Optimal Design and Actuator Sizing of Redundantly Actuated Omni-directional Mobile Robots

Tae Bum Park¹, Jae Hoon Lee¹, Byung-Ju Yi¹, Whee Kuk Kim², Bum Jae You¹, Sang-Rok Oh³

¹School of Electrical Engineering and Computer Science, Hanyang University
²Department of Control & Instrumentation Engineering, Korea University, Korea
³Intelligent System Control Research Center, KIST, Korea

bj@hanyang.ac.kr

Abstract—Despite that omni-directional mobile robots have been employed popularly in several application areas, effort on optimal design of such mobile robots has been few in literature. Thus, this paper investigates the optimal design of omni-directional mobile robots. Particularly, optimal design parameters such as one or double offset distance of wheel mechanism and the wheel radius are identified with respect to isotropic characteristic of mobile robots. In addition, the force transmission characteristics and actuator-sizing problem of mobile robots are investigated. Analysis has been performed for three actuation sets. It is shown that the redundantly actuated mobile robot with three active caster wheels represents the best performance among them.

1. INTRODUCTION

For the mobile robot to have omni-directional characteristics on the plane, only wheels with three degrees of freedom must be employed in mobile robots. Either the caster wheel or Swedish wheel could be modeled kinematically as a three degrees-of-freedom serial chain. However, it is known that either Swedish wheels or most of other type of “omni directional wheels” are very sensitive to road conditions, or thus its operational performance is more or less limited, compared to conventional wheels. However, the active caster wheel is not sensitive to road conditions and also is able to overcome a sort of steps encountered in uneven floors by using the active driving wheel. Kinematic modeling [1-6], singularity analysis and load distribution algorithm [2,3] for this type of mobile robots have been addressed recently. However, optimal design issue of mobile robots has not been deeply explored in literature so far.

Thus, in this paper optimal design of redundantly actuated omni-directional mobile robots equipped with three caster wheels will be mainly investigated.

2. KINEMATIC MODELING

Assume that the motion of the mobile robot is constrained to the plane and there exists no sliding and skidding friction, but that rotation of the wheel about the axis vertical to the ground is allowed. Denote (x̂, ŷ,ẑ̂) and (x̂, ŷ,ẑ̂) as the reference frame fixed to ground and the body frame fixed to the body of the mobile robot, respectively. O denotes the origin of (x̂, ŷ,ẑ̂) and ẑ̂=ẑb . (x̂, ŷ,ẑ̂) represents the wheel contact frame, the origin of which is located at the contact point between the wheel and the ground as shown in Fig. 1.

Define the output velocity of the mobile robot as

\[ \dot{u} = (v_x, v_y, \omega)^T, \]  

(1)

where \( \dot{u} = (v_x, v_y)^T \) and \( \omega \) represents the translational velocity of the point \( O_b \) and the angular velocity of the body frame about vertical axis.

Consider one of the caster wheels attached to the mobile robot. Let \( \theta \) and \( \eta \) represent the rotational angular velocity about the wheel axis \( \hat{z}_b \) and that about \( \hat{Z} \), respectively. \( r \), \( d \), and \( \phi \) denotes the radius of the wheel, the length of the steering link and the relative angular velocity between the steering link and the body frame, respectively. \( v_w \) represents the translational velocity of the point \( O_c \). \( O_cO_b \) and \( O_cO_k \) represents the position vector from \( O_c \) to \( O_b \) on the steering axis and that from \( O_c \) to \( O_k \) in the body frame, respectively.

Now, the translational velocity and the rotational angular velocity for each chain can be expressed as

\[ v_b = v_w + \eta \hat{Z} \times O_bO_c + \omega \hat{Z}_b \times O_cO_k, \]  

\[ \omega = \eta + \phi, \]  

(2)

(3)

where

\[ v_w = \theta \hat{z}_b \times r \hat{z}_b, \]  

\[ \hat{z}_w = -\cos \phi \hat{x}_w + \sin \phi \hat{y}_w, \]  

\[ O_cO_k = d \sin \phi \hat{x}_w + d \cos \phi \hat{y}_w, \]  

(4)

(5)

(6)
\[ \ddot{O}G_b = -(x\dot{x}_b + y\dot{y}_b + z\dot{z}_b), \quad (7) \]

\[ \dot{x}_b = \cos \phi \dot{x}_b - \sin \phi \dot{y}_b, \quad (8) \]

\[ \dot{y}_b = \sin \phi \dot{x}_b - \cos \phi \dot{y}_b. \quad (9) \]

Now, let \( \dot{\phi} = (\dot{\theta}_x, \dot{\theta}_y, \dot{\theta}_z)^T \) \( (i = 1, 2, 3) \) denotes the joint angular velocity vector of the \( i^{th} \) wheel. And \( x \) and \( y \) positions of the three wheels in the body frame are given as \((-\frac{l}{2}, -a)\), \((\frac{l}{2}, -a)\), and \((0, b)\), respectively. Then, the velocity relationship between the output vector of the mobile robot and the joint variables of each of the three wheels can be written in a matrix form as

\[ u = [G^*_b] \dot{\phi} \quad i = 1, 2, 3, \quad (10) \]

where

\[
\begin{bmatrix}
-d \cos \phi_1 - a \\
-d \cos \phi_2 - a \\
-d \cos \phi_3 - a
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2 \\
\dot{\phi}_3
\end{bmatrix}
\begin{bmatrix}
r \sin \phi_1 \\
r \sin \phi_2 \\
r \sin \phi_3
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
 1
\end{bmatrix}
\]

\[ \left[ G^*_b \right] = \begin{bmatrix}
-d \cos \phi_1 - a \\
-d \cos \phi_2 - a \\
-d \cos \phi_3 - a
\end{bmatrix}
\begin{bmatrix}
r \sin \phi_1 \\
r \sin \phi_2 \\
r \sin \phi_3
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
 1
\end{bmatrix}, \quad (11)
\]

\[ \left[ G^*_b \right] = \begin{bmatrix}
-d \cos \phi_1 - a \\
-d \cos \phi_2 - a \\
-d \cos \phi_3 - a
\end{bmatrix}
\begin{bmatrix}
r \sin \phi_1 \\
r \sin \phi_2 \\
r \sin \phi_3
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
 1
\end{bmatrix} \quad (12)
\]

\[
\begin{bmatrix}
-d \cos \phi_1 + b \\
-d \cos \phi_2 + b \\
-d \cos \phi_3 + b
\end{bmatrix}
\begin{bmatrix}
r \sin \phi_1 \\
r \sin \phi_2 \\
r \sin \phi_3
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
 1
\end{bmatrix}
\]

\[
\left[ G^*_b \right] = \begin{bmatrix}
-d \cos \phi_1 + b \\
-d \cos \phi_2 + b \\
-d \cos \phi_3 + b
\end{bmatrix}
\begin{bmatrix}
r \sin \phi_1 \\
r \sin \phi_2 \\
r \sin \phi_3
\end{bmatrix}
\begin{bmatrix}
 1 \\
 0 \\
 1
\end{bmatrix} \quad (13)
\]

Taking inverse of equations (11)-(13) and selecting any arbitrary three rows corresponding to three input variables, the velocity relationship between the output of the mobile robot and the active joint variables of the mobile robot can be obtained as [2]

\[ \dot{\phi}_i = [G^*_u] \dot{u}, \quad (14) \]

where \([G^*_u]\) represents the inverse Jacobian matrix of the mobile robot and \( \dot{\phi}_i \) denotes the active input vector consisting of three independent joint variables such as two driving variables \((\dot{\theta}_i, \dot{\theta}_j)\) and one steering variable \((\phi_i)\).

### 3. KINEMATIC OPTIMAL DESIGN

#### 3.1 Kinematic Performance Index

Kinematic optimization is performed to obtain optimal kinematic parameters and locations of the active joint set to enhance kinematic characteristics of the omni-directional mobile robot. For that purpose, the kinematic isotropic index given by

\[ \sigma_j = \frac{\sigma_{\text{min}}([G^*_u])}{\sigma_{\text{max}}([G^*_u])} \quad (15) \]

will be employed in simulation. It denotes the uniformity of the input-to-output velocity transmission ratios of the mobile robot. And \( \sigma_{\text{min}}([G^*_u]) \) and \( \sigma_{\text{max}}([G^*_u]) \) represents the minimum and the maximum singular value of the matrix \([G^*_u]\), respectively.

In order to represent the global kinematic characteristic, a global kinematic isotropic index defined by

\[ \sigma_{\text{G}} = \frac{\int \sigma_j([G^*_u])dW}{\int dW} \quad (16) \]

will be used to compare the kinematic characteristics of mobile robot for various active joint sets. In (16), \( \sigma_j([G^*_u]) \) and \( W = \int dW \) represents the local isotropic index of the Jacobian matrix and the workspace of the mobile robot, respectively. Thus, global kinematic characteristic of the mobile robot is evaluated by computing the average value of kinematic isotropic indices at all the configurations formed by iterating the three steering angles \((i.e., \phi_1, \phi_2, \phi_3)\).

Depending on the capacity of the actuators attached to the mobile robot, the magnitude of the maximum joint velocity may be different. And depending on the operational conditions of the mobile robot, the magnitude of maximum velocity along various output motion direction may be different. Taking these conditions into account, the velocity relationship given in (14) can be normalized as follows

\[ \dot{\phi}^* = [G^*_u] \dot{u}^*, \quad (17) \]

where

\[ \dot{\phi}^* = \begin{bmatrix}
\phi_1^* \\
\phi_2^* \\
\phi_3^*
\end{bmatrix}, \quad \phi_1^* = \frac{\dot{\phi}_1}{\phi_{\text{max}}}, \quad \phi_2^* = \frac{\dot{\phi}_2}{\phi_{\text{max}}}, \quad \phi_3^* = \frac{\dot{\phi}_3}{\phi_{\text{max}}}, \quad (18) \]

\[ \dot{u}^* = \begin{bmatrix}
\dot{u}_x \\
\dot{u}_y
\end{bmatrix}, \quad \dot{u}_x = \frac{\dot{u}_x}{u_{\text{max}}}, \quad \dot{u}_y = \frac{\dot{u}_y}{u_{\text{max}}}, \quad (19) \]

\[
\left[ G^*_u \right] = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\phi_{\text{max}} \\
u_{\text{max}} \\
u_{\text{max}} \\
\phi_{\text{max}}
\end{bmatrix} \quad \phi_{\text{max}} \quad (20)
\]

Here, \( \phi_{\text{max}}, v_{\text{max}}, v_{\text{max}}, \) and \( \omega_{\text{max}} \) represents the maximum rotational velocity of \( i^{th} \) joint, the maximum velocities along the \( x \) and \( y \) direction, and the maximum rotational velocity of the mobile robot, respectively.

#### 3.2 Optimal Design of Omni-directional Mobile Robot

##### 3.2.1 Kinematic optimization

As the operational conditions of the mobile robot, both the ratio of maximum and minimum joint velocities and the ratio of maximum translational velocity and maximum angular velocity of the mobile robot are selected. Assume that we employ the same actuator for driving and steering of the wheel mechanisms and that maximum translational velocities of the mobile robot along the direction of \( x \) and \( y \) are the same. Thus, we have

\[ v_{x\text{max}} = v_{y\text{max}}, \quad \phi_{\text{max}} / \dot{\theta}_{\text{max}} = 1. \quad (21) \]

There are three design parameters; the radius of the wheel \((r)\), the offset distance of the steering link \((d)\), and the
lateral length of the equilateral triangle \((l)\) formed by connecting three connection points of the wheel to the body of the mobile robot. In simulation, the normalized matrix of (20) is employed to calculate the value of global isotropic index of the mobile robot. Each of the design parameters considered are normalized by the lateral length of the equilateral triangle \((l)\) for convenience. In the following consideration, the values of \((l)\) are normalized by the lateral length of the index of the mobile robot. Each of the design parameters \((20)\) is employed to calculate the value of global isotropic index of the mobile robot. In simulation, the normalized matrix of connecting three connection points of the wheel to the body of the robot.

In simulation, the values of \(r\) and \(d\) range from \(r = 0.02m\) to \(r = 0.12m\) and from \(d = 0.02m\) and \(d = 0.12m\), respectively. Table 1 shows the simulation result with respect to three independent input joints \((\theta_1, \theta_2, \phi_1)\), four input joints \((\theta_1, \theta_2, \phi_1, \phi_2)\), and six input joints \((\theta_1, \theta_2, \phi_1, \phi_2, \phi_3)\) respectively. Five different values of the ratio \(v_{max}/\omega_{max}\) are considered in simulation. It turns out that the case employing six actuators has the best kinematic performance in terms of the global isotropic index and that it is optimal when the ratio of \(r\) to \(d\) is equal to 1. Also, the isotropic index becomes maximum when \(v_{max}/\omega_{max} = 0.5\).

<table>
<thead>
<tr>
<th>Table 1. Optimization Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{v_{max}}{\omega_{max}})</td>
</tr>
<tr>
<td>(r : d)</td>
</tr>
<tr>
<td>(r : d)</td>
</tr>
<tr>
<td>(r : d)</td>
</tr>
</tbody>
</table>

On the other hand, it is possible to include other offset distance as another design parameter as shown in Fig. 2. Thus, we consider kinematic optimization specifically for three active caster wheels (i.e., the case of six actuators) since this case shows the best kinematic performance in Table 1. Note that as the angle \(\alpha\) gets larger, the length of the second offset distance \((p)\) gets longer. The case that the angle \(\alpha\) is zero represents the configuration of the mobile robot of Fig. 1. Thus, there are four design parameters; the radius of the wheel \((r)\), two offset distances of the steering link \((d\ and\ p)\), the lateral length of the equilateral triangle \((l)\) formed by connecting three connection points of the wheel to the body of the mobile robot.

In simulation, the offset angle varies from 0 to 360 degree while the other conditions are fixed as \(v_{max}/\omega_{max} = 0.5\), \(\theta_{max}/\phi_{max} = 1\). And Fig. 3 shows the contour plot of isotropic index for the fixed value of \(d\) while two parameters \(r\) and \(\alpha\) are varying.

It is observed that the optimal region is widespread. The ratio between \(r\) and \(d\) is not necessarily equal to one, and the offset angle can have any specific angle.

Similarly, consider another case in which \(r\) is fixed as 0.075m, while two parameters \(d\ and\ \alpha\) are varying. Fig. 4 shows the contour plot for this case. In this case, it is optimal when \(d\) is equal to \(r\). Also, the steering angle becomes zero. Specifically, the case of Fig. 3 is analyzed for three different sets of \(r\ and\ d\) : (0.035m, 0.075m), (0.075m, 0.075m), and
(0.1m, 0.075m). Fig. 5, 6, and 7 show the results of the three cases, respectively.

Fig. 5 and Fig. 6 clearly show that the isotropic index has the maximum value at zero $\alpha$ when $r$ is equal to $d$. On the other hand, the optimal point exists around $\pm 60$ degrees of $\alpha$ when $r$ is greater than $d$ as shown in Fig. 7.

In general, we come up with the following conclusion.
(i) when the wheel radius $r$ is greater than $d$, the optimal point is widespread and $\alpha \neq 0$.
(ii) when the wheel radius $r$ is not greater than $d$, and $r$ is smaller than $b/2$, it is optimal when $r/d = 1$ and $\alpha = 0$ (where $b$ denotes the radius of the mobile platform).

The case (i) is sometimes impractical due to large sized wheel. However, it may be applicable to outdoor application that requires large traction force allowed by large-sized wheel. In fact, the decision of the wheel radius is associated with the maximum operational velocity, the maximum rpm of motor, and gear ratio of reducer. We will visit this problem in section 4 where actuator sizing for the given operational specifications such as maximum velocity and maximum acceleration of the mobile robot is treated.

3.2.2 Maximum force transmission ratio

As another advantage of employing three caster wheels can be shown through maximum force transmission ratio. The maximum force transmission ratio represents the ratio of the torque vector $\|T_e\|$ of the operational space to the torque vector $\|T_a\|$ of the joint space. And Based on Rayleigh Quotient [8], where the maximum force transmission ratio is defined as $1/\sigma_{\text{min}}$, its relationship can be written as

$$\frac{1}{\sigma_{\text{max}}}\|T_e\| \leq \|T_a\| \leq \frac{1}{\sigma_{\text{min}}}\|T_e\|.$$  (22)

This value implies the maximum required norm $\|T_e\|$ of the joint torque for an unit magnitude of operational force norm $\|T_e\| = 1$. Thus, the smaller $\|T_e\|$ is, the better the force transmission characteristic is. Table 2 shows the maximum and average force transmission ratios of the mobile robots with four active inputs and with six active inputs, respectively.

![Fig. 5. Isotropic index when $r = 0.035m, d = 0.075m$](image1)

![Fig. 6. Isotropic index when $r = 0.075m, d = 0.075m$](image2)

![Fig. 7. Isotropic index when $r = 0.1m, d = 0.075m$](image3)

<table>
<thead>
<tr>
<th>$v_a/v_{\text{max}}$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 caster wheel</td>
<td>Max.</td>
<td>4.4074</td>
<td>0.7384</td>
<td>0.7188</td>
<td>0.7140</td>
</tr>
<tr>
<td></td>
<td>Avg.</td>
<td>2.0372</td>
<td>0.2742</td>
<td>0.2568</td>
<td>0.2527</td>
</tr>
<tr>
<td>3 caster wheel</td>
<td>Max.</td>
<td>2.6955</td>
<td>0.4217</td>
<td>0.4133</td>
<td>0.4112</td>
</tr>
<tr>
<td></td>
<td>Avg.</td>
<td>1.1144</td>
<td>0.1478</td>
<td>0.1419</td>
<td>0.1405</td>
</tr>
</tbody>
</table>

It can be seen from the table that as the number of actuator is increased, average values of maximum force transmission ratio is decreased. Thus, it can be confirmed that redundantly actuated mobile robot with three active caster wheels (i.e., six actuators) has the best performance.

4. ACTUATOR SIZING

The dynamic equation of robot manipulator in operational space is given by [2]

$$T_a = [I_{\text{inv}}]\ddot{u} + [P_{\text{inv}}]\dot{u}$$  (23)

and the dual relation of (14) denoting an effective force relation between the end-effector's torque ($T_e$) and the active joint torque ($T_a$) can be written as

$$T_e = [G_a^T]T_a.$$  (24)

where $[G_a]$ is the first-order KIC between the active joint vector $\phi_a$ and the output velocity vector $\dot{u}$, and it can be obtained as

$$[G_a] = [G_a^T][G_a].$$  (25)

In (25), $[G_a]$ is the Jacobian that relates the active joints to

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the three independent joints. When the dimension of $T_d$ is larger than the one of $T_u$, it is an underdetermined case. In such a case, the general solution of $T_u$ is expressed as

$$T_u = (G_u^*)' T_u + (\{I\} - [G_u^*]' [G_u^*])k,$$  \hspace{1cm} (26)

where $[G_u^*]'$ given by

$$[G_u^*]' = (G_u^*)'$$

\hspace{1cm} (27)

denotes the pseudo-inverse matrix of $[G_u^*]'$. In the following analysis, we only consider the particular solution of (26).

4.1 Actuator sizing by maximum acceleration

Maximum hand acceleration is defined as the smallest magnitude of maximum velocity at the end-effector such that none of the actuation limits are not exceeded regardless of the direction of the acceleration vector, when the end-effector acceleration is zero. Methodology to be described in the following is based on Thomas, et al. [9]. We extend their algorithm to the case with redundant actuators.

When $\dot{u} = 0$, the inertial torque vector at the actuated joints is obtained by substituting (23) into the first term of (26)

$$T_u = [G_u^*]' T_u = [G_u^*]' [T_u']_u \dot{u} = [T_u']_u \dot{u}.$$  \hspace{1cm} (28)

The given optimization problem is to solve for the actuator size which minimizes $\|T_u\|_2$ while satisfying $\|T_u\|_2 = \text{const}$ for the given configuration. Thus, Lagrangian is defined as

$$L = \|T_u\|_2 + 2k(\|T_u\|_2 - \text{const}),$$  \hspace{1cm} (29)

where $\|T_u\|_2$ is define as

$$\|T_u\|_2^2 = \dot{u}' [W] \dot{u}$$  \hspace{1cm} (30)

and taking derivative $L$ with respect to $\dot{u}_u$ yield

$$(\ddot{u}_u)_{\text{max}} = -k[W]^{-1}[T_{u',0}]_{u,0},$$

\hspace{1cm} (31)

where $[T_{u',0}]_{u,0}$ is the $n$-th row vector of $[T_{u'}]_{u}$. If the maximum output velocity due to $n$-th joint actuator is defined as $(\ddot{u}_u)_{\text{max}}$, we have

$$(T_{u,0})_{\text{max}} = [T_{u',0}]_{u,0} (\ddot{u}_u)_{\text{max}}.$$  \hspace{1cm} (32)

Then, $k$ is found from (31) and (32)

$$k = -\frac{(T_{u,0})_{\text{max}}}{[T_{u',0}]_{u,0} [W]^{-1}[T_{u',0}]_{u,0}},$$  \hspace{1cm} (33)

and substituting (33) into (31) yields

$$(\ddot{u}_u)_{\text{max}} = \frac{(T_{u,0})_{\text{max}}}{[T_{u',0}]_{u,0} [W]^{-1}[T_{u',0}]_{u,0}} [W]^{-1}[T_{u',0}]_{u,0}.$$  \hspace{1cm} (34)

From the definition of $\|T_u\|_2$, the maximum acceleration $(\ddot{u})_{\text{max}}$ is obtained as

$$(\ddot{u})_{\text{max}} = \frac{1}{2} [W]^{-1}[T_{u',0}]_{u,0} (\ddot{u})_{\text{max}}.$$  \hspace{1cm} (35)

When the maximum acceleration of the end-effector is $(\ddot{u})_{\text{max}}$, the torque of the $n$-th active joint actuator is

$$(T_{u,0})_{\text{max}} = \frac{1}{2} [W]^{-1}[T_{u',0}]_{u,0} (\ddot{u})_{\text{max}}.$$  \hspace{1cm} (36)

$(\ddot{u})_{\text{max}}$ must be evaluated for each actuated joint and the smallest value of them is defined as the maximum hand acceleration.

4.2 Actuator sizing by maximum velocity

Maximum hand velocity is defined as the smallest magnitude of maximum velocity at the end-effector such that none of the actuation limits are not exceeded regardless of the direction of the velocity vector, when the end-effector acceleration is zero.

When $\ddot{u} = 0$, the inertial torque vector at the actuated joints is

$$T_u = [G_u^*]' [T_u']_u \ddot{u} = [T_u']_u \ddot{u}.$$  \hspace{1cm} (37)

where

$$[T_u']_u = [G_u^*]' [P_{um}]_u.$$  \hspace{1cm} (38)

The 2-norm of the end-effector velocity vector $\|\dot{u}\|_2$ is defined as

$$\|\dot{u}\|_2 = \dot{u}' [W]^{1/2}$$  \hspace{1cm} (39)

and the velocity vector $\dot{u}$ of the end-effector is written as the product of the direction normal vector $\epsilon$ and the absolute value $\dot{u}$

$$\dot{u} = \dot{u}\epsilon,$$  \hspace{1cm} (40)

so, from (39)

$$\dot{u} = (\dot{u}\epsilon [W]^{1/2})^{1/2} = \dot{u}([W]^{1/2})^{1/2},$$  \hspace{1cm} (41)

and

$$[W]^{1/2} = 1.$$  \hspace{1cm} (42)

The torque of the $n$-th active joint is expressed as

$$T_{u,0} = [T_{u',0}]_{u,0} \ddot{u}.$$  \hspace{1cm} (43)

where $[P_{um}]_u$ is the $n$-th plane of $[P_{um}]$. Using (46) through (42), (44) can be written as

$$T_{u,0} = \frac{\dot{u}^2}{\dot{u}} = \frac{\dot{u}}{\epsilon}.$$  \hspace{1cm} (44)

Since (43) is a form of Rayleigh quotient, we have

$$(\lambda_u)_{\text{max}} \leq \frac{\dot{u}^2}{\epsilon} \leq (\lambda_u)_{\text{min}},$$  \hspace{1cm} (45)

where $\lambda_u$ denotes the $n$-th eigenvalue of (46)

$$\frac{1}{2} [W]^{-1} [P_{um}]_u + [P_{um}]_u'.$$  \hspace{1cm} (46)

Thus, the torque of the $n$-th active joint is

$$\dot{u}^2 (\lambda_u)_{\text{max}} \leq T_{u,0} \leq \dot{u}^2 (\lambda_u)_{\text{min}}.$$  \hspace{1cm} (47)

Rearranging (47) yields the maximum velocity $(\ddot{u})_{\text{max}}$ as

$$(\ddot{u})_{\text{max}} = \min \left[ \frac{(T_{u,0})_{\text{max}}}{(\lambda_u)_{\text{max}}}, \frac{(T_{u,0})_{\text{max}}}{(\lambda_u)_{\text{min}}} \right]^{1/2}.$$  \hspace{1cm} (48)

when the required velocity of end-effector is $\dot{u} = (\ddot{u})_{\text{max}}$, the torque $T_{u,0}$ of the $n$-th joint is obtained as

$$(T_{u,0})_{\text{max}} = (\ddot{u})_{\text{max}} \cdot \max \left[ (\lambda_u)_{\text{min}}, (\lambda_u)_{\text{max}} \right].$$  \hspace{1cm} (49)

Similar to the acceleration case, $(\ddot{u})_{\text{max}}$ must be evaluated for
each actuated joint and the smallest value of them is defined as the maximum hand velocity.

4.3 Simulation examples

Simulation is performed for two cases; two active caster wheels and three active caster wheels. In simulation, the radius of the wheel \( r \), length of the steering link \( a \), and the radius of the platform \( b \) is set as 0.075m, 0.075m, and 0.25m, respectively, and specifically the maximum end-effector acceleration and velocity are set as \( 1\text{m/s}^2 \) and \( 0.7\text{m/s} \), respectively. Table 3 shows the maximum required torque of each actuator with respect to the given maximum velocity and maximum acceleration over the workspace spanned by three steering angles \((\phi_1, \phi_2, \phi_3)\). It can be observed that the actuator size decreases as the number of actuators increases. It is concluded that the case of three active caster wheels not only can benefit in aspect of employing multiple small-sized actuators instead of three large-sized actuators, but also is able to employ many subtasks utilizing the redundant actuation.

Though not included in the Table, the case of employing three actuators (minimum actuators) requires much bigger actuator size due to several singular configurations, which does not happen in redundantly actuated cases shown in Table 3.

<table>
<thead>
<tr>
<th>Actuator location</th>
<th>( \theta_1 )</th>
<th>( \phi_1 )</th>
<th>( \theta_2 )</th>
<th>( \phi_2 )</th>
<th>( \theta_3 )</th>
<th>( \phi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 active actuators</td>
<td>Torque (Nm)</td>
<td>5.14</td>
<td>7.43</td>
<td>5.14</td>
<td>7.43</td>
<td>*</td>
</tr>
<tr>
<td>6 active actuators</td>
<td>Torque (Nm)</td>
<td>3.08</td>
<td>5.50</td>
<td>3.08</td>
<td>5.00</td>
<td>3.17</td>
</tr>
<tr>
<td>veloicty</td>
<td>2.00</td>
<td>3.13</td>
<td>1.96</td>
<td>5.17</td>
<td>3.65</td>
<td>4.08</td>
</tr>
</tbody>
</table>

Table 3. Actuator sizing by consideration of maximum acceleration and velocity

Based on the above results on optimal kinematic parameters such as the offset distance and the radius of the wheel and actuator sizing, a case study can be executed. The actuator to be selected should satisfy the kinematic performances such as maximum acceleration and maximum velocity and also satisfy the required torque to satisfy those performances. The only design parameter that has not been considered in the above design process is the gear ratio of the reducer attached to each electric motor. Finally, we could choose a Maxon EC motor that has the following specifications:

- Stall torque: 206 mNm
- Continuous torque at 2000 rpm: 100mNm
- Maximum motor speed: 35700 rpm
- Reducer: 1: [driving], 231: 1: [steering].

The stall torque was used as a threshold value to maximum acceleration, and the continuous torque was used as that to maximum speed. The gear ratios of the driving motor and steering motor have been selected such that the maximum speed of the mobile robot and the calculated motor size are simultaneously satisfied. Finally, the values of \( r \) and \( d \) are decided identically as 0.075m. Fig. 5 shows the omni-directional mobile robot developed by reflecting the optimization result.

Fig. 5. Omni-directional mobile robot with three caster wheels

5. CONCLUSIONS

In this paper, kinematic optimal design of the omni-directional mobile robot with three active caster wheels has been performed. From the simulation results based on the kinematic isotropic index of the mobile robots, it is shown that when the two design parameters \( r \) and \( d \) are equal to each other, the mobile robot has the best performance. The case of double offset of the steering link was also analyzed. The merits of the redundantly actuated omni-directional mobile robot equipped with three active caster wheels have been shown in terms of the isotropic characteristic and the force transmission ratio. Furthermore, actuator-sizing problem based on the dynamic model of the mobile robot is also conducted. It is believed that the design methodology employed in this paper can be beneficially applied to design of any type of mobile robots under investigation in literature.

REFERENCES