SENSITIVITY OF THE BAYESIAN RELIABILITY
ESTIMATES FOR THE MODIFIED GUMBEL
FAILURE MODEL

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The classical Gumbel probability distribution is modified in order to study the failure
times of a given system. Bayesian estimates of the reliability function under five dif-
ferent parametric priors and the square error loss are studied. The Bayesian reliability
estimate under the non-parametric kernel density prior is compared with those under
the parametric priors and numerical computations are given to study their effectiveness.

Keywords: Bayesian inference; extreme value distribution; kernel density estimation;
minimum variance unbiased estimates.

1. Introduction

Extreme value probability distributions have been used effectively to model various
problems in engineering, environment, business, etc. Examples include return peri-
ods of extreme precipitation, accelerated corrosion tests, system reliability, carbon
dioxide levels in the atmosphere, and high return levels of wind speeds in the design
of structures among others. Some key references are Refs. 1–13. A book by Kotz
and Nadarajah14 lists applications ranging from accelerated life testing to earth-
quakes, floods, rainfall, sea currents, and wind speeds. The objective of this paper

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is to modify the classical Gumbel distribution and study the reliability function under the inverse Gaussian, inverted gamma, gamma, general uniform, and diffuse priors. The modification is necessary in order to obtain analytically tractable Bayesian estimates of the reliability function under the assumption that the location parameter of the classical Gumbel distribution behaves as a random variable. When an engineer or scientist studies reliability for a given system and is unable to identify a well-defined probability distribution function for the prior density, the non-parametric kernel density estimate of the prior provides for a good alternative. The main difference between the non-parametric kernel density and parametric priors is that the former is distribution free and most flexible in modeling the probabilistic structure of prior information. In addition to the analytic framework, we perform an extensive numerical analysis to compare the Bayesian reliability estimates. We intend to show that the non-parametric kernel density prior performs well compared with its parametric counterparts.

2. The Gumbel Model

For the classical Gumbel model, the probability distribution function (PDF) and the cumulative distribution function (CDF) of the failure time $T$ are given, respectively, by

$$f(t) = \frac{1}{\sigma} e^{-\frac{t-\mu}{\sigma}} - e^{-\frac{\mu}{\sigma}}, \quad -\infty < t, \mu < \infty, \sigma > 0$$

(1)

and

$$F(t) = \exp\left\{-\exp\left(-\frac{(t-\mu)}{\sigma}\right)\right\},$$

(2)

where $\mu$ and $\sigma$ are the location and scale parameters, respectively. The likelihood function $L(t; \mu, \sigma)$, is given by

$$L(t; \mu, \sigma) = \sigma^{-n} \exp\left\{-\sum_{i=1}^{n} \frac{t_i - \mu}{\sigma} - \sum_{i=1}^{n} \exp\left(-\frac{t_i - \mu}{\sigma}\right)\right\}$$

(3)

If we let

$$g(t; \sigma) = e^{-e^{-\frac{t}{\sigma}}},$$

and

$$\theta(\mu, \sigma) = e^{\frac{\mu}{\sigma}},$$

then the CDF of the classical Gumbel failure model can be written as

$$F(t; \theta) = [g(t)]^{\theta}$$

(4)

If the scale parameter $\sigma$ is assumed to be fixed, the modification transforms the location parameter $\mu$, so that $\theta(\mu, \sigma)$ is defined over the positive real line and is thus more suitable for reliability analysis. Note that $g(t)$ is monotone increasing.
Bayesian Reliability Estimates for the Modified Gumbel Failure Model

and $F(t; \theta)$ is bounded from above and below. Using the new parameterization, the probability density and the likelihood functions can be written as

$$f(t; \theta) = \frac{\partial F(t; \theta)}{\partial \theta} = [g(t)]^{(\theta-1)}g(t)'$$  \hspace{1cm} (5)

and

$$L(t; \theta) = \theta^n \Pi g'(t_i)[g(t_i)]^{(\theta-1)}, \quad -\infty < t < \infty, \quad \theta > 0, \quad \sigma > 0$$  \hspace{1cm} (6)

Reliability estimates of the modified Gumbel probability distribution are discussed next.

3. Reliability Modeling

Let $t_1, t_2, t_3, \ldots, t_n$ be the failure times that follow the modified Gumbel PDF given by (5). The reliability at time $t$ of a system whose life follows the probability law $f(x; \theta)$ is given by

$$R(t; \theta) = \int_t^\infty f(x; \theta)dx = 1 - [g(t)]^\theta$$  \hspace{1cm} (7)

The ordinary estimates of reliability are presented next.

3.1. The maximum likelihood estimates

The maximum likelihood estimates (MLE) for $\sigma$ and $\theta$ can be derived from equation (6) by solving $\frac{\partial \ln L}{\partial \sigma} = 0$ and $\frac{\partial \ln L}{\partial \theta} = 0$ and obtaining the following equations

$$\hat{\sigma}_{ML} + \frac{\sum_i e^{-t_i/\hat{\sigma}_{ML}}}{\sum_i e^{-t_i/\hat{\sigma}_{ML}}} = \bar{t}$$  \hspace{1cm} (8)

and

$$\hat{\theta}_{ML} = \frac{n}{\bar{G}},$$  \hspace{1cm} (9)

where

$$\bar{G} = -\sum_{i=1}^{n} \ln g(t_i).$$  \hspace{1cm} (10)

Equations (8) and (9) are not analytically tractable and must be solved numerically. By the invariance property, the MLE of the reliability function can be obtained by replacing parameters $\sigma$ and $\theta$ with $\hat{\sigma}_{ML}$ and $\hat{\theta}_{ML}$.

3.2. The minimum variance unbiased estimates

In complex systems, the cumulative effect of bias may be quite considerable and a system might prove unsatisfactory during operation time. Given a sample of failure times, $t_1, t_2, \ldots, t_n$, and CDF in (4), Miladinovic and Tsokos have shown that
the minimum variance unbiased (MVU) estimate of the reliability function \( R(t; \theta) \) is given by
\[
\hat{R}_{MVU}(t) = 1 - \left( 1 + \frac{\ln g(t)}{G} \right)^{n-1}, -\ln g(t) < G < \infty
\]
This result follows by noting that \( G = -\sum_{i=1}^{n} \ln g(t_i) \) is a complete sufficient statistic for \( \theta \) and that the expected value of the MVU reliability estimate equals the true reliability estimate, that is
\[
E\hat{R}_{MVU}(t) = 1 - \int_{-\ln g(t)}^{\infty} \left( 1 + \frac{\ln g(t)}{G} \right)^{n-1} \frac{G^{n-1}}{\Gamma(n)} e^{-\theta G} dG
\]
so that
\[
E\hat{R}_{MVU}(t) = R(t)
\]
In the following section, we proceed to derive the Bayesian estimates of reliability for the modified Gumbel probability distribution function.

4. Bayes Estimators of Reliability

In this section, we consider a Bayesian analysis of reliability for the modified Gumbel failure model under the influence of square error loss and six different priors for \( \theta \). In practice, prior selection must be done subjectively by an engineer or scientist conducting the experiment. The issues and procedures related to the selection of priors in Bayesian analysis are given an excellent treatment by Gelman et al.\(^{16}\) and the reader is referred to the text for further information. As a side note, since the scale parameter \( \sigma \) is assumed to be fixed, we recommend that the method of maximum likelihood be used in its estimation.

4.1. Reliability estimate under the natural conjugate gamma prior

Since the statistic \( G = -\sum_{i=1}^{n} \ln g(t_i) \) is sufficient and complete for \( \theta \) and is distributed as a Gamma random variable,\(^{15}\) we may assume that the natural conjugate prior has the form
\[
g(\theta; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta \theta}, \quad \theta > 0, \quad \alpha, \beta > 0.
\]
Given \( n \) failure times \( t_1, \ldots, t_n \), the joint PDF is given by
\[
L(t; \theta) = \int f(t \mid \theta) g(\theta; \alpha, \beta) d\theta
\]
\[
L(t; \theta) = \int_{0}^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{n+\alpha-1} e^{-\beta \theta} g(t_i)[g(t_i)]^{\theta-1} d\theta,
\]
which gives

\[
L(t; \theta) = \frac{\beta \Gamma(n + \alpha)}{\Gamma(\alpha)(\beta + G)^{n+\alpha}} \prod g'(t) g(t).
\]  

Similarly, the posterior distribution of \( \theta \) is given by

\[
f(\theta | t) = \frac{L(t, \theta) g(\theta)}{\int L(t, \theta) g(\theta) d\theta}
\]
or

\[
f(\theta | t) = \frac{(\beta + G)^{n+\alpha}}{\Gamma(n + \alpha)} \theta^{n+\alpha-1} e^{-\theta(\beta + G)}.
\]

Therefore, the Bayesian estimate of \( R(t) \) under the natural conjugate prior and square error loss is:

\[
\hat{R}_{GM}(t) = 1 - \int_0^\infty g(t) \frac{1}{\Gamma(n + \alpha)} \theta^{n+\alpha-1} e^{-\theta(\beta + G)} d\theta
\]
or

\[
\hat{R}_{GM}(t) = 1 - \left( \frac{\beta + G}{\beta + G - \ln g(t)} \right)^{n+\alpha}
\]

### 4.2. Reliability under the inverse gaussian prior

The inverse Gaussian prior with parameters \( \mu \) and \( \lambda \) is given by

\[
g(\theta; \mu, \lambda) = \left( \frac{\lambda}{2\pi \theta^3} \right)^{\frac{1}{2}} e^{-\frac{\lambda(\theta - \mu)^2}{2 \theta^2}}, \quad \theta > 0, \quad \mu, \lambda > 0.
\]

The posterior probability distribution for the inverse Gaussian prior is given by

\[
h_{IGS}(\theta | t) = \frac{L(t, \theta) g(\theta; \mu, \lambda)}{\int_{0}^{\infty} L(t, \theta) g(\theta; \mu, \lambda) d\theta}
\]

\[
h_{IGS}(\theta | t) = \frac{\left( \frac{\lambda}{2\pi \theta^3} \right)^{\frac{1}{2}} e^{-\theta G - \frac{\lambda(\theta - \mu)^2}{2 \theta^2}}}{0.798(2\mu^2 + \lambda)0.25 - \mu - 0.5n \mu - 0.5 + n \lambda 0.25 + 0.5n e^\mu}
\]

\[
\times \left( BK \left( n - 0.5, \frac{\sqrt{2\mu^2 + \lambda \sqrt{\lambda}}}{\mu} \right) \right)^{-1}
\]

\[
h_{IGS}(\theta | t) = 0.5\theta^{-1.5+n} e^{-\frac{\lambda 0.25 + 0.5n - 0.5n \lambda 0.25 + 0.5n - 0.5n}{\mu}}
\]

\[
\times \left( BK \left( n - 0.5, \frac{\sqrt{2\mu^2 + \lambda \sqrt{\lambda}}}{\mu} \right) \right)^{-1}
\]

\[
BK(m, n) \text{ represents the second order Bessel function that satisfies the differential equation}
\]

\[
n^2 y'' + ny' - (n^2 + m^2)y = 0, \quad y \geq 0.
\]
The Bayesian reliability estimate corresponding to the inverse Gaussian prior with the square error loss function is given by

\[
\hat{R}_{IGS}(t) = 1 - \int_0^\infty g(t)^\beta h_{IGS}(\theta \mid t)d\theta
\]

\[
\hat{R}_{IGS}(t) = 1 - (2G\mu^2 + \lambda - 2\ln g(t)\mu^2)^{0.25-0.5n}(2G\mu^2 + \lambda)^{-0.25+0.5n}BK(t; n, \mu, \lambda),
\]

where

\[
BK(t; n, \mu, \lambda) = \frac{G^{n-0.5}}{BK(n-0.5, \sqrt{2G\mu^2 + \lambda})}.
\]

4.3. Reliability under the inverted gamma prior

The inverted gamma probability distribution is defined as follows:

\[
g(\theta; \alpha, \beta) = \frac{1}{\alpha \Gamma(\beta)} \left( \frac{\alpha}{\theta} \right)^{\beta+1} e^{-\frac{\alpha}{\theta}}, \quad \theta > 0, \quad \alpha, \beta > 0.
\]

For the inverted gamma prior the posterior density is given by

\[
h_{IGM}(\theta \mid t) = \frac{0.5^n(\alpha G)^{\beta+1}e^{-\theta G - \frac{\mu}{\theta}}}{\alpha^{n+1}(\alpha G)^{\frac{1}{2}}BK(n-\beta, 2\sqrt{\alpha G})},
\]

and the Bayesian reliability estimate by

\[
\hat{R}_{IGM}(t) = 1 - \frac{G^{0.5n-0.5\beta}(G - \ln g(t))^{-0.5n+0.5\beta}BK(n-\beta, 2\sqrt{\alpha G})}{BK(n-\beta, 2\sqrt{\alpha G})}.
\]

4.4. Reliability under the general uniform and diffuse priors

The general uniform probability distribution is given by

\[
g(\theta; \alpha, \beta, b) = \frac{(b-1)(\alpha \beta)^{b-1}}{\theta^b(\beta^{b-1} - \alpha^{b-1})}, \quad 0 \leq \alpha \leq \theta \leq \beta.
\]

The diffuse distribution is obtained from the general uniform distribution by setting \(b = 0\) and letting \(\alpha \to 0\) and \(\beta \to \infty\). For the general uniform prior the posterior density is given by

\[
h_{GU}(\theta \mid t) = \frac{e^{-\theta G}}{\theta^{b-n}} \left( \int_\alpha^\beta e^{-\theta G \theta^{n-b}}d\theta \right)^{-1},
\]

and the Bayesian reliability estimate by

\[
\hat{R}_{GU}(t) = 1 - \frac{\int_\alpha^\beta g(t)\theta e^{-\theta G \theta^{n-b}}d\theta}{\int_\alpha^\beta e^{-\theta G \theta^{n-b}}d\theta}.
\]
The general uniform PDF restricts the domain of the parameter $\theta$ to an interval $[\alpha, \beta]$. If we lack knowledge to define $\alpha$ and $\beta$ we may let $b = 0$ and $[\alpha, \beta] \rightarrow [0, \infty)$ in the general uniform density. Parameter $\theta$ then has a diffuse prior over the nonnegative real line, and the Bayesian estimate of reliability becomes

$$\hat{R}_D(t) = 1 - \left( \frac{G}{G - \ln(g(t))} \right)^{n+1}$$

(25)

We note that the choice of the general uniform density of $\theta$ is surely a realistic one if one considers the possibility of some prior information concerning the range of the parameter.

4.5. Reliability under kernel density prior

The kernel density prior for sample size $n$ and bandwidth $h$ is given by

$$g(\theta; n, h) = \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{\theta - \theta_i}{h} \right),$$

(26)

For the discussion on the selection of the kernel function and optimal bandwidth in kernel density estimation, we refer the reader to Refs. 17 and 18. For the kernel function we shall use the Gaussian kernel, given by

$$K(u) = \frac{1}{\sqrt{2\pi}} e^{-0.5u^2}, \quad -\infty < u < \infty$$

and the optimal bandwidth. For the kernel density prior, the posterior distribution is given by

$$h_K(\theta | t) = \frac{L(t, \theta) \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{\theta - \theta_i}{h} \right)}{\int_0^\infty L(t, \theta) \frac{1}{nh} \sum_{i=1}^{n} K \left( \frac{\theta - \theta_i}{h} \right) d\theta}$$

(27)

and the Bayesian reliability estimate by

$$\hat{R}_K(t) = 1 - \int_0^\infty [g(t)]^\beta h_K(\theta | t)d\theta.$$ (28)

The Bayes reliability estimate given by (28) does not have an analytic form and must be evaluated numerically. The kernel density prior is empirical in nature and does not assume any particular distribution for the parameter $\theta$. We note that the Bayesian reliability estimates satisfy the basic properties of the reliability function

$$\lim_{t \rightarrow \infty} \hat{R}(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow 0} \hat{R}(t) = 1.$$ 

If $BK(m, n)$ represents the second order Bessel function that satisfies the differential equation

$$n^2 y'' + ny' - (n^2 + m^2)y = 0, \quad y \geq 0$$
then it satisfies the following properties:

\[ \lim_{n \to \infty} B_K(m, n) = 0 \quad \text{and} \quad \lim_{n \to 0} B_K(m, n) = \infty \]

and also

\[ \lim_{t \to \infty} \ln(t) = 0 \quad \text{and} \quad \lim_{t \to 0} \ln(t) = -\infty \]

In the next section, we present a numerical study to pairwise compare the effectiveness of the five different priors and the kernel density prior in estimating the reliability function.

5. Numerical Comparison of Priors

A schematic diagram of the complete step by step process of the numerical analysis is presented in the Appendix (Fig. 1). Our numerical study was conducted in the following manner:

(i) Assuming the values of parameter \( \theta \) follow each of the parametric priors under study, we generated \( k = 30 \) values of \( \theta \) from the inverse Gaussian distribution with parameters \( \mu = 10 \) and \( \lambda = 3 \), the inverted gamma distribution with parameters \( \alpha = 10 \) and \( \beta = 5 \), gamma probability distribution with parameters \( \alpha = 3 \) and \( \beta = 1 \), the general uniform prior with \( \alpha = \beta = 20 \) and \( b = 1 \), and the diffuse prior. The kernel density prior is estimated for each of the \( k \) samples using the Gaussian kernel and the optimal bandwidth.

(ii) Reliability estimates are calculated for samples of size \( n = 30, 50, \) and \( 100 \). The samples are generated from the Gumbel distribution, using values of \( \theta \) from step (i) and scale parameters \( \sigma = 1 \) (small variance), \( \sigma = 2 \) (medium variance) and \( \sigma = 4 \) (large variance).

(iii) For comparison purposes, the mean integrated square error (MISE) is calculated between the true modified Gumbel reliability and the corresponding Bayes estimates across all samples of size \( n \), where

\[ \text{MISE}(R(t), \hat{R}_n(t)) = \frac{1}{N} \sum_{i=1}^{N} E_I((\hat{R}_n(t) - R(t))^2) dt \]

across all failure times \( t \), for the reliability function \( R(t) \), reliability function estimate \( \hat{R}_n(t) \), and \( N \) the number of simulations performed. The results are summarized in Tables 1–3. As expected, all parametric priors, except for the diffuse prior, produced closer estimates of the true reliability than the kernel density prior. However, the reliability estimates obtained through the kernel density prior were very close to those obtained through the parametric priors. The kernel density prior performed especially well for larger values of the scale parameter \( \sigma \). As expected, the increase in sample size \( n \) produced estimates that are closer to the true Gumbel reliability, irrespective of the size of the scale parameter \( \sigma \). For the fixed sample size \( n \), the increase in \( \sigma \) increased the MISE. It is clear that for all \( n \) tested kernel density prior gives us good results without any distributional assumptions.
6. Conclusion

We obtained Bayesian reliability estimates for the Gumbel failure model whose modified location parameter is characterized by the inverse Gaussian, inverted gamma, gamma, general uniform, and diffuse priors. Additionally, we assumed a prior structure based on kernel density estimation. We also performed an extensive numerical simulation. Based on our analytical developments and computer simulation, we conclude that the Bayesian reliability estimates are sensitive to the choice of the prior distribution. The natural conjugate prior, the gamma probability distribution, does not always lead to the closest estimates of reliability. The inverse Gaussian, inverted gamma and gamma priors produce almost identical estimates of Bayesian reliability and are asymptotically efficient. The general uniform and diffuse priors produce reliability estimates that are not as close to the true reliability, but they do provide more flexibility if we are uncertain about the prior choice. The kernel density prior performed very well when compared with its parametric counterparts. It generally performed better than the diffuse prior. We recommend its application if one is uncertain about the prior distribution of the location parameter, as it is distribution free and relatively easy to implement.

### Table 1. MISE for $\sigma = 1$

<table>
<thead>
<tr>
<th>Error</th>
<th>$\hat{R}_{IGS}$, $\hat{R}_K$</th>
<th>$\hat{R}_{IGM}$, $\hat{R}_K$</th>
<th>$\hat{R}_{GM}$, $\hat{R}_K$</th>
<th>$\hat{R}_{GU}$, $\hat{R}_K$</th>
<th>$\hat{R}_{D}$, $\hat{R}_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MISE (n = 30)</td>
<td>0.0022, 0.02</td>
<td>0.0023, 0.03</td>
<td>0.0021, 0.06</td>
<td>0.004, 0.06</td>
<td>0.121, 0.08</td>
</tr>
<tr>
<td>MISE (n = 50)</td>
<td>0.0002, 0.08</td>
<td>0.0002, 0.09</td>
<td>0.0002, 0.09</td>
<td>0.005, 0.04</td>
<td>0.005, 0.004</td>
</tr>
<tr>
<td>MISE (n = 100)</td>
<td>0.0002, 0.001</td>
<td>0.0002, 0.001</td>
<td>0.0002, 0.005</td>
<td>0.0002, 0.01</td>
<td>0.005, 0.004</td>
</tr>
</tbody>
</table>

### Table 2. MISE for $\sigma = 2$

<table>
<thead>
<tr>
<th>Error</th>
<th>$\hat{R}_{IGS}$, $\hat{R}_K$</th>
<th>$\hat{R}_{IGM}$, $\hat{R}_K$</th>
<th>$\hat{R}_{GM}$, $\hat{R}_K$</th>
<th>$\hat{R}_{GU}$, $\hat{R}_K$</th>
<th>$\hat{R}_{D}$, $\hat{R}_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MISE (n = 30)</td>
<td>0.012, 0.015</td>
<td>0.014, 0.017</td>
<td>0.017, 0.029</td>
<td>0.02, 0.02</td>
<td>0.12, 0.1</td>
</tr>
<tr>
<td>MISE (n = 50)</td>
<td>0.009, 0.01</td>
<td>0.01, 0.011</td>
<td>0.02, 0.025</td>
<td>0.032, 0.03</td>
<td>0.1, 0.07</td>
</tr>
<tr>
<td>MISE (n = 100)</td>
<td>0.001, 0.01</td>
<td>0.01, 0.01</td>
<td>0.014, 0.02</td>
<td>0.06, 0.06</td>
<td>0.09, 0.08</td>
</tr>
</tbody>
</table>

### Table 3. MISE for $\sigma = 4$

<table>
<thead>
<tr>
<th>Error</th>
<th>$\hat{R}_{IGS}$, $\hat{R}_K$</th>
<th>$\hat{R}_{IGM}$, $\hat{R}_K$</th>
<th>$\hat{R}_{GM}$, $\hat{R}_K$</th>
<th>$\hat{R}_{GU}$, $\hat{R}_K$</th>
<th>$\hat{R}_{D}$, $\hat{R}_K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MISE (n = 30)</td>
<td>0.08, 0.01</td>
<td>0.012, 0.09</td>
<td>0.08, 0.08</td>
<td>0.04, 0.06</td>
<td>0.12, 0.06</td>
</tr>
<tr>
<td>MISE (n = 50)</td>
<td>0.005, 0.05</td>
<td>0.008, 0.01</td>
<td>0.07, 0.06</td>
<td>0.0031, 0.07</td>
<td>0.11, 0.06</td>
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<tr>
<td>MISE (n = 100)</td>
<td>0.001, 0.03</td>
<td>0.002, 0.07</td>
<td>0.004, 0.03</td>
<td>0.0026, 0.09</td>
<td>0.086, 0.09</td>
</tr>
</tbody>
</table>
Appendix

Fig. 1. Numerical study of the modified Gumbel reliability.

References


About the Authors

Branko Miladinovic is a research statistician at the University of South Florida in Tampa, Florida. His research interests include extreme value distributions, kernel density estimation, meta-analysis, mixed models, Bayesian modeling.

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