Lie-algebraic approach for pricing zero-coupon bonds in single-factor interest rate models

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Abstract

The Lie-algebraic approach has been applied to solve the bond pricing problem in single-factor interest rate models. Four of the popular single-factor models, namely the Vasicek model, Cox-Ingersoll-Ross model, the double square-root model, and the Ahn-Gao model, are investigated. By exploiting the dynamical symmetry of their bond pricing equations, analytical closed-form pricing formulae can be derived in a straightforward manner. Except in the Cox-Ingersoll-Ross model, time-varying model parameters could also be incorporated in the derivation of the bond price formulae, and this has the added advantage of allowing yield curves to be fitted. Furthermore, this Lie-algebraic approach can be easily extended to the pricing of other standard European interest rate derivatives, e.g. swaptions, captions and floortions.

Key words: Bond pricing equation, Zero-coupon bond, Interest rate model, Lie algebra
1 Introduction

Recently a Lie-algebraic method was introduced by Lo and Hui [1-3] to the field of finance for the pricing of financial derivatives with time-dependent model parameters. This new method is very simple and consists of two basic ingredients: (1) identifying the dynamical symmetries of the given pricing partial differential equations, and (2) applying the Wei-Norman theorem [4] to solve the equations and obtain analytical closed-form pricing formulae.\(^1\) For demonstration, the Lie-algebraic approach has already been applied to price European options for the constant elasticity of variance processes, corporate discount bonds with default risk, multi-asset financial derivatives, and etc. In this paper we extend the Lie-algebraic method to tackle the bond pricing problem in single-factor interest rate models, and four of the popular single-factor models, namely the Vasicek model [5], the Cox-Ingersoll-Ross model [6], the double square-root model [7], and the Ahn-Gao model [8], are studied. The bond pricing equations of these four models are given by

1. Vassieck model
   \[
   \frac{1}{2} \sigma (t)^2 \frac{\partial^2 B(r,t)}{\partial r^2} + \kappa (t) \left[ \theta (t) - r \right] \frac{\partial B(r,t)}{\partial r} - rB(r,t) + \frac{\partial B(r,t)}{\partial t} = 0
   \]

2. Cox-Ingersoll-Ross model
   \[
   \frac{1}{2} \sigma (t)^2 r \frac{\partial^2 B(r,t)}{\partial r^2} + \kappa (t) \left[ \theta (t) - r \right] \frac{\partial B(r,t)}{\partial r} - rB(r,t) + \frac{\partial B(r,t)}{\partial t} = 0
   \]

3. Double square-root model
   \[
   \frac{1}{2} \sigma (t)^2 r \frac{\partial^2 B(r,t)}{\partial r^2} + \left[ \frac{1}{4} \sigma (t)^2 - \kappa (t) \sqrt{r} - 2\lambda (t) r \right] \frac{\partial B(r,t)}{\partial r} - rB(r,t) + \frac{\partial B(r,t)}{\partial t} = 0
   \]

\(^1\)Our Lie-algebraic method is different from the Lie group analysis which was introduced by Gazizov and Ibragimov [9] to tackle partial differential equations occurring in financial problems. The Lie group analysis is a mathematical theory developed by Sophus Lie and classifies partial differential equations in terms of their symmetry groups, thereby identifying the set of equations which could be integrated or reduced to low-order equations by group theoretic algorithms. A recent review of the applications of the Lie group analysis to problems in mathematical finance and economics can be found in the Ref.10.
where $B (r,t)$ denotes the price of a zero-coupon bond which matures at $t = T$ with a value of unity, i.e. $B (r,T) = 1$. Despite that the four bond pricing equations look seemingly different, our analysis indicates that the Lie symmetry $SU (1,1)$ plays a key role in all these four equations and thus their solutions exhibit similar mathematical structures. This interesting feature might provide some insights upon how to formulate new analytically tractable single-factor interest rate models.

The remainder of this paper is organized as follows. Section 2 outlines the Wei-Norman theorem and its applications. Section 3 applies the Lie-algebraic technique to the valuation problem of a zero-coupon bond for the four single-factor interest rate models: the Vasicek model, the Cox-Ingersoll-Ross model, the double square-root model, and the Ahn-Gao model. Analytical closed-form pricing formulae are derived. Finally, concluding remarks are presented in section 4.

2 Wei-Norman Theorem

Consider the linear operator differential equation of the first order

$$
\frac{dU (t)}{dt} = H (t) U (t) , \quad U (0) = 1
$$

where $H$ and $U$ are both time-dependent linear operators in a Banach space or a finite-dimensional space. According to the Wei-Norman theorem [4], if the operator $H$ can be expressed as

$$
H (t) = \sum_{n=1}^{N} a_n (t) L_n
$$

where $a_n$’s are scalar functions of time and $L_n$ are the generators of an $N$-dimensional solvable Lie algebra or a real split 3-dimensional simple Lie algebra, then the operator $U$ can assume the following form

$$
U (t) = \prod_{n=1}^{N} \exp \left[ g_n (t) L_n \right] .
$$
Here the $g_n$’s are time-dependent scalar functions to be determined. To find the $g_n$’s, we simply substitute Eq.(6) and Eq.(7) into Eq.(5), and compare the two sides term by term to obtain a set of coupled nonlinear differential equations

$$
\frac{dg_n(t)}{dt} = \sum_{m=1}^{N} \eta_{nm} a_m(t) , \quad g_n(0) = 0
$$

where $\eta_{nm}$ are nonlinear functions of $g_n$’s. Thus, we have transformed the linear operator differential equation in Eq.(5) to a set of coupled nonlinear differential equations of scalar functions in Eq.(8).

For illustration, we consider the special case that the generators $L_n$’s form the Heisenberg-Weyl Lie algebra $h(4)$ defined by the commutation relations [11]:

$$
[L_1, L_2] = L_3 , \quad [L_1, L_3] = [L_2, L_3] = 0 .
$$

Then $H$ is given by

$$
H(t) = a_1(t) L_1 + a_2(t) L_2 + a_3(t) L_3 .
$$

According to the Wei-Norman theorem, $U(t)$ can be expressed as

$$
U(t) = \exp [g_1(t) L_1] \cdot \exp [g_2(t) L_2] \cdot \exp [g_3(t) L_3] .
$$

By differentiation, we obtain

$$
\frac{dU(t)}{dt} U(t)^{-1} = \frac{dg_1(t)}{dt} L_1 + \frac{dg_2(t)}{dt} L_2 + \frac{dg_3(t)}{dt} L_3 + \frac{dg_1(t)}{dt} \cdot \exp [g_1(t) L_1] \cdot \exp [g_2(t) L_2] \cdot \exp [g_3(t) L_3]
$$

Comparing Eq.(10) and Eq.(12) gives a set of three coupled nonlinear differential equations:

$$
\frac{dg_1(t)}{dt} = a_1(t) , \quad \frac{dg_2(t)}{dt} = a_2(t) , \quad \frac{dg_3(t)}{dt} + g_1(t) \frac{dg_2(t)}{dt} = a_3(t) .
$$
It is obvious that the set of differential equations can be easily solved by quadrature:

\begin{align*}
    g_1(t) &= \int_0^t d\tau a_1(\tau) \\
    g_2(t) &= \int_0^t d\tau a_2(\tau) \\
    g_3(t) &= \int_0^t d\tau [a_3(\tau) - a_2(\tau) g_1(\tau)] .
\end{align*}

(14)

As a result, the operator \( U(t) \) is thus determined.

## 3 Bond Pricing Equations

Now we try to apply the Wei-Norman theorem to solve the bond pricing equations given in Eqs.(1-4) by exploiting their algebraic structures. It is interesting that the Lie symmetry \( SU(1, 1) \) plays a key role in the four pricing equations although these equations are very different in appearance. Thus, these four equations can be solved in a unified manner and their solutions exhibit similar mathematical structures.

### 3.1 Vasicek model

We first examine the bond pricing equation of the Vasicek model. In terms of the generators

\begin{align*}
    K_- &= \frac{1}{2} \frac{\partial^2}{\partial r^2} , \\
    K_0 &= \frac{1}{2} \left( r \frac{\partial}{\partial r} + \frac{1}{2} \right) , \\
    K_+ &= \frac{1}{2} r^2
\end{align*}

(15)

of the Lie algebra \( SU(1, 1) \) defined by the commutation relations \([11]::

\begin{align*}
    [K_+, K_-] &= -2K_0 , \\
    [K_0, K_\pm] &= \pm K_\pm ,
\end{align*}

(16)

the bond pricing equation in Eq.(1) can be written as

\[
\frac{\partial B(r, \tau)}{\partial \tau} = \left\{ \sigma(\tau)^2 K_- - 2\kappa(\tau) K_0 + \kappa(\tau) \theta(\tau) \frac{\partial}{\partial r} - r + \frac{1}{2} \kappa(\tau) \right\} B(r, \tau)
\]

(17)

where \( \tau \equiv T - t \) denotes the time-to-maturity. We define the bond price \( B(r, \tau) \) by\(^2\)

\[
B(r, \tau) = U_0(\tau) U_I(\tau) B(r, 0) , \quad B(r, 0) = 1
\]

(18)

\(^2\)According to the **Levi’s Theorem**: “If \( L \) is a finite-dimensional Lie algebra with radical \( R \), then there exists a semi-simple subalgebra \( S \) of \( L \) such that \( L \) is the semi-direct sum \( L = S \oplus R \).” in the equation \( dU(t)/dt = H(t)U(t) \), where \( H(t) \) generates \( L \), the decomposition \( L = S \oplus R \) gives
where the operators $U_0 (\tau)$ and $U_I (\tau)$ satisfy the evolution equations:

\[
\frac{\partial U_0 (\tau)}{\partial \tau} = \left\{ \sigma (\tau)^2 K_- - 2\kappa (\tau) K_0 \right\} U_0 (\tau) , \quad U_0 (0) = 1 \tag{19}
\]

\[
\frac{\partial U_I (\tau)}{\partial \tau} = H_I (\tau) U_I (\tau) , \quad U_I (0) = 1 \tag{20}
\]

with

\[
H_I (\tau) \equiv U_0 (\tau)^{-1} \left\{ \kappa (\tau) \theta (\tau) \frac{\partial}{\partial r} - r + \frac{1}{2} \kappa (\tau) \right\} U_0 (\tau) . \tag{21}
\]

Since the $SU(1, 1)$ algebra is a real “split 3-dimensional” simple Lie algebra, the Wei-Norman theorem states that the operator $U_0 (\tau)$ can be expressed in the form [4]

\[
U_0 (\tau) = \exp \left[ c_1 (\tau) K_+ \right] \cdot \exp \left[ c_2 (\tau) K_0 \right] \cdot \exp \left[ c_3 (\tau) K_- \right] \tag{22}
\]

where the coefficients $c_i (\tau)$ are to be determined. Substituting Eq.(22) into Eq.(19), we obtain, after direct differentiation and simplification, (see the Appendix for details)

\[
c_1 (\tau) = 0
\]

\[
c_2 (\tau) = -2 \int_0^\tau \kappa (\tau') \, d\tau'
\]

\[
c_3 (\tau) = \int_0^\tau \sigma (\tau')^2 \exp \left[ c_2 (\tau') \right] \, d\tau' . \tag{23}
\]

Then, using the above explicit form of the operator $U_0 (\tau)$, we can apply the Baker-Hausdorff formula [12] to derive the operator $H_I (\tau)$:

\[
H_I (\tau) = \left[ \kappa (\tau) \theta (\tau) \exp \left\{ \frac{1}{2} c_2 (\tau) \right\} + c_3 (\tau) \exp \left\{ -\frac{1}{2} c_2 (\tau) \right\} \right] \frac{\partial}{\partial r} - \exp \left\{ -\frac{1}{2} c_2 (\tau) \right\} r + \frac{1}{2} \kappa (\tau) . \tag{24}
\]

rise to the corresponding decomposition $H (t) = H_S (t) + H_R (t)$ where $H_S (t) \in S$ and $H_R (t) \in R$. Then it is easy to verify that $U (t) = U_S (t) U_R (t)$ where $U_S (t)$ and $U_R (t)$ satisfy the equations:

\[
\frac{\partial U_S (\tau)}{\partial \tau} = H_S (t) U_S (t)
\]

\[
\frac{\partial U_R (t)}{\partial t} = \left\{ U_S (t)^{-1} H_R (t) U_S (t) \right\} U_R (t) .
\]

Since $R$ is an ideal in $L$, we can easily see that $U_S (t)^{-1} H_R (t) U_S (t)$ is in $R$. The fact that $R$ is solvable makes it easy to find $U_R (t)$ once $U_S (t)$ has been found. (See the Ref.4 for details.)
It is obvious that the operators
\[ L_1 = \frac{\partial}{\partial r} \quad , \quad L_2 = r \quad , \quad L_3 = 1 \tag{25} \]
form the Heisenberg-Weyl Lie algebra \( h(4) \). As shown in Section 2, the operator \( U_1(\tau) \) is given by
\[
U_1(\tau) = \exp[\, g_1(\tau) L_1 \cdot \exp[\, g_2(\tau) L_2 \cdot \exp[\, g_3(\tau) L_3] \tag{26} \]
where
\[
g_1(\tau) = \int_0^\tau \left[ \kappa(\tau') \theta(\tau') \exp\left\{ \frac{1}{2}c_2(\tau') \right\} + c_3(\tau') \exp\left\{ \frac{1}{2}c_2(\tau') \right\} \right] d\tau'
\]
\[
g_2(\tau) = -\int_0^\tau \exp\left\{ -\frac{1}{2}c_2(\tau') \right\} d\tau'
\]
\[
g_3(\tau) = \int_0^\tau \left[ \frac{1}{2}\kappa(\tau') + g_1(\tau') \exp\left\{ -\frac{1}{2}c_2(\tau') \right\} \right] d\tau' \tag{27}.
\]
As a result, the bond price \( B(r, \tau) \) can be expressed as
\[
B(r, \tau) = \exp\left\{ \frac{1}{4}c_2(\tau) + g_3(\tau) + g_2(\tau) \left( g_1(\tau) + \frac{1}{2}c_3(\tau) g_2(\tau) \right) \right\} \times \exp\left\{ g_2(\tau) \exp\left[ \frac{1}{2}c_2(\tau) \right] \cdot r \right\} \tag{28}.
\]
In the special case of constant model parameters, \( \sigma(t) = \sigma_0, \kappa(t) = \kappa_0 \) and \( \theta(t) = \theta_0 \), we can determine the \( c_i(\tau) \) and \( g_i(\tau) \) analytically as
\[
c_1(\tau) = 0 \quad , \quad c_2(\tau) = -2\kappa_0\tau \quad , \quad c_3(\tau) = \frac{\sigma_0^2}{2} \left( \frac{1 - \exp\left\{ -2\kappa_0\tau \right\}}{\kappa_0} \right)
\]
\[
g_1(\tau) = \left( \kappa_0\theta_0 - \frac{\sigma_0^2}{2\kappa_0} \right) \left( \frac{1 - \exp\left\{ -\kappa_0\tau \right\}}{\kappa_0} \right) + \frac{\sigma_0^2}{2\kappa_0} \left( \frac{\exp\left\{ \kappa_0\tau \right\} - 1}{\kappa_0} \right)
\]
\[
g_2(\tau) = -\left( \frac{\exp\left\{ \kappa_0\tau \right\} - 1}{\kappa_0} \right)
\]
\[
g_3(\tau) = \frac{1}{2}\kappa_0\tau - \left( \theta_0 - \frac{\sigma_0^2}{2\kappa_0^2} \right) \tau + \left( \theta_0 - \frac{\sigma_0^2}{2\kappa_0^2} \right) \left( \frac{\exp\left\{ \kappa_0\tau \right\} - 1}{\kappa_0} \right) + \frac{\sigma_0^2}{4\kappa_0} \left( \frac{\exp\left\{ \kappa_0\tau \right\} - 1}{\kappa_0} \right)^2 \tag{29}.
\]
Accordingly, the bond price \( B(r, \tau) \) is reduced to the well-known closed-form expression \([5]\)
\[
B(r, \tau) = \exp\left\{ \left( \theta_0 - \frac{\sigma_0^2}{2\kappa_0^2} \right) \left( \frac{1 - \exp\left\{ -\kappa_0\tau \right\}}{\kappa_0} - \tau \right) - \frac{\sigma_0^2}{4\kappa_0} \left( \frac{1 - \exp\left\{ -\kappa_0\tau \right\}}{\kappa_0} \right)^2 - \left( \frac{1 - \exp\left\{ -\kappa_0\tau \right\}}{\kappa_0} \right) \cdot r \right\} \tag{30}.
\]
3.2 Cox-Ingersoll-Ross model

Next we try to solve the bond pricing equation of the Cox-Ingersoll-Ross model. Introducing the operators

\[ K_- = r \frac{\partial^2}{\partial r^2} + \frac{2\kappa \theta}{\sigma^2} \frac{\partial}{\partial r} \quad , \quad K_0 = r \frac{\partial}{\partial r} + \frac{\kappa \theta}{\sigma^2} \quad , \quad K_+ = r \quad , \]

we can re-write the bond pricing equation in Eq.(2) as

\[ \frac{\partial B(r, \tau)}{\partial \tau} = \left\{ \frac{1}{2} \sigma^2 K_- - \kappa K_0 - K_+ + \frac{\kappa^2 \theta}{\sigma^2} \right\} B(r, \tau) \quad . \]

Here \( \tau \equiv T - t \) denotes the time-to-maturity, and all the model parameters do not depend upon time.\(^3\) It is not difficult to verify that the operators \( \{K_+, K_0, K_-\} \) form the \( SU(1, 1) \) Lie algebra. Defining the bond price \( B(r, \tau) \) by

\[ B(r, \tau) = U(\tau) \exp \left\{ \frac{\kappa^2 \theta}{\sigma^2} \tau \right\} B(r, 0) \quad , \quad B(r, 0) = 1 \quad (33) \]

Eq.(32) gives the evolution equation of the operator \( U(\tau) \):

\[ \frac{\partial U(\tau)}{\partial \tau} = \left\{ \frac{1}{2} \sigma^2 K_- - \kappa K_0 - K_+ \right\} U(\tau) \quad , \quad U(0) = 1 \quad . \]

By the Wei-Norman theorem \([4]\), the operator \( U(\tau) \) can be represented by

\[ U(\tau) = \exp [c_1(\tau) K_+] \cdot \exp [c_2(\tau) K_0] \cdot \exp [c_3(\tau) K_-] \quad (35) \]

with (see the Appendix for details)

\[ \frac{dc_1(\tau)}{d\tau} = -1 - \kappa c_1(\tau) + \frac{1}{2} \sigma^2 c_1(\tau)^2 \quad , \quad c_1(0) = 0 \]

\[ c_2(\tau) = -\kappa \tau + \sigma^2 \int_0^\tau c_1(\tau') d\tau' \]

\[ c_3(\tau) = \frac{1}{2} \sigma^2 \int_0^\tau \exp \{ c_2(\tau') \} d\tau' \quad . \]

Then the bond price \( B(r, \tau) \) can be easily determined as

\[ B(r, \tau) = \exp \{ c_1(\tau) r \} \exp \left\{ \frac{\kappa \theta}{\sigma^2} [\kappa \tau + c_2(\tau)] \right\} \quad . \]

\(^3\)Strictly speaking, the model parameters \( \kappa, \theta \) and \( \sigma \) could be time-varying with the constraint that \( \kappa \theta/\sigma^2 \) is independent of time \( t \).
Furthermore, the Ricatti equation with constant coefficients in Eq.(36) can be straightforwardly solved to yield

\[
c_1 (\tau) = -\frac{2 (\exp \{\gamma \tau\} - 1)}{\gamma + \kappa} \left( \exp \{\gamma \tau\} - 1 \right) + 2\gamma
\]  

(38)

with \( \gamma = \sqrt{\kappa^2 + 2\sigma^2} \). Once the \( c_1 (\tau) \) has been found, we are also able to obtain

\[
c_2 (\tau) = -\kappa \tau + 2 \ln \left\{ \frac{2\gamma \exp \left[ \frac{1}{2} (\gamma + \kappa) \tau \right]}{(\gamma + \kappa) \left( \exp \{\gamma \tau\} - 1 \right) + 2\gamma} \right\}
\]

\[
c_3 (\tau) = -\frac{1}{2} \sigma^2 c_1 (\tau)
\]

(39)

via analytical integrations. Beyond question, our finding is in agreement with the well-known closed-form result [6]:

\[
B (r, \tau) = \left\{ \frac{2\gamma \exp \left[ \frac{1}{2} (\gamma + \kappa) \tau \right]}{(\gamma + \kappa) \left( \exp \{\gamma \tau\} - 1 \right) + 2\gamma} \right\}^{2e^\theta/\sigma^2} \times
\]

\[
\exp \left\{ -\frac{2 (\exp \{\gamma \tau\} - 1)}{(\gamma + \kappa) \left( \exp \{\gamma \tau\} - 1 \right) + 2\gamma} \cdot r \right\}
\]

(40)

3.3 Double square-root model

By drawing an analogy to Eq.(31), we can easily identify the generators of the Lie algebra \( SU(1, 1) \) in the double square-root model as

\[
K_- = r \frac{\partial^2}{\partial r^2} + \frac{1}{2} \frac{\partial}{\partial r} \quad , \quad K_0 = r \frac{\partial}{\partial r} + \frac{1}{4} \quad , \quad K_+ = r
\]

(41)

in terms of which the bond pricing equation in Eq.(3) can be cast in the form

\[
\frac{\partial B (r, \tau)}{\partial \tau} = \left\{ \frac{1}{2} \sigma^2 (\tau)^2 K_- - 2\lambda (\tau) K_0 - K_+ - \kappa (\tau) \sqrt{r} \frac{\partial}{\partial r} 
\right.
\]

\[
+ \left. \frac{1}{2} \lambda (\tau) \right\} B (r, \tau)
\]

(42)

where \( \tau \equiv T - t \) denotes the time-to-maturity. As in the Vasicek model, we define the bond price \( B (r, \tau) \) by

\[
B (r, \tau) = U_0 (\tau) U_f (\tau) B (r, 0)
\]

(43)
where the operators $U_0(\tau)$ and $U_I(\tau)$ satisfy the evolution equations:

\[
\frac{\partial U_0(\tau)}{\partial \tau} = \left\{ \frac{1}{2} \sigma(\tau)^2 K_- - 2\lambda(\tau) K_0 - K_+ \right\} U_0(\tau) , \quad U_0(0) = 1 \tag{44}
\]
\[
\frac{\partial U_I(\tau)}{\partial \tau} = H_I(\tau) U_I(\tau) , \quad U_I(0) = 1 \tag{45}
\]

with

\[
H_I(\tau) \equiv U_0(\tau)^{-1} \left\{ -\kappa(\tau) \sqrt{r} \frac{\partial}{\partial r} + \frac{1}{2} \lambda(\tau) \right\} U_0(\tau) . \tag{46}
\]

Then, by the Wei-Norman theorem [4], the operator $U_0(\tau)$ can be represented by (see the Appendix for details)

\[
U_0(\tau) = \exp \left[ c_1(\tau) K_+ \right] \cdot \exp \left[ c_2(\tau) K_0 \right] \cdot \exp \left[ c_3(\tau) K_- \right] \tag{47}
\]

where

\[
\frac{dc_1(\tau)}{d\tau} = -1 - 2\lambda(\tau) c_1(\tau) + \frac{1}{2} \sigma(\tau)^2 c_1(\tau)^2 \quad , \quad c_1(0) = 0
\]
\[
c_2(\tau) = \int_{\tau}^{0} \left\{ -2\lambda(\tau') + \sigma(\tau')^2 c_1(\tau') \right\} d\tau' 
\]
\[
c_3(\tau) = \frac{1}{2} \int_{\tau}^{0} \sigma(\tau')^2 \exp \left[ c_2(\tau') \right] d\tau' . \tag{48}
\]

Since no analytical closed-form solution is generally available for the Riccati equation with time-varying coefficients in Eq.(48), one usually needs to resort to numerical methods. Once the $c_1(\tau)$ has been determined, the $c_2(\tau)$ and $c_3(\tau)$ can be readily evaluated by quadrature.

Moreover, using the above explicit form of the operator $U_0(\tau)$, we can express the operator $H_I(\tau)$ as

\[
H_I(\tau) = \frac{1}{\sqrt{2}} \kappa(\tau) \exp \left\{ -\frac{1}{2} c_2(\tau) \right\} \left[ c_1(\tau) c_3(\tau) - \exp \left\{ c_2(\tau) \right\} \right] \sqrt{2r} \frac{\partial}{\partial r} \\
- \frac{1}{\sqrt{2}} \kappa(\tau) c_1(\tau) \exp \left\{ -\frac{1}{2} c_2(\tau) \right\} \sqrt{2r} + \frac{1}{2} \lambda(\tau) \tag{49}
\]

where the operators

\[
L_1 = \sqrt{2r} \frac{\partial}{\partial r} , \quad L_2 = \sqrt{2r} , \quad L_3 = 1 \tag{50}
\]

form the Heisenberg-Weyl Lie algebra $h(4)$. As shown in Section 2, the operator $U_I(\tau)$ is given by

\[
U_I(\tau) = \exp \left[ g_1(\tau) L_1 \right] \cdot \exp \left[ g_2(\tau) L_2 \right] \cdot \exp \left[ g_3(\tau) L_3 \right] \tag{51}
\]
where
\[
g_1(\tau) = \frac{1}{\sqrt{2}} \int_0^\tau \kappa(\tau') \exp \left\{ -\frac{1}{2} c_2(\tau') \right\} \left[ c_1(\tau') c_3(\tau') - \exp \{ c_2(\tau') \} \right] d\tau'
\]
\[
g_2(\tau) = -\frac{1}{\sqrt{2}} \int_0^\tau \kappa(\tau') c_1(\tau') \exp \left\{ -\frac{1}{2} c_2(\tau') \right\} d\tau'
\]
\[
g_3(\tau) = \int_0^\tau \left[ \frac{1}{2} \lambda(\tau') + \frac{1}{\sqrt{2}} \kappa(\tau') c_1(\tau') \exp \left\{ -\frac{1}{2} c_2(\tau') \right\} g_1(\tau') \right] d\tau' .
\]

(52)

Accordingly, making use of the relations:
\[
K_- = \frac{1}{2} L_1^2 , \quad K_0 = \frac{1}{4} (L_1 L_2 + L_2 L_1) , \quad K_+ = \frac{1}{2} L_2^2 ,
\]
we are able to determine the bond price \( B(r, \tau) \) as
\[
B(r, \tau) = \exp \left\{ g_3(\tau) + g_1(\tau) g_2(\tau) + \frac{1}{2} c_3(\tau) g_2^2(\tau) + \frac{1}{4} c_2(\tau) \right\} \times \\
\exp \left\{ g_2(\tau) \exp \left[ \frac{1}{2} c_2(\tau) \right] \sqrt{2r} + c_1(\tau) r \right\} .
\]

(54)

In the special case of constant model parameters, \( i.e. \sigma(t) = \sigma_0, \kappa(t) = \kappa_0 \) and \( \lambda(t) = \lambda_0 \), Eq.(44) resembles Eq.(34) very closely, and by inspection we obtain
\[
c_1(\tau) = -\frac{2 (\exp \{ \gamma \tau \} - 1)}{(\gamma + 2 \lambda_0) (\exp \{ \gamma \tau \} - 1) + 2 \gamma}
\]
\[
= \frac{2 \lambda_0 - \gamma}{\sigma_0^2} + \frac{2 \gamma}{\sigma_0^2 [1 - C_0 \exp \{ \gamma \tau \}]}
\]
\[
c_2(\tau) = -2 \lambda_0 \tau + 2 \ln \left\{ \frac{2 \gamma \exp \left[ \frac{1}{2} (\gamma + 2 \lambda_0) \tau \right]}{(\gamma + 2 \lambda_0) (\exp \{ \gamma \tau \} - 1) + 2 \gamma} \right\}
\]
\[
= \gamma \tau + 2 \ln \left\{ \frac{2 \gamma}{(\gamma - 2 \lambda_0) [1 - C_0 \exp \{ \gamma \tau \}]} \right\}
\]
\[
c_3(\tau) = -\frac{1}{2} \sigma_0^2 c_1(\tau)
\]

(55)

with
\[
\gamma = \sqrt{4 \lambda_0^2 + 2 \sigma_0^2} \quad \text{and} \quad C_0 = \frac{2 \lambda_0 + \gamma}{2 \lambda_0 - \gamma} .
\]

(56)

Then, by straightforward analytical integrations we can also derive
\[
g_1(\tau) = \frac{\sqrt{2} \kappa_0 \lambda_0}{\gamma^2} \exp \left\{ -\frac{1}{2} \gamma \tau \right\} \left( 1 - \exp \left\{ \frac{1}{2} \gamma \tau \right\} \right)^2 - \\
\frac{\kappa_0}{\sqrt{2} \gamma} \left( \exp \left\{ \frac{1}{2} \gamma \tau \right\} - \exp \left\{ -\frac{1}{2} \gamma \tau \right\} \right)
\]

(57)
\[ g_2(\tau) = \frac{\sqrt{2} \kappa_0}{\gamma^2} \exp \left\{ -\frac{1}{2} \gamma \tau \right\} \left( 1 - \exp \left\{ \frac{1}{2} \gamma \tau \right\} \right)^2 \]

\[ g_3(\tau) = \left( \frac{1}{2} \lambda_0 - \frac{\kappa_0^2}{\gamma^2} \right) \tau - \frac{\kappa_0^2 \lambda_0}{\gamma^4} \exp \left\{ -\gamma \tau \right\} \left( 1 - \exp \left\{ \frac{1}{2} \gamma \tau \right\} \right)^4 + \frac{\kappa_0^3}{2 \gamma^3} (\exp \{\gamma \tau\} - \exp \{-\gamma \tau\}) \]  

(57)

Consequently, substituting Eqs.(55-57) into Eq.(54) yields the well-known closed-form result [7]:

\[ B(r, \tau) = A(\tau) \exp \left\{ B(\tau) r + C(\tau) \sqrt{r} \right\} \]  

(58)

where

\[ A(\tau) = \sqrt{\frac{1 - C_0}{1 - C_0 \exp \{\gamma \tau\}}} \exp \left( \alpha_1 + \alpha_2 \tau + \frac{\alpha_3 + \alpha_4 \exp \{\frac{1}{2} \gamma \tau\}}{1 - C_0 \exp \{\gamma \tau\}} \right) \]

\[ B(\tau) = \frac{2 \lambda_0 - \gamma}{\sigma_0^2} + \frac{2 \gamma}{\sigma_0^2 [1 - C_0 \exp \{\gamma \tau\}]} \]

\[ C(\tau) = \frac{2 \kappa_0 (2 \lambda_0 + \gamma) \left( 1 - \exp \left\{ \frac{1}{2} \gamma \tau \right\} \right)^2}{\gamma \sigma_0^2 [1 - C_0 \exp \{\gamma \tau\}]} \]  

(59)

with

\[ \alpha_1 = -\frac{\kappa_0^2}{\gamma^2 \sigma_0^2} \left( 4 \lambda_0 + \gamma \right) \left( 2 \lambda_0 - \gamma \right) \quad , \quad \alpha_2 = \frac{2 \lambda_0 + \gamma}{4} - \frac{\kappa_0^2}{\gamma^2} \]

\[ \alpha_3 = \frac{4 \kappa_0^2}{\gamma^2 \sigma_0^2} \left( 2 \lambda_0^2 - \sigma_0^2 \right) \quad , \quad \alpha_4 = -\frac{8 \kappa_0^2 \lambda_0}{\gamma^3 \sigma_0^2} \left( 2 \lambda_0 + \gamma \right) \]  

(60)

### 3.4 Ahn-Gao model

Finally, we tackle the bond pricing equation of the Ahn-Gao model, which includes nonlinearity in the drift term and a realistic \( r^{3/2} \) dependence in the volatility.\(^4\) We first define the generators of the Lie algebra \( SU(1, 1) \) associated with this model:

\[ K_- = \frac{1}{2} r^3 \frac{\partial^2}{\partial r^2} - qr^2 \frac{\partial}{\partial r} - \frac{r}{\sigma^2} \quad , \quad K_0 = -r \frac{\partial}{\partial r} + q + 1 \quad , \quad K_+ = \frac{2}{r} \]  

(61)

\(^4\)Not only the nonlinear drift term of the Ahn-Gao model is consistent with the empirical findings of Aït-Sahalia [13], but the chosen \( r^{3/2} \) dependence of volatility is also the best fit power law for volatility [14,15].
where the model parameters \( q \) and \( \sigma \) do not depend upon time. In terms of these generators, the bond pricing equation in Eq.(4) can be expressed as

\[
\frac{\partial B (r, \tau)}{\partial \tau} = \left\{ \sigma^2 K_- - \sigma^2 a (\tau) K_0 + \sigma^2 (q + 1) a (\tau) \right\} B (r, \tau) \quad .
\]

(62)

Then, representing the bond price \( B (r, \tau) \) by

\[
B (r, \tau) = U (\tau) \exp \left\{ \sigma^2 (q + 1) \int_0^\tau a (\tau') \, d\tau' \right\} B (r, 0) \quad , \quad B (r, 0) = 1
\]

(63)

we obtain the evolution equation of the operator \( U (\tau) \):

\[
\frac{\partial U (\tau)}{\partial \tau} = \left\{ \sigma^2 K_- - \sigma^2 a (\tau) K_0 \right\} U (\tau) \quad , \quad U (0) = 1
\]

(64)

which resembles Eq.(19) very closely. By analogy, it is clear that Eq.(64) admits the solution

\[
U (\tau) = \exp \left\{ c_1 (\tau) K_+ \right\} \cdot \exp \left\{ c_2 (\tau) K_0 \right\} \cdot \exp \left\{ c_3 (\tau) K_- \right\}
\]

(65)

where

\[
\begin{align*}
c_1 (\tau) &= 0 \\
c_2 (\tau) &= -\sigma^2 \int_0^\tau a (\tau') \, d\tau' \\
c_3 (\tau) &= \sigma^2 \int_0^\tau \exp \left\{ c_2 (\tau') \right\} \, d\tau'
\end{align*}
\]

(66)

Now we try to derive the bond price \( B (r, \tau) \) defined by Eq.(63). Without loss of generality, we suppose that \( B (r, 0) = x^{-(2q + 1)} V (x) \) where \( x = 2/\sqrt{r} \),

\[
V (x) = \int_0^\infty d\nu J_p (x \nu) \int_0^\infty dyy J_p (y \nu) V (y) \quad .
\]

(67)

and \( p = \sqrt{(2q + 1)^2 + 8/\sigma^2} \). Then it is not difficult to show that the bond price \( B (r, \tau) \) is given by

\[
B (r, \tau) = \int_0^\infty dx' G (x, \tau; x', 0) B (r', 0)
\]

(68)

with \( x' = 2/\sqrt{r'} \) and

\[
G (x, \tau; x', 0) = x' \left[ \frac{x'}{x \exp \left\{ c_2 (\tau) / 2 \right\}} \right]^{2q+1} \int_0^\infty d\nu J_p (x \nu \exp \left\{ c_2 (\tau) / 2 \right\}) \times J_p (x' \nu) \exp \left\{ -\frac{c_3 (\tau)}{2} \nu^2 \right\} \quad .
\]

(69)
The function \( J_p(\xi) \) is the Bessel function of the first kind of order \( p \). Here we have made use of the fact that \( x^{-(2q+1)}J_p(x\nu) \) is an eigenfunction of the operator \( K_\nu \) with the eigenvalue \(-\nu^2/2\). The integral over \( \nu \) can be evaluated to give \( [16] \)

\[
\frac{1}{c_3(\tau)} \exp \left\{-\frac{x'^2 + x^2 \exp [c_2(\tau)]}{2c_3(\tau)} \right\} I_p \left( \frac{x'x \exp \{c_2(\tau)/2\}}{c_3(\tau)} \right) \tag{70}
\]

for \( p > -1, \ x' > 0, \ x \exp \{c_2(\tau)/2\} > 0 \) and \( |\arg [c_2(\tau)/2]^{1/2}| < \pi/4 \). The function \( I_p(\xi) \) is the modified Bessel function of the first kind of order \( p \). As a result, \( G(x, \tau; x', 0) \) is found to be

\[
G(x, \tau; x', 0) = \frac{x'}{c_3(\tau) \exp \{(2q + 1)c_2(\tau)/2\}} \left( \frac{x'}{x} \right)^{2q+1} \times \\
\exp \left\{-\frac{x'^2 + x^2 \exp [c_2(\tau)]}{2c_3(\tau)} \right\} I_p \left( \frac{x'x \exp \{c_2(\tau)/2\}}{c_3(\tau)} \right). \tag{71}
\]

Since \( B(r, 0) = 1 \), we can readily derive the bond price \( B(r, \tau) \) as follows:

\[
B(r, \tau) = \frac{x^{-(2q+1)}}{c_3(\tau) \exp \{(2q + 1)c_2(\tau)/2\}} \exp \left\{-\frac{x^2 \exp [c_2(\tau)]}{2c_3(\tau)} \right\} \\
\int_0^\infty dx' x'^{2q+1} \exp \left\{-\frac{x'^2}{2c_3(\tau)} \right\} I_p \left( \frac{x'x \exp \{c_2(\tau)/2\}}{c_3(\tau)} \right) \\
= \frac{\Gamma(p + 1 - \omega)}{\Gamma(p + 1)} M \left( \omega, p + 1, -\frac{2 \exp [c_2(\tau)]}{c_3(\tau) \cdot r} \right) \times \\
\left\{ \frac{2 \exp [c_2(\tau)]}{c_3(\tau) \cdot r} \right\}^{\omega} \tag{72}
\]

where \( \omega = -(2q + 1 - p)/2 \), \( \Gamma(\xi) \) denotes the Gamma function, and \( M(\xi, \chi, \rho) \) is the standard confluent hypergeometric function \([16,17]\). Furthermore, Eq.(72) will reproduce the well-known closed-form result if the model parameter \( a(t) \) is independent of time \([8]\).

4 Conclusion

In this paper the Lie-algebraic method has been applied to solve the bond pricing problem in single-factor interest rate models. Four of the popular single-factor models, namely the Vasicek model, Cox-Ingersoll-Ross model, the double square-root model, and the Ahn-Gao model, are investigated, and analytical closed-form pricing formulae
are derived. Since all the four bond pricing equations exhibit the dynamical symmetry $SU(1, 1) \oplus h(4)$ or its subgroup, their solutions can be derived in a unified manner and have very similar mathematical structures. This interesting feature might shed light upon the formulation of new analytically tractable single-factor interest rate models. Except in the Cox-Ingersoll-Ross model, time-varying model parameters could also be incorporated in the derivation of the bond price formulae without difficulty. This has the added advantage of allowing yield curves to be fitted, and thus a “no-arbitrage” yield curve model can be developed to match the current market data. Hence, we believe that the Lie-algebraic method will provide an easy-to-use analytical tool for the bond pricing problem. Furthermore, this Lie-algebraic approach can be easily extended to the pricing of other standard European interest rate derivatives, e.g. swaptions, captions and floortions, for they differ from the zero-coupon bonds in the final payoff conditions only [18].
Appendix:

Consider the evolution equation of the operator $U(t)$:

$$\frac{dU(t)}{dt} = \{a_1(t) K_+ + a_2(t) K_0 + a_3(t) K_-\} U(t) \quad , \quad U(0) = 1 \quad (A.1)$$

where the operators $\{K_+, K_0, K_-\}$ form the $SU(1,1)$ Lie algebra defined by the commutation relations [11]:

$$[K_+, K_-] = -2K_0 \quad , \quad [K_0, K_-] = \pm K_+ . \quad (A.2)$$

According to the Wei-Norman theorem [4], the operator $U(t)$ can be expressed in the product form

$$U(\tau) = \exp [c_1(\tau) K_+] \cdot \exp [c_2(\tau) K_0] \cdot \exp [c_3(\tau) K_-] \quad , \quad c_i(0) = 0 . \quad (A.3)$$

By straightforward differentiation, we obtain

$$\frac{dU(t)}{dt} U(t)^{-1} = \left\{ \frac{dc_1(t)}{dt} K_+ + \frac{dc_2(t)}{dt} \exp [c_1(t) K_+] K_0 \exp [-c_1(t) K_+] + \right.$$  

$$\left. + \frac{dc_3(t)}{dt} \exp [c_1(t) K_+] \exp [c_2(t) K_0] K_- \exp [-c_2(t) K_0] \times \right.$$  

$$\exp [-c_1(t) K_+] \right\} \exp [-c_2(t)] \left\{ \frac{dc_3(t)}{dt} \right\} K_+ +$$  

$$= \left\{ \frac{dc_1(t)}{dt} - c_1(t) \frac{dc_2(t)}{dt} + c_1(t)^2 \exp [-c_2(t)] \frac{dc_3(t)}{dt} \right\} K_+ +$$  

$$- \frac{dc_2(t)}{dt} - 2c_1(t) \exp [-c_2(t)] \frac{dc_3(t)}{dt} \right\} K_0 +$$  

$$\exp [-c_2(t)] \frac{dc_3(t)}{dt} K_- . \quad (A.4)$$

Comparing Eq.(A.1) and Eq.(A.4) gives a set of three coupled nonlinear ordinary differential equations:

$$\frac{dc_1(t)}{dt} - c_1(t) \frac{dc_2(t)}{dt} + c_1(t)^2 \exp [-c_2(t)] \frac{dc_3(t)}{dt} = a_1(t)$$

$$\frac{dc_2(t)}{dt} - 2c_1(t) \exp [-c_2(t)] \frac{dc_3(t)}{dt} = a_2(t)$$

$$\exp [-c_2(t)] \frac{dc_3(t)}{dt} = a_3(t) . \quad (A.5)$$
After further simplification, these three differential equations become

\[
\frac{dc_1(t)}{dt} = a_1(t) + a_2(t) c_1(t) + a_3(t) c_1(t)^2, \quad c_1(0) = 0
\]

\[
c_2(t) = \int_0^t \{a_2(t') + 2a_3(t) c_1(t')\} \, dt'
\]

\[
c_3(t) = \int_0^t a_3(t') \exp[c_2(t')] \, dt'
\]

(A.6)

Hence, once the \( c_1(t) \) is found by solving the Ricatti equation, the \( c_2(t) \) and \( c_3(t) \) can be readily determined by quadrature.
References:


