Bell polynomials and generalized Blissard problems

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We introduce two possible generalizations of the classical Blissard problem and we show how to solve them by using the second order and multi-dimensional Bell polynomials, whose most important properties are recalled.

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1. Introduction

The importance and utility of ordinary Bell [1] polynomials – which are a classical mathematical tool for representing the nth derivative of a composite function – in many different frameworks of mathematics is well known.

They are often used in Combinatorial Analysis [2,3] and Statistics [4,5] and also in mathematical applications, such as the Blissard problem, the representation of Lucas polynomials of the first and second kind [6,7], the representation of symmetric functions of a countable set of numbers, i.e. the generalization of the Newton–Girard formulas [8].

Recently, some extensions of the Bell polynomials to the case of multi-variable composite functions and to the functions with several variables have been carried out (see [9–11]).

Still open is the issue of determining whether with such extensions it is possible to generalize the applications already known for the ordinary Bell polynomials.

This paper takes the applications of second order Bell polynomials and multi-dimensional Bell polynomials to an extension of the Blissard problem into consideration.

In Section 2 we recall the ordinary, the second order and multi-dimensional Bell polynomials.

In Sections 3 and 4 we apply respectively second order Bell polynomials and multi-dimensional Bell polynomials to generalized Blissard problem.

This problem occurs in the symbolic calculus in order to find the inverse of exponential series.

2. Second order and multi-dimensional Bell polynomials

In [9] a first extension of the classical Bell polynomials into Bell polynomials of order r was achieved.

For the applications considered in Sections 3, 4, the extension of the Bell polynomials is analyzed in the case r = 2.

Consider $\Phi(t) := f(\phi^1(\phi^2(t)))$ i.e. the composition of functions $x = \phi^2(t)$, $y = \phi^1(x)$ defined in suitable intervals of the real axis and assume the functions $\phi^1$, $\phi^2$, $f$ are n times differentiable with respect to the relevant independent variables so that, by using the chain rule $\Phi(t)$ can be differentiated n times with respect to t.
We use the following notations:

\[ \Phi_h := D_h^1 \Phi(t) \]
\[ f_h := D_h^1 f \big|_{y = \varphi^1(\varphi^2(t))} \]
\[ \phi_h^1 := D_h^1 \phi^2(t) \]
\[ \phi_h^2 := D_h^1 \phi^2(t) \]

and

\[ \left( [f, \phi^1, \phi^2]_n \right) := \left( f_1, \phi^1_1, \phi^2_1; \ldots; f_n, \phi^1_n, \phi^2_n \right). \]

Then the \( n \)th derivative of the function \( \Phi \) allows us to define the second order Bell polynomials \( Y_n^2 \) as follows:

\[ Y_n^2 ([f, \phi^1, \phi^2]_n) := \Phi_n. \]

The first polynomials have the following explicit expressions:

\[ Y_1^2 ([f, \phi^1, \phi^2]) = f_1 \phi^1_1 \]
\[ Y_2^2 ([f, \phi^1, \phi^2]) = f_2 \left( \phi^1_1 \right)^2 + f_1 \phi^1_1 (\phi^1_1)^2 + f_1 \phi^1_1. \]

In [10] was introduced a further generalization of Bell polynomials by means of the so-called multi-dimensional ones which are briefly recalled as follows.

Let

\[ \Psi(t) = f \left( \varphi^1(t), \varphi^2(t), \ldots, \varphi^m(t) \right) \]

be a composite function of \( m \) variables \( x^i = \varphi^i(t) \) \( (i = 1, \ldots, m) \), defined in suitable intervals of real axis \( t \). Moreover, suppose that the functions \( f \) and \( \varphi^i(t) \) \( (i = 1, \ldots, m) \) are \( n \) times differentiable with respect to the relevant independent variables so that \( \Psi(t) \) can be differentiated \( n \) times with respect to \( t \), by using the differentiation rule of multi-dimensional functions.

Set \( m = 2 \) and \( \Psi(t) = f \left( \varphi^1(t), \varphi^2(t) \right) \) with \( x = \varphi^1(t) \) and \( y = \varphi^2(t) \).

By using the following notations

\[ \Psi_h := D_h^1 \Psi(t) \]
\[ f_{s_1, s_2} := \frac{d^{s_1+s_2}}{(dx)^{s_1} (dy)^{s_2}} f(x, y) \bigg|_{x = \varphi^1, y = \varphi^2} \]
\[ \phi_h^1 := D_h^1 \phi^1(t) \quad \phi_h^2 := D_h^1 \phi^2(t) \]

and

\[ \left( [f, \varphi^1, \varphi^2]_n \right) := \left( \left( f_{s_1, s_2} \right)_{s_1 + s_2 = n}, \varphi^1_1, \varphi^2_1; \ldots; \left( f_{s_1, s_2} \right)_{s_1 + s_2 = n}, \varphi^1_n, \varphi^2_n \right) \]

(2.1)

where \( \left\{ f_{s_1, s_2} \right\}_v \) is the set of all partial derivatives of \( f \) of order \( v \) with respect to its independent variables (i.e. such that \( h_1 + h_2 = v \)), we can define the \( n \)th two-dimensional Bell polynomial (of first order) of two variables by means of the derivative of order \( n \) of the function \( \Psi(t) \) and we use the following notation:

\[ Y_n^2 \left( [f, \varphi^1, \varphi^2]_n \right) := \Psi_n. \]

The first polynomial has the following explicit expression:

\[ Y_1^2 \left( [f, \varphi^1, \varphi^2]_1 \right) = f_{s_1} \varphi^1_1 + f_{s_2} \varphi^2_1. \]

(2.2)

It is also known that for every positive integer \( n \), the following recurrence relation between the \((n + 1)\)th and the \((n - k)\)th Bell polynomials of two variables \((k = 0, 1, \ldots, n)\) holds true:

\[ Y_{n+1}^2 \left( [f, \varphi^1, \varphi^2]_{n+1} \right) = f_{s_1} \varphi^1_{n+1} + f_{s_2} \varphi^2_{n+1} + \sum_{k=0}^{n-1} \binom{n}{k} \left( Y_{n-k}^2 \left( [f_{s_1}, \varphi^1, \varphi^2]_{n-k} \right) \varphi^1_{k+1} + \right. \]
\[ \left. + \ Y_{n-k}^2 \left( [f_{s_2}, \varphi^1, \varphi^2]_{n-k} \right) \varphi^2_{k+1} \right) \]

(2.3)

where:

\[ f_{s_1} := \frac{d}{dx} f(x, y), \]
\[ f_{s_2} := \frac{d}{dy} f(x, y) \]
and:
\[
\left[ \left[ f_{i_1}, \varphi^1, \varphi^2 \right]_{n-k} \right] := \left( \left( f_{i_1 + 1} \varphi_{2_2} \right)_{n-k} \cdot \varphi^1_{n-k}, \varphi^2_{n-k} \right) \\
\left[ \left[ f_{i_2}, \varphi^1, \varphi^2 \right]_{n-k} \right] := \left( \left( f_{i_2 + 1} \varphi_{2_2+1} \right)_{n-k} \cdot \varphi^1_{n-k}, \varphi^2_{n-k} \right)
\]

where \( f_{i_1}, f_{i_2}, \varphi^1, \varphi^2 \) are the set of all the partial derivatives of the functions, respectively, \( f_x \) and \( f_y \) of order \( v \) such that \( h_1 + h_2 = v \).

3. Second order Bell polynomials and a first generalized Blissard problem

The Blissard problem occurs in the symbolic calculus in order to find the inverse of exponential series. Let us consider a sequence \( a = \{a_k\} \) associated to the formal power series
\[
e^{\psi(t)} = \sum_{k=0}^{\infty} \frac{a_k (\psi(t))^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{a_k (\psi(t))^k}{k!} = 1 + e^{\tilde{a}(t)}
\]
where \( a_k \in \mathbb{N} \) with \( a^k := a_k \forall k \geq 0 \) and \( a_0 := 1 \) and \( \psi(t) \) is an analytic function defined in a suitable interval of the real axis.

The generalized Blissard problem consists in determining a sequence \( b = \{b_k\} \) satisfying the equation:
\[
e^{\psi(t)}e^{bt} = 1.
\]  

(3.1)

**Theorem 3.1.** The solution of Eq. (3.1) can be expressed as the sequence
\[
b_k := 1 \\
b_k = Y_k^2 (f_1, g_1, \varphi_1; \ldots; f_k, g_k, \varphi_k) = Y_k^2 (-1, a_1, \varphi_1; 2!, a_2, \varphi_2; \ldots; (-1)^k k!, a_k, \varphi_k)
\]
where \( Y_k^2 \) denotes the \( k \)th second order Bell polynomial.

**Proof.** For the function \( e^{\psi(t)} \) the following formula applies:
\[
e^{\tilde{a}(t)} = \sum_{k=0}^{\infty} \frac{a_k (\psi(t))^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{a_k (\psi(t))^k}{k!} = 1 + e^{\tilde{a}(t)}
\]

(3.2)

Likewise, for the function \( e^{bt} \) we set
\[
e^{bt} = \sum_{k=0}^{\infty} b_k e^{t k} = \sum_{k=0}^{\infty} b_k \frac{k!}{k!} = 1 + \sum_{k=1}^{\infty} \frac{b_k \cdot k!}{k!} = 1 + e^{\tilde{b}(t)}
\]

(3.2)

with \( b_0 = 1 \) and \( b = \{b_k\}_{k=1,2,\ldots} \). From Eq. (3.1) we have therefore
\[
e^{\tilde{a}(t)}e^{bt} = \left( 1 + e^{\tilde{a}(t)} \right) \left( 1 + e^{\tilde{b}(t)} \right) = 1 + e^{\tilde{a}(t)} + e^{\tilde{b}(t)} + e^{\tilde{a}(t)}e^{\tilde{b}(t)} = 1
\]

which implies
\[
e^{\tilde{b}(t)} \left( 1 + e^{\tilde{a}(t)} \right) = -e^{\tilde{a}(t)}
\]

(3.3)

and
\[
e^{\tilde{b}(t)} = -\frac{e^{\tilde{a}(t)}}{1 + e^{\tilde{a}(t)}}
\]

(3.4)

Let us define now the function \( g \) putting
\[
g(t) := e^{\tilde{b}(t)} = X
\]

(3.5)

and set
\[
\varphi(t) = t.
\]

(3.6)

In such a way, for Eqs. (3.4) and (3.5) we can write the function \( f \) as follows
\[
e^{bt} = \frac{X}{1 + X} := f(X).
\]
where
\[ f(X) = f(g(\varphi(t))). \]

This implies that the coefficients \( b_k \) of the Taylor expansion of \( e^{tX} \) (which can be written as \( e^{tX} = \sum_{k=1}^{\infty} b_k \frac{X^k}{k!} = b_1 t + b_2 \frac{t^2}{2} + b_3 \frac{t^3}{3} + \cdots \)), coincide with the coefficients of the Taylor series of the function \( f(g(\varphi(t))) \), for \( t = 0 \).

Therefore we have:
\[ b_k = D^k_f \left( g(\varphi(t)) \right) |_{t=0} \quad (k = 1, 2, \ldots) . \quad (3.7) \]

The computation of the \( k \)th order derivatives of the functions \( \varphi, g \) and \( f \) in the respective points \( t = 0, \varphi(0) = \tau_0 = 0, g(\tau_0) = 0 \) carries the following results:
\[ \varphi_k = D^k \varphi(t) |_{t=0} = \varphi^{(k)}(0), \quad g_k = D^k g(t) |_{t=0} = a_k, \quad f_k = D^k f(X) |_{X=0} = (-1)^k k! . \]

Therefore Eq. (3.7) can be written through the second order Bell polynomials as follows:
\[ b_k = Y^2_k \left( f_1, g_1, \varphi_1; \ldots; f_k, g_k, \varphi_k \right) \]
\[ = Y^2_k \left( -1, a_1, \varphi_1; 2!, a_2, \varphi_2; \ldots; (-1)^k k!, a_k, \varphi_k \right) . \]

4. Two-dimensional Bell polynomials and a second generalized Blissard problem

Let us consider a sequence \( a = \{a_k\} \) associated to the following power series:
\[ e^{at} = \sum_{k=0}^{\infty} a^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \]
where \( a_k \in \mathbb{N} \) with \( a^k := a_k \forall k \geq 0 \) and \( a_0 := 1 \) and a sequence \( c = \{c_k\} \) associated analogously to the following power series:
\[ e^{ct} = \sum_{k=0}^{\infty} c^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} c_k \frac{t^k}{k!} \]
where \( c_k \in \mathbb{N} \) with \( c^k := c_k \forall k \geq 0 \) and \( c_0 := 1 \). A second generalization of the Blissard problem consists in determining a sequence \( b = \{b_k\} \) solving the equation:
\[ e^{at} e^{bt} = e^{ct} . \quad (4.1) \]

**Theorem 4.1.** The solution of Eq. (4.1) is given by the sequence
\[ b_0 := 1 \]
\[ b_k = Y^2_k \left( -1, 1, a_1, c_1; 2!, -1, 0, a_2, c_2; \ldots; (-1)^k k!, (-1)^{k-1} (k - 1)!, 0, 0, \ldots, 0, a_k, c_k \right) , \]
where \( Y^2_k \) denotes the \( k \)th two-dimensional Bell polynomial.

**Proof.** In analogy with what has been carried out in Section 3, we can state now for the functions \( e^{at}, e^{bt}, e^{ct} \) the following relations:
\[ e^{at} = 1 + e^{\hat{a}t}, \]
with \( \hat{a} = \{a_k\}_{k=1,2,\ldots} \); 
\[ e^{bt} = 1 + e^{\hat{b}t}, \]
with \( \hat{b} = \{b_k\}_{k=1,2,\ldots} \); 
\[ e^{ct} = 1 + e^{\hat{c}t}, \]
with \( \hat{c} = \{c_k\}_{k=1,2,\ldots} \).

Therefore, from Eq. (4.1) we have:
\[ \left( 1 + e^{\hat{a}t} \right) \left( 1 + e^{\hat{b}t} \right) = \left( 1 + e^{\hat{c}t} \right) . \]
which implies:
\[ e^{\delta t} \left( 1 + e^{\delta t} \right) = e^{\delta t} - e^{\delta t} \]
and
\[ e^{\delta t} = \frac{e^{\delta t} - e^{\delta t}}{1 + e^{\delta t}}. \]

Let us define now the functions \( x_1, x_2 \) by setting:
\[ x_1 := x_1(t) = e^{\delta t}, \quad (4.2) \]
\[ x_2 := x_2(t) = e^{\delta t}, \quad (4.3) \]
with \( x_1(0) = 0 \) and \( x_2(0) = 0 \).

In such a way, from Eqs. (4.2), (4.3) we can also define the function \( F \) as follows:
\[ e^{\delta t} = \frac{x_2 - x_1}{1 + x_1} := F(x_1, x_2). \quad (4.4) \]

This implies that the coefficients \( b_k \) of the Taylor expansion of \( e^{\delta t} \) (which can be written as \( e^{\delta t} = \sum_{k=1}^{\infty} \frac{b_k}{k!} = b_1 t + b_2 \frac{t^2}{2} + b_3 \frac{t^3}{3!} + \cdots \)), coincide with the coefficients of the Taylor series of the function \( F(x_1(t), x_2(t)) \) for \( t = 0 \).

Therefore we have:
\[ b_k = D^k_1 [F(x_1(t), x_2(t))] \big|_{t=0}. \quad (4.5) \]

Furthermore the (4.5) can be written through the kth two-dimensional Bell polynomials of two variables
\[ b_k = Y_k^2 ([F, x_1, x_2]_n), \]
with \([F, x_1, x_2]_n\) given by Eq. (2.1).

Computing all the derivatives which appear in Eq. (2.1) for \( t = 0 \), we find
\[
\begin{align*}
\left. \frac{dF}{(dx_1)^i} \right|_{t=0} &= (-1)^i! \quad (i = 1, 2, \ldots, k) \\
\left. \frac{dF}{dx_2} \right|_{t=0} &= 1, \quad \left. \frac{dF}{(dx_2)^i} \right|_{t=0} = 0 \quad (i = 2, \ldots, k) \\
\left. \frac{d^{i+1}F}{(dx_1)^i (dx_2)} \right|_{t=0} &= (-1)^i! \quad (i = 1, \ldots, k - 1) \\
\left. \frac{d^{i+j}F}{(dx_1)^i (dx_2)^j} \right|_{t=0} &= 0 \quad (2 \leq i + j \leq k, j \geq 2) \\
\left. \frac{d^2F}{dx_1 dx_2} \right|_{t=0} &= a_i, \quad \left. \frac{d^2F}{dx_1 dt^i} \right|_{t=0} = c_i \quad (i = 1, \ldots, k).
\end{align*}
\]

Then the sequence \( \{b_k\} \) can be written as follows:
\[ b_k = Y_k^2 \left( -1, 1, a_1, c_1; 2, -1, 0, a_2, c_2; \ldots; (-1)^k k!, (-1)^k - 1 (k - 1)!; 0, 0, 0, \ldots, 0, a_k, c_k \right). \]

With the recurrence relation (2.3) it is possible to calculate the kth multi-dimensional Bell polynomial bearing in mind that:
\[ Y_1^2 \left( [F, x_1, x_2]_1 \right) \big|_{t=0} = 2a_1 - c_1 \quad \text{and} \quad Y_1^2 \left( [F, x_1, x_2]_2 \right) \big|_{t=0} = -a_1. \]

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**References**