Computing the first passage time density of a time-dependent Ornstein–Uhlenbeck process to a moving boundary

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Abstract

In this paper we use the method of images to derive the closed-form formula for the first passage time density of a time-dependent Ornstein–Uhlenbeck process to a parametric class of moving boundaries. The results are then applied to develop a simple, efficient and systematic approximation scheme to compute tight upper and lower bounds of the first passage time density through a fixed boundary.

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In this paper we derive the closed-form formula for the first passage time (or first hitting time) density of a time-dependent Ornstein–Uhlenbeck process\textsuperscript{1} (abbreviated as OU-process) to a parametric class of moving boundaries by means of the method of images. We also apply the results to develop a simple, efficient and systematic approximation scheme to compute tight upper and lower bounds of the first passage time density through a fixed boundary. The first passage time density of an OU-process has been of great interest in different areas as diverse as neuroscience\textsuperscript{[1]}, CD4 cell counts modelling for HIV-1 infected patients\textsuperscript{[2,3]} and quantitative finance\textsuperscript{[4,5]}, and a large amount of literature has been dedicated to finding the first passage time density (for an overview, see the Ref.\textsuperscript{[6]}). Despite the importance and wide applications of the first passage time density, explicit analytic solutions to such problems are not known except for a very few instances. As summarized by Alili, Patie and Pedersen in the Ref.\textsuperscript{[6]}, three representations of analytical nature have been obtained for the first passage time density of an OU-process through a constant threshold. The first one is based on an eigenfunction expansion involving zeros of the parabolic cylinder functions, the second one is an integral representation involving some special functions, and the third one is given in terms of a functional of a three-dimensional Bessel bridge. In addition to the numerical methods, e.g. the finite-difference approach and the direct Monte Carlo simulation, these three representations suggest alternative ways to approximate the first passage time density. Nevertheless, these three representations are valid for an OU-process with constant model parameters only.

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\textsuperscript{1} This is a generalization of the Ornstein–Uhlenbeck process with time-dependent model parameters.

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To begin with, we consider the Fokker–Planck equation (FPE) associated with a time-dependent OU-process [7]:

$$\frac{1}{2} \sigma(t)^2 \frac{\partial^2 P(x, t)}{\partial x^2} - \left[\mu(t) x + v(t)\right] \frac{\partial P(x, t)}{\partial x} - \mu(t) P(x, t) = \frac{\partial P(x, t)}{\partial t}$$  (1)

where $\sigma(t)$, $\mu(t)$, $v(t)$ are arbitrary functions of time $t$. It is straightforward to show that its solution corresponding to the so-called natural boundary condition is given by

$$P(x, t) = \int_{-\infty}^{\infty} K(x, t; x', 0) P(x', 0) \, dx'$$  (2)

where

$$K(x, t; x', 0) = \frac{1}{\sqrt{4\pi \eta(t)}} \exp \left\{ -\frac{[xe^{\alpha(t)} + \gamma(t) - x']^2}{4\eta(t)} + \alpha(t) \right\}$$  (3)

with

$$\alpha(t) = -\int_0^t \mu(t') \, dt'$$

$$\gamma(t) = -\int_0^t v(t')e^{\alpha(t')} \, dt'$$

$$\eta(t) = \int_0^t \frac{1}{2} \sigma(t')^2 e^{2\alpha(t')} \, dt'.$$  (4)

Moreover, by the method of images we are able to derive the solution

$$P(x, t) = \int_{-\infty}^0 \left\{ K(x, t; x', 0) - K(x, t; -x', 0)e^{-2\beta x'} \right\} P(x', 0) \, dx'$$  (5)

or

$$P(x, t) = \int_0^{\infty} \left\{ K(x, t; x', 0) - K(x, t; -x', 0)e^{-2\beta x'} \right\} P(x', 0) \, dx'$$  (6)

which vanishes at $x = x^*(t) \equiv -[\gamma(t) + 2\beta \eta(t)]e^{-\alpha(t)}$ at any time $t \geq 0$. Here, $\beta$ is a real adjustable parameter. The former solution is valid for the interval $-\infty < x \leq x^*(t)$, whereas the latter is valid for $x^*(t) \leq x < \infty$. Hence, we have obtained a parametric class of closed-form solutions of Eq. (1) with a moving absorbing boundary whose movement is controlled by the parameter $\beta$. For simplicity, we shall concentrate on the solution given in Eq. (5) in the following.

As a result, the corresponding first passage time density conditional to $P(x, 0) = \delta(x - x_0)$ can be analytically obtained in closed form as follows:

$$P_{fp}(x_0, t) = 1 - \int_{-\infty}^{x^*(t)} \left\{ K(x, t; x_0, 0) - K(x, t; -x_0, 0)e^{-2\beta x_0} \right\} \, dx$$

$$= N \left( \frac{2\beta \eta(t) + x_0}{\sqrt{2}\eta(t)} \right) + N \left( \frac{-2\beta \eta(t) - x_0}{\sqrt{2}\eta(t)} \right) \exp(-2\beta x_0)$$  (7)

where $N(\cdot)$ is the cumulative normal distribution function. In order to approximate the first passage time density through a fixed boundary at $x = 0$, we shall choose an optimal value of the adjustable parameter $\beta$ in such a way that the integral

$$\int_0^\tau [x^*(t)]^2 \, dt$$

is a minimum. In other words, we try to minimize the deviation of the moving boundary from the fixed boundary by varying the parameter $\beta$. Here, $\tau$ denotes the time at which the solution of the FPE is evaluated. Simple algebraic
manipulations then yield the optimal value of $\beta$ as follows:

$$\beta_{opt} = \frac{\int_0^\tau \gamma(t) \eta(t) e^{-2\alpha(t)} dt}{2 \int_0^\tau \eta^2(t) e^{-2\alpha(t)} dt}$$

(8)

An illustrative example of such an optimal moving boundary (represented by the solid curve) is shown in Fig. 1. Making use of the maximum principle for parabolic partial differential equations [8], we can also determine the upper and lower bounds for the exact solution associated with the fixed boundary. It is not difficult to show that the upper bound can be provided by the solution of the FPE associated with a moving boundary whose $x^*(t)$ is always larger than or equal to zero for the duration of interest. Similarly, the solution of the FPE associated with a moving boundary whose $x^*(t)$ is always smaller than or equal to zero for the duration of interest can serve as the lower bound. Furthermore, the upper and lower bounds can be optimized by adjusting the corresponding values of the parameter $\beta$. In this example, the best lower bound can be obtained by choosing an appropriate value of $\beta$ such that $x^*(t = 0) = x^*(t = \tau) = 0$. That is, at time $t = \tau$ the moving boundary will return to its initial position at the origin and merge with the fixed boundary. This requirement gives the following value of $\beta$:

$$\beta_L = \frac{\gamma(t = \tau)}{2\eta(t = \tau)}$$

(9)

In Fig. 1, an example of such a boundary movement is denoted by the dashed curve. On the other hand, the best upper bound can be obtained by choosing an appropriate value of $\beta$ such that $x^*(t = 0) = x^*(t = \tau) = 0$. That is, at time $t = \tau$ the moving boundary will return to its initial position at the origin and merge with the fixed boundary. This requirement gives the following value of $\beta$:

$$\beta_U = \frac{\nu(t = 0)}{\sigma^2(t = 0)}$$

(10)

An example of such a moving boundary is represented by the dotted curve in Fig. 1. Furthermore, the first passage time density corresponding to the “upper-bound” solution is smaller than the exact first passage time density, whilst the one derived from the “lower-bound” solution is larger than the exact value. For simplicity, in the following we shall call these estimates the “lower bound” and “upper bound” of the first passage time density through a fixed boundary, respectively.

Now, we propose a multistage approximation scheme to systematically tighten the upper and lower bounds of the exact solution of the FPE with a fixed absorbing boundary at $x = 0$. This multistage approximation scheme has been applied successfully to compute tight upper and lower bounds of barrier option prices with time-dependent parameters very efficiently, where the underlying asset prices follow the lognormal process and the constant elasticity of the variance process [9,10]. First of all, we consider the estimate of the upper bound and perform the evaluation in two stages.

**Stage 1: The time interval $[0, \tau/2]$**

Following the same procedure as that discussed above, we choose an appropriate value of the parameter $\beta$, denoted by $\beta_{L1}$, such that $x^*(t = 0) = x^*(t = \tau/2) = 0$. This determines the movement of the boundary within the time interval $[0, \tau/2]$. The corresponding solution is given by

$$P(x, 0 \leq t \leq \tau/2) = \int_{-\infty}^0 G(x, t; x', 0; \beta_{L1}) P(x', 0) dx',$n$$

(11)

where

$$G(x, t; x', 0; \beta_{L1}) = K(x, t; x', 0) - K(x, t; -x', 0) \exp(-2\beta_{L1} x').$$

(12)
Stage 2: The time interval $[\tau/2, \tau]$

We repeat the procedure in stage 1 such that $x^*(t = \tau/2) = x^*(t = \tau) = 0$. This will give us another value of $\beta$, denoted by $\beta_{L2}$, and determine the moving boundary’s trajectory for the time interval $[\tau/2, \tau]$. Then, the corresponding solution is evaluated as follows:

$$P(x, \tau/2 \leq t \leq \tau) = \int_{-\infty}^{0} \tilde{G}(x, t; x', \tau/2; \beta_{L2}) P(x', \tau/2) dx',$$

where

$$\tilde{G}(x, t; x', \tau/2; \beta) = \tilde{K}(x, t; x', \tau/2) - \tilde{K}(x, t; -x', \tau/2) \exp(-2\beta x'),$$

$$\tilde{K}(x, t; x', \tau/2) = \frac{1}{\sqrt{4\pi \eta'(t)}} \exp \left\{ -\frac{[xe^{\alpha'(t)} + \gamma'(t) - x']^2}{4\eta'(t)} + \alpha'(t) \right\},$$

and

$$\alpha'(t) = -\int_{\tau/2}^{t} \mu(t') dt',$$

$$\gamma'(t) = -\int_{\tau/2}^{t} \nu(t')e^{\alpha'(t')} dt',$$

$$\eta'(t) = \int_{\tau/2}^{t} \frac{1}{2} \sigma(t)^2 e^{2\alpha'(t')} dt'.$$

As a result, the associated first passage time density is found to be

$$P_{fp}(x_0, 0 \leq t \leq \tau/2) = 1 - \int_{-\infty}^{0} [\gamma(t) + 2\beta_{L1} \eta(t)] e^{-u(t)} G(x, t; x_0, 0; \beta_{L1}) dx$$

$$= N \left( \frac{2\beta_{L1} \eta(t) + x_0}{\sqrt{2\eta(t)}} \right) + N \left( -\frac{2\beta_{L1} \eta(t) - x_0}{\sqrt{2\eta(t)}} \right) \exp(-2\beta_{L1} x_0)$$
Fig. 2. Moving boundaries within the two-stage approximation scheme: “lower bound” track with $\beta_{L1} = \beta_{L2} = 1.2449$ (dashed curve), and “upper bound” track with $\beta_{U1} = 1.0$ and $\beta_{U2} = 1.4906$ (dotted curve). Other input parameters are: $\mu = \nu = \sigma = \tau = 1.0$.

and

$$P_{fp}(x_0, \tau/2 \leq t \leq \tau) = 1 - \int_{-\infty}^{0} \int_{-\infty}^{\infty} \tilde{G}(x, t; x', \tau/2; \beta_{L2}) G(x', \tau/2; x_0, 0; \beta_{L1}) \, dx' \, dx$$

$$= 1 - \int_{-\infty}^{0} \left\{ N \left( -\frac{2\beta_{L2} \eta'(t) + x'}{\sqrt{2} \eta'(t)} \right) - N \left( -\frac{2\beta_{L2} \eta'(t) + x'}{\sqrt{2} \eta'(t)} \exp(-2\beta_{L2} x') \right) \right\} \times G(x', \tau/2; x_0, 0; \beta_{L1}) \, dx'. \quad (18)$$

The integration can be performed analytically and the result can be expressed in closed form in terms of the cumulative bivariate normal distribution function $N_2(\cdot)$. However, in practice it is also very efficient to calculate the integral numerically, e.g. using the Gauss quadrature method.

In Fig. 2, the dashed curve gives the moving boundary’s trajectory within the two-stage approximation scheme. It is clear that the deviation from the fixed boundary is much smaller in this two-stage approximation. Apparently, one can further improve the estimate by splitting the evaluation process into four stages instead, namely $[0, \tau/4], [\tau/4, \tau/2], [\tau/2, 3\tau/4]$ and $[3\tau/4, \tau]$. Then, what one needs to do is to determine the corresponding values of $\beta$ for these four different stages and perform successive integrations similar to the one in the two-stage approximation. The final expression of the associated first passage time density can be expressed in closed form in terms of the $N_4(\cdot)$ function. Again, the successive integrations can also be efficiently calculated using the Gauss quadrature method.

Next, we discuss how to implement the multistage approximation to improve the lower bound of the first passage time density. For the two-stage approximation, $\beta_U$ in Eq. (10) is used for the time interval $[0, \tau/2]$, i.e. we set $\beta_{U1} = \beta_U$. At $t = \tau/2$, another value of $\beta$, denoted by $\beta_{U2}$, is selected so that the moving boundary will then start moving back to its initial position at the origin, i.e. $x = 0$, and merge with the fixed boundary at $t = \tau$. The corresponding solution is then given by

$$P(x, 0 \leq t \leq \tau/2) = \int_{-\infty}^{0} G(x, t; x', 0; \beta_{U1}) P(x', 0) \, dx' \quad (19)$$

and

$$P(x, \tau/2 \leq t \leq \tau) = \int_{-\infty}^{0} \tilde{G}(x - x^*(\tau/2), t; x', \tau/2; \beta_{U2}) P(x' + x^*(\tau/2), \tau/2) \, dx'. \quad (20)$$
As a result, the associated first passage time density is found to be

\[ P_f(x_0, 0 \leq t \leq \tau/2) = 1 - \int_{-\infty}^{-[\gamma(t)+2\beta U_1 \eta(t)]} G(x, t; x_0; \beta U_1) \, dx \]

\[ = N \left( \frac{2\beta U_1 \eta(t) + x_0}{\sqrt{2\eta(t)}} \right) + N \left( \frac{-2\beta U_1 \eta(t) - x_0}{\sqrt{2\eta(t)}} \right) \exp(-2\beta U_1 x_0) \]  

(21)

and

\[ P_f(x_0, \tau/2 \leq t \leq \tau) = 1 - \int_{-\infty}^{0} \int_{-\infty}^{B(t)} \tilde{G}(x - x^*(\tau/2), t; x', \tau/2; \beta U_2) \times G(x' + x^*(\tau/2), \tau/2; x_0, 0; \beta U_1) \, dx' \]

\[ \times \left\{ N \left( \frac{-2\beta U_2 \eta'(t) + x'}{\sqrt{2\eta'(t)}} \right) - N \left( \frac{-2\beta U_2 \eta'(t) - x'}{\sqrt{2\eta'(t)}} \right) \exp(-2\beta U_2 x') \right\} \]

\[ \times G(x' + x^*(\tau/2), \tau/2; x_0, 0; \beta U_1) \, dx' \]  

(22)

where \( B(t) = -[\gamma'(t) + 2\beta U_2 \eta'(t)]e^{-\beta(t)} + x^*(\tau/2) \). The integration can be performed analytically to yield a closed-form expression in terms of the cumulative bivariate normal distribution function \( N_2(\cdot) \). Further improvement in the estimation of the first passage time density can be achieved easily using the approximation involving \( 2^N \) stages with \( N > 2 \). In Fig. 2, the dotted curve gives the trajectory of the moving boundary for the two-stage approximation.

Finally, for illustration, we apply the multistage approximation to estimate the first passage time density of the example depicted in Figs. 1 and 2. In this example, all model parameters are assumed to be constant. Table 1 exhibits the numerical results up to the two-stage approximation. Obviously, both the upper and lower bounds of the first passage time density improve dramatically as we go from the single-stage approximation to the two-stage approximation. As a result, the gap between the upper and lower bounds diminishes by 45%–75%. It is expected that, as the number of stages involved increases, the multistage approximation for both the upper and lower bounds would keep improving significantly. In fact, the gap between the bounds is asymptotically reduced to zero. Moreover, even a rather low-order approximation can yield very tight upper and lower bounds to the exact result. For comparison, we also include in Table 1 the (numerically) exact results obtained using the Crank–Nicolson method. It is clear that the exact results do lie between the upper and lower bounds.

In summary, we have presented a simple, efficient and systematic method for computing tight upper and lower bounds of the first passage time density of a time-dependent OU-process through a fixed boundary, based on a parametric class of closed-form solutions of the associated FPE with a moving absorbing boundary whose movement is controlled by an adjustable parameter \( \beta \). Unlike previous attempts, it does not involve any sophisticated special functions or numerical inversion of Laplace transforms. It is natural that, by tuning the parameter \( \beta \), the method can be applied to those cases with specified moving absorbing boundaries too. Furthermore, by the method of multiple images, this new approach can be straightforwardly generalized to the case with two absorbing boundaries.

References


