G)/3
\]
\[
= \lambda(n - r)(n - 2r - 1)/6 + \varepsilon(G)/3,
\]
so we need that 
\[
\lambda(n - 2r - 1)/2 + \gamma_n(n - r)/(n - r) - y_0 + y_1
\]
\[
\leq \lambda(n - r)(n - 2r - 1)/6 + \varepsilon(G)/3.
\]
(5)

Since both sides of (5) are integers, this inequality is true if and only if 
\[
\lambda(n - 2r - 1)/2 + (\varepsilon(G) + 3y_0)/(n - r) - y_0 + y_1
\]
\[
\leq \lambda(n - r)(n - 2r - 1)/6 + \varepsilon(G)/3,
\]
or, in other words, 
\[
\lambda(2r + 1 - n)(n - r - 3)/6
\]
\[
\leq \varepsilon(G)/3 - \varepsilon(G)/(n - r) - 3y_0/(n - r) + y_0 - y_1.
\]
This can be written as 
\[
\lambda(n - r)(2r + 1 - n)(n - r - 3)/2 \leq (n - r - 3) \varepsilon(G) + 3k_0 - 3k_1.
\]
If \(n - r > 3\) this is equivalent to 
\[
\lambda(n - r)(2r + 1 - n)/2 \leq \varepsilon(G) + 3k_0/(n - r - 3) - 3k_1/(n - r - 3).
\]
But condition (iii) is equivalent to the stronger inequality 
\[
\lambda(n - r)(2r + 1 - n)/2 \leq \varepsilon(G) - k_0 - k_1^*.
\]
and so (5) follows in this case. When \( n - r = 3 \) then \( y_0 = y_1 = 0 \). The right-hand side of (5) is an integer (it is the number of 3-triangles), and so is \( \lambda(n - r)(n - 2r - 1)/6 \). Therefore \( 3|e(G) \) and so (5) is true in this case also. This completes the demonstration that the numerical manoeuvres described in the procedure for constructing \( S' \) are possible.

We now go on to show that \( S' \) is in fact a pseudo \( TS(n, \lambda) \). It is apparent from the description of the procedure that (Biv) is satisfied and furthermore that each edge of \( S' \) is in a colour class (a 0-, 1-, 2-, or 3-triangle).

First consider a vertex \( v \in V(G) \). For each edge in \( F \) incident with \( v \), a 1-triangle is removed and a new 0-triangle involving \( v \) and \( u \) is formed. Thus the number of edges from \( v \) to \( Q \) decreases for this reason by the number, say \( e(F, v) \), of edges of \( F \) incident with \( v \); the number of (new) edges from \( v \) to \( u \) equals this amount. For each half-loop in \( F \) incident with \( v \), a 2-triangle is removed and replaced by a new 1-triangle involving \( v \) and \( u \); each loop on \( v \) in \( G^* \) corresponds to a 2-triangle on \( v \), and this process replaces \( h(F, v) \) such 2-triangles by \( h(F, v) \) (new) 1-triangles. Therefore the number of edges from \( v \) to \( Q \) decreases for this second reason by \( h(F, v) \), and the further number of (new) edges from \( v \) to \( u \) equals \( h(F, v) \). Therefore altogether the number of edges from \( v \) to \( Q \) decreases by \( e(F, v) + h(F, v) = \lambda(n - 1) \) and the number of (new) edges from \( v \) to \( u \) is similarly \( \lambda \). So \( u \) also has degree \( \lambda(n - 1) \) in \( S' \).

Now consider \( u \). We have seen that the number of edges from \( v \) to \( u \) is \( \lambda \). The number of new 1-triangles involving \( u \) is \( t(u) \), so this accounts for \( t(u) \) edges between \( u \) and \( Q \). \( (\lambda(n - r - 1) - t(u))/2 \) new 2-triangles are placed on \( u \), which accounts for \( \lambda(n - r - 1) - t(u) \) further edges between \( u \) and \( Q \). Thus the total number of edges between \( u \) and \( Q \) is \( \lambda(n - r - 1) \).

Finally consider \( Q \). The total number of edges between \( Q \) and other vertices is \( \lambda(n - r - 1)(r + 1) \). For each half-loop in \( F \), a 2-triangle is removed and replaced by a 1-triangle on \( u \); thus a loop is removed from \( Q \), the total number of such loops being \( t(u) \). However, \( (\lambda(n - r - 1) - t(u))/2 \) 2-triangles are placed on \( u \); each such 2-triangle contains a loop on \( Q \), and so this increases the number of loops on \( Q \) by \( (\lambda(n - r - 1) - t(u))/2 \). Finally \( (\lambda(n - r - 1) - t(u))/2 \) 3-triangles are removed from \( Q \). The final number of loops on \( Q \) is therefore

\[
\lambda \binom{n-r}{2} - 3(\lambda(n - r - 1) - t(u))/2 + (\lambda(n - r - 1) - t(u))/2 - t(u)
= \lambda \binom{n-r-1}{2}.
\]

From all this, (Bi)-(Biii) now follow (with \( r \) replaced by \( (r + 1) \)).

We now show that condition (iv) of Theorem 7.1 is satisfied. In fact we show that choosing \( z \) suitably gives the stronger condition that each com-
ponent $(G')^\circ$ has an even number of loops. Lemma 6.3 tells us that every component of $(G')^\circ$, except possibly the component containing $u$, has an even number of loops. We have seen that if we change the value of $z$ by one, then the number of loops in $(G')^\circ$ changes by one. Therefore whenever we have a choice of two consecutive values of $z$, then we can choose the one which makes $(G')^\circ$ have an even number of loops. Since in $(G')^\circ$ there is only one component where there is a possibility of having an odd number of loops, it follows that in this case all components of $(G')^\circ$ have an even number of loops.

There is only one case in which we do not have the choice of two consecutive values of $z$, and in that case (4) is true, and each component $C_e$ with an even number of loops satisfies $2(n-r)|e(C_e)$ and each component $C_0$ with an odd number of loops satisfies $2(n-r)|(e(C_0)+3)$. Thus $2(n-r)|(e(G)+3y_0)$. To examine this situation, let us count the number of loops in $(G')^\circ$. The number of loops in $G^\circ$ is $(\lambda r(n-r)-2\kappa(G))/2 = \lambda r(n-r)/2 - \varepsilon(G)$. The number of loops in $(G')^\circ$ on $u$ is $(\lambda(n-r-1)-t(u))/2 = \lambda(n-2r-1)/2 + z$, from above. Therefore the number of loops in $(G')^\circ$ is

$$(\lambda r(n-r)/2 - \varepsilon(G)) - h(F) + \lambda(n-2r-1)/2 + z.$$ 

The number $h(F)$ is even, and so it follows that the number of loops in $(G')^\circ$ is even if and only if $\varepsilon(G) \equiv z \pmod{2}$. But $2(n-r)|(\varepsilon(G)+3y_0)$, so it follows from (4) that $z \equiv y_0 \pmod{2}$. Therefore the number of loops in $(G')^\circ$ is even if and only if $\varepsilon(G) \equiv y_0 \pmod{2}$. But $2|(\varepsilon(G)+3y_0)$, so this congruence is clearly true, and thus it follows that the number of loops in each component of $(G')^\circ$ is even, as required.

Lastly consider the procedure in forming $S'$ when $n-r=2$. By Theorem 5.3 and condition (iv), $G^\circ$ has an equalized skew-free $\lambda$-half-loop factorization. Let the half-loop factors be $H_\alpha$ and $H_\beta$. Then we replace $Q$ and the edges and loops on $Q$ by two vertices $v_\alpha$ and $v_\beta$. Corresponding to each edge $w_1w_2$ of $H_\alpha$ or $H_\beta$ we have a triangle $\{v_\alpha, w_1, w_2\}$ or $\{v_\beta, w_1, w_2\}$, respectively. Corresponding to each loop of $G^\circ$ on a vertex $w$ we have a triangle $\{w, v_\alpha, v_\beta\}$. We retain the $0$-triangles of $S$. It is easy to check that $S'$ thus formed is a pseudo $TS(n, \lambda)$ (in fact it is an actual $TSh_2$).

8. THE MAIN THEOREM

Finally we prove our main result, namely Theorem 1.1.

**Proof of Theorem 1.1.** Let $n$ be $\lambda$-admissible, $n \geq 2r+1$, and let $T$ be a $PTS(r, \lambda)$. We may assume that $T$ is maximal. If $r \leq 3$ the result follows
from the known existence of a $TS(n, \lambda)$ whenever $n$ is $\lambda$-admissible [16], so assume that $r \geq 4$. From $T$ form a $PTS(r + 1, \lambda) T'$ by adjoining one further vertex $v_0$, but no further triples so that $v_0$ is in no triples of $T')$. Let $G'$ be the missing-edge graph of $T'$. Then $G'$ has maximum degree $\lambda r = \lambda((2r + 1) - (r + 1))$, is connected, and has $r + 1$ vertices. If $(G')^0$ has an even number of loops, put $G'' = G'$; if $(G')^0$ has an odd number of loops (so $T$ is not a $TS(r, \lambda)$), then form $T''$ from $T'$ by adding in one triangle incident with $v_0$ and let $G''$ be the missing edge-graph of $T''$. Then $(G'')^0$ has an even number of loops, has maximum degree at most $\lambda r$, is connected, and has $r + 1$ vertices.

We now show that $T''$ satisfies the conditions of Theorem 7.1 (with $r + 1$ replacing $r$). Condition (i) is satisfied by assumption. $T''$ satisfies condition (ii) (with $r$ replaced by $r + 1$), since $2(r + 1) - n - 1 = 2r - n + 1 \leq 0$. Since $G''$ consists of one component with an even number of loops, $k_0 = k_1 = 0$. The sum $\sum_{v \in V(T''')} N(v)$ is at most $\lambda \binom{r}{2}$, which gives the bound in (iii) (with $r + 1$ replacing $r$). Finally, (iv) is satisfied since $(G'')^0$ has an even number of loops and is connected.

Thus $T'$ satisfies the conditions of Theorem 7.1 (with $r + 1$ replacing $r$), and so $T$ can be embedded in a $TS(n, \lambda)$, as required.

9. A Final Remark

When $4 \mid \lambda$ one can use Theorem 7.1 to give an alternative proof of the value of $\mu(n, \lambda)$. Note that (Bvii) can only fail to be satisfied if $G^0$ contains exactly one loop, which would contradict (Bvi) and condition (iv) of Theorem 7.1.

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References

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