Time-Frequency Distributions based on Compact Support Kernels: Properties and Performance Evaluation

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Abstract—This paper presents two new time-frequency distributions (TFDs) based on kernels with compact support (KCS) namely the separable (CB) (SCB) and the polynomial CB (PCB) TFDs. The implementation of this family of TFDs follows the method developed for the Cheriet-Belouchrani (CB) TFD. The mathematical properties of these three TFDs are analyzed and their performance is compared to the best classical quadratic TFDs using several tests on multi-component signals with linear and nonlinear frequency modulation (FM) components including the noise effects. Instead of relying solely on visual inspection of the time-frequency domain plots, comparisons include the time slices’ plots and the evaluation of the Boashash-Sucic’s normalized instantaneous resolution performance measure that permits to provide the optimized TFD using a specific methodology. In all presented examples, the KCS-TFDs show a significant interference rejection, with the component energy concentration around their respective instantaneous frequency laws yielding high resolution measure values.

Index Terms—Time-frequency analysis, compact support kernel, separable compact support kernel, polynomial compact support kernel, performance evaluation, instantaneous frequency, quadratic TFDs.

1. INTRODUCTION

The majority of real-life signals are generally classified as nonstationary, i.e. as signals with time-varying spectra. In addition, signals in practice are often multi-component. Because of this, time-frequency distributions (TFDs) are the natural choice to analyze and process nonstationary signals accurately and efficiently by performing a mapping of one-dimensional signal $x(t)$ into a two dimensional function of time and frequency $TFD_x(t, f)$.

Herein, we are interested in the quadratic TFDs, also known in the literature as kernel-based transform [1]

$$TFD_x(t, f) = \int \int_{-\infty}^{+\infty} e^{j2\pi(s-t)} \phi(\eta, \tau)x(s + \tau/2) x^*(s - \tau/2)e^{-j2\pi f \tau} d\eta d\tau$$

where $\phi(\eta, \tau)$ is a two-dimensional kernel. This class of distributions could also be expressed as

$$TFD_x(t, f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J(s-t, \tau)x(s + \tau/2) x^*(s - \tau/2)e^{-j2\pi f \tau} dsd\tau$$

where

$$J(s', \tau) = \int_{-\infty}^{+\infty} \phi(\eta, \tau)e^{j2\pi s' \eta} d\eta$$

The advantage of expression 1 is to facilitate the computation and the analysis of the considered TFD by reducing the number of integrals. On the other hand, the quantity $J$ can be viewed as simply the inverse Fourier transform of $\phi(\eta, \tau)$ with respect to $\eta$. Moreover, if we note by $C_{Jx}(t, \tau)$ the convolution of the instantaneous autocorrelation function $y(t, \tau) = x(t + \tau/2)x^*(t - \tau/2)$ with $G(t, \tau) = J(-t, \tau)$, i.e.

$$C_{Jx}(t, \tau) = \int_{-\infty}^{+\infty} y(s, \tau)G(t - s)ds$$

$$= \int_{-\infty}^{+\infty} x(s + \tau/2)x^*(s - \tau/2)J(s-t, \tau)ds$$

then any quadratic TFD can be expressed as the Fourier transform of $C_{Jx}(t, \tau)$ with respect to $\tau$. It is known in the art that the use of a quadratic class of distributions permits the definition of kernels whose main property is to reduce the interference patterns induced by the distribution itself. In [2], it was shown that kernels with compact support (KCS), derived from the Gaussian kernel, allow a tradeoff between a good autoterm resolution and a high cross term rejection. The Gaussian kernel suffers from information loss due to reduction in accuracy when the Gaussian is cut off to compute the time-frequency distribution, and the prohibitive processing time due to the mask’s width which is increased to minimize the accuracy loss [2]. On the contrary, kernels with compact support are found to recover this information loss and improve processing time and, at the same time, retains the most important properties of the Gaussian kernel [3]. These features are achieved thanks to the compact support analytical property of this type of kernels since they vanish themselves outside a given compact set. It turns out that through a control parameter of the kernel width, the corresponding time-frequency distributions allow a better elimination of cross-terms while providing good resolution in both time and frequency.

Motivated by these interesting properties, we propose in...
This contribution the use of two new kernels with compact support derived from the Gaussian kernel for time-frequency analysis namely the separable KCS (SKCS) [4] and the polynomial KCS (PKCS) [5]. Similarly to the CB TFD [6], the induced TFDs referred to as SCB TFD and PCB TFD, respectively are generated following a specific method that uses first the Hilbert transform for producing analytical signals from real samples of the original signal then computes the convolutions of the proposed compact support kernels and the instantaneous autocorrelation functions and finally applies a Fourier transform to determine information related to the energy of the original signal with respect to time and frequency. In order to provide an objective assessment, the established comparisons between the KCS based TFDs and the most commonly used time-frequency representations are based on the Boashash-Sucic performance measure [7]. In this context, it is shown through several tests that the compact support kernels outperform the other ones even for the hard case of closely spaced noisy multi-component signals in the \( t-f \) plane.

The paper is organized as follows. In the next section, we analyze the mathematical characteristics that the CB kernel satisfies in the time-frequency domain. In Sections III and IV, we detail the construction and the main properties of the two new proposed classes of quadratic distributions based on the SCB and PCB kernel respectively. Section V describes the performance evaluation of TFDs with special attention to the Boashash-Sucic objective performance measure used to select the optimum time-frequency representation in each studied case. Section VI is devoted to presenting comparative experimental results obtained by applications involving energy estimation of linear and nonlinear multi-component frequency modulated signals including the influence of noise. Finally, concluding remarks are given in Section VII.

II. MATHEMATICAL PROPERTIES OF THE CB TFD

The choice of the two-dimensional kernel is crucial in the definition of a quadratic TFD and it determines the properties of the generated distribution e.g. real-valued, marginal conditions, instantaneous frequency (IF) as well as its overall performance in terms of energy concentration and resolution. In general the purpose of the kernel is to reduce the interference terms in the time-frequency distribution. However, Eq. (1) shows that the reduction of the interference patterns involves smoothing and thus results in a reduction of time-frequency resolution. Moreover, depending on the type of kernel, some of the desired properties of the time-frequency distribution are preserved while others are lost [8]. In what follows, we consider the main desirable properties verified by the CB kernel defined as [6]

\[
\phi_{CB}(\eta, \tau) = \begin{cases} 
\frac{C}{A} \exp\left(\frac{\eta^2 + \tau^2}{2D^2} - 1\right) & \text{if } \frac{\eta^2 + \tau^2}{2D^2} < 1 \\
0 & \text{Otherwise}
\end{cases}
\]

where \( D \) and \( A = e^{C} \) are control parameters. The CB TFD is thus expressed as

\[
CB_{\phi}(t, f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_{CB}(s-t, \tau) y(s, \tau) e^{-j2\pi f \tau} ds d\tau
\]

where

\[
J_{CB}(s', \tau) = A \int_{-\infty}^{+\infty} \left\{ \frac{C}{\eta^2 + \tau^2 + \frac{1}{D^2}} \right\} e^{\frac{i2\pi \eta s'}{D}} d\eta
\]

As all quadratic time-frequency distributions, the CB TFD verifies translation covariance with respect to time and frequency. Furthermore, as shown in the Appendices A-E respectively, the CB TFD is always real-valued, conserves energy and does not satisfy the marginal properties, dilation covariance and perfect localization on linear chirp signals property. Moreover, from the definitions [9], [10], [11], unitarity, compatibility with filterings and compatibility with modulations cannot be satisfied by any smoothed version of the WVD.

III. MODIFICATION OF THE CB KERNEL: THE SEPARABLE CB (SCB)

Recent results in the field of time-frequency signal analysis have shown that quadratic TFDs with separable kernels outperform many other popular TFDs in resolving closely spaced components [12], [13], [14]. This type of kernels takes the following general form

\[
\phi(\eta, \tau) = \phi_1(\eta) \phi_2(\tau)
\]

In [4], a separable kernel family with compact support (SKCS) applied to image processing was introduced. The later is a separable version of the compact support kernel. Hence, the CB kernel also referred to as KCS can be modified to the separable form that we will call Separable Cheriit-Belouchrani (SCB) kernel yielding to a new time-frequency distribution of quadratic class referred to as SCB TFD. The derived SCB kernel is given by

\[
\phi_{SCB}(\eta, \tau) = \left\{ \begin{array}{ll}
\phi_{CB}(\eta, 0) \phi_{CB}(0, \tau) & \text{if } \eta^2 < D^2 \\
0 & \text{Otherwise}
\end{array} \right.
\]

Thus

\[
\phi_{SCB}(\eta, \tau) = \begin{cases} 
\frac{CD^2}{A^2 \eta^2 - D^2 + \tau^2 - D^2} & \text{if } \frac{\eta^2}{2D^2} < 1 \\
0 & \text{Otherwise}
\end{cases}
\]

The separable CB (SCB) TFD is thus expressed as

\[
SCB_{\phi}(t, f) = \int_{-D}^{+D} \int_{-\infty}^{+\infty} J_{SCB}(s-t, \tau) y(s, \tau) e^{-j2\pi f \tau} ds d\tau
\]

Note that the SCB TFD satisfies all the mathematical properties verified by the CB TFD.
IV. THE POLYNOMIAL KCS BASED TFD

Because of its infinite support, the Gaussian kernel must be truncated to a finite window when implemented in a computer. Regardless of the size of the window, a discontinuity will be introduced at its borders that could lead to serious errors in the derivatives [15]. As a result, the use of the Gaussian kernel presents two practical limitations: information loss and derivative border effects owing to diminished accuracy, and the prohibitive processing time due to the mask size [5]. In order to avoid these drawbacks, two approaches exist: approximating the Gaussian kernel by a finite support kernel, or defining new kernels with properties close to the Gaussian. In [5], a new compact support kernel of polynomial form was proposed and applied to scale-space image processing. The new kernel, called PKCS, is not obtained by approximating the Gaussian, though it is derived from it. This compact support nature together with the possibility of controlling the kernel’s window led us to propose a new quadratic time-frequency distribution referred to as PCB TFD. The latter is implemented following the same procedure as for the CB TFD and the SCB TFD.

The PCB kernel is defined as
\[
\phi_{PCB}(\eta, \tau) = \begin{cases} 
\frac{\sqrt{\pi} \lambda^{\gamma}}{\gamma^{\gamma+1}} \left( \frac{\lambda^2 - (\eta^2 + \tau^2)^2}{\gamma^{\gamma+1}} \right) & \text{if } (\eta^2 + \tau^2) < \lambda^2 \\
0 & \text{Otherwise}
\end{cases}
\]

where \( \lambda \) is the radius of the kernel support and \( \gamma \) is considered to be a positive integer so that the resulting kernel has a polynomial form.

The polynomial CB (PCB) TFD is thus formulated as
\[
PCB_x(t, f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J_{PCB}(s-t, \tau) y(s, \tau) e^{-j2\pi ft} dsd\tau
\]

The PCB TFD is real-valued and satisfies translation covariance with respect to time and frequency. Table I gives comparisons between the most known kernel-based transforms and the KCS TFDs in terms of mathematical properties. It is important to note that there is a trade-off between the quantity of interferences and the number of good properties. In fact, many popular and valuable TFDs (e.g., the spectrogram) do not satisfy the marginal and the IF moment condition. What is more important in many practical applications is to maximize the energy concentration about the IF for mono-component signals and improve the resolution for multi-component signals [7]. The powerful point of KCS based TFDs is that they have by definition a limited width extend since they have a compact support. The kernel width is controlled through the parameter \( D \) for both the CB TFD and the SCB TFD and \( \lambda \) for the PCB TFD; and its peak is adjusted through the parameter \( A = e^C \) for the CB and SCB TFDs and \( \gamma \) for the PCB TFD allowing a tradeoff between a good autoterm resolution and a sufficient cross-term suppression.

V. PERFORMANCE EVALUATION OF TIME-FREQUENCY DISTRIBUTIONS

Just like some spectral estimates are better than others, some time-frequency distributions outperform others when used to analyze certain classes of signals [16]-[19]. For example, the Wigner-Ville distribution (WVD) [1], [16] is known to be optimal for linear frequency modulated mono-component signals since it achieves the best energy concentration around the signal IF law. The spectrogram [1], [16], on the other hand, results in an undesirable smoothing of the signal energy around its IF [16]. Consequently, the choice of the right TFD to analyze the given signal is not straightforward. An illustration example is shown in Fig. 1 where the bat echo location signal is represented in the t-f plane using the WVD, the spectrogram, the Born-Jordan distribution [1], the Choi-Williams distribution [20], the Zhao-Atlas-Marks distribution [21], the CB TFD, the SCB TFD and the PCB TFD. According to the common practice, determination of the best representing TFD is based on visual inspection of the eight plots so that the most appealing one is chosen. From Fig. 1, we can see that the KCS based TFDs and the spectrogram have cleaner plots (less interference and better component’s concentration) than the other distributions. Hence, efficient TFD concentration and resolution measurement can provide a quantitative criterion to evaluate performances of different distributions and can be used for adaptive and automatic parameters selection in t-f analysis [1]. Among the various objective measures that are discussed in literature, our attention is focused particularly on the Boashash-Sucic performance measure [7]

\[
P_1 = 1 - \frac{1}{3} A_x \frac{A_x}{2A_m} + (1 - S)
\]

| TABLE I |
|-----------------|-----------------|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Mathematical properties verified by WDM, CMD, BJD, ZAMD, CB TFD, SCB TFD and PCB TFD. |
| WVD | CMD | BJD | ZAMD | CB TFD | SCB TFD | PCB TFD |
| Real-valued | x | x | x | x | x | x |
| Marginal properties | x | x | x | x | x | x |
| Energy conservation | x | x | x | x | x | x |
| Translation covariance | x | x | x | x | x | x |
| Dilatation covariance | x | x | x | x | x | x |
| Perfect localization on linear chirp signals | x | x | x | x | x | x |
| Unitarity | x | x | x | x | x | x |
| Compatibility with filterings | x | x | x | x | x | x |
| Compatibility with modulations | x | x | x | x | x | x |
Fig. 1. TFDs of the bat echo location signal. (a) WVD, (b) Spectrogram (Hanning, $L = 55$), (c) BJD, (d) CWD ($\sigma = 0.6$), (e) ZAMD ($\alpha = 0.8$), (f) CB TFD ($D = 2.5, A = 1.4$), (g) SCB TFD ($D = 4, A = 1.4$) and (h) PCB TFD ($\lambda = 3, \gamma = 2$).
where $A_m$, $A_s$, $A_2$ are respectively the average amplitudes of the main-lobes, side-lobes and cross-terms of two consecutive signal components, with $S = (B_1 + B_2)/(2[f_2 - f_1])$ being a measure of the components’ separation in frequency ($B_k$ and $f_k, k = 1, 2$, are respectively the instantaneous bandwidth and the instantaneous frequency (IF) of the $k$th component).

The components’ main lobes average instantaneous bandwidth is defined by the quantity $B_i(t) = (B_1(t) + B_2(t))/2$. $P_i$ is close to 1 for well-performing TFDs and 0 for poorly-performing ones. An overall measure $P$ is taken to be the median of the instantaneous measures $P_i$ corresponding to different time slices in the relevant sections of the signals. The parameters in (14) can be computed automatically using the methodology described in [22].

VI. EXPERIMENTAL RESULTS

The performance of the KCS based TFDs is compared to the classical best known time-frequency distributions. Four examples are considered and discussed in detail in order to evaluate each TFD and determine the best one in terms of concentration and resolution. The TFDs with smoothing parameters are first optimized and their relative overall performance measure $P$ values are recorded in tables where

$$ P = \frac{1}{N} \sum_{j=1}^{N} P_i(t_0 = j); \quad (15) $$

and $N$ is the full range of time instants. Then, the maximum value among them is selected and it corresponds to the best performing TFD in representing the multi-component test signal.

A. Example 1: Sum of 2 crossing linear FM signals

Here, we deal with a multi-component signal $s_1(t)$ of duration $T = 128$ composed of two noiseless crossing chirps of frequency ranges $f = [0.1 - 0.2]$ Hz and $f = [0.2 - 0.1]$ Hz, respectively. The time-frequency representations of the signal $s_1(t)$ are given in Fig. 2 using several popular TFDs together with the CB TFD, the SCB TFD and the PCB TFD. It can be seen that the KCS based TFDs and the spectrogram have the greatest ability to remove the cross terms and present all clear curves in contrast to the other representations. Let us examine in depth the performance of each distribution. For this purpose, the considered TFDs are optimized with respect to the Boashash-Sucic’s criterion over the time interval $[1, T]$ except for the WVD and the BJD that have no smoothing parameters and then they cannot be optimized.

The resulting $P$’s values are recorded in Table II and they clearly reveal that the KCS based TFDs produce the best performance compared with the other time-frequency representations. Moreover, the CB TFD with control parameters $D = 4$ and $A = 1.44$ gives the largest value of $P$ and hence is selected as the best performing TFD of the signal $s_1(t)$.

<table>
<thead>
<tr>
<th>TFD</th>
<th>Optimal kernel Parameters</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>$N/A$</td>
<td>0.6581</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>Hanning, $L = 85$</td>
<td>0.8588</td>
</tr>
<tr>
<td>BJD</td>
<td>$\sigma/\alpha$</td>
<td>0.7072</td>
</tr>
<tr>
<td>CWVD</td>
<td>$\alpha = 0.6$</td>
<td>0.7269</td>
</tr>
<tr>
<td>ZAMD</td>
<td>$\alpha = 0.8$</td>
<td>0.6822</td>
</tr>
<tr>
<td>CB TFD</td>
<td>$D = M/B = 4, A = e^{c_1} = 1.44$</td>
<td>0.8708</td>
</tr>
<tr>
<td>SCB TFD</td>
<td>$D = M/B = 5, A = e^{c_2} = 2.1$</td>
<td>0.8678</td>
</tr>
<tr>
<td>PCB TFD</td>
<td>$\lambda = 2, \gamma = 2$</td>
<td>0.8626</td>
</tr>
</tbody>
</table>

B. Example 2: Sum of 2 parallel FM signals

As a second illustration test, we consider a multicomponent signal $s_2(t)$ of length $N = 128$ that consists of two closely spaced parallel linear FMs with frequencies increasing from 0.15 to 0.25 Hz and from 0.2 to 0.3 Hz, respectively. The signal $s_2(t)$ is analyzed in the $t-f$ domain using the same selection of TFDs as in example 1. The time-frequency plots of the optimized TFDs according to Boashash-Sucic’s performance measure are shown in Fig. 3, where we can see that the CB TFD, the SCB TFD and the PCB TFD have all clear plots since the two time-varying components of the signal $s_2(t)$ are well concentrated in their respective frequency ranges and the interferences between them are largely attenuated by the effects of the compact support nature of the three investigated kernels.

In this example, we first compare the TFDs’ resolution performance at time instant $t_0 = 64$; the middle of the signal duration, including the Modified B-distribution [23] as well. Table III reports the related performed measurements by referring to the Boashash-Sucic’s methodology that is used to compute the parameters of (14), whereas Fig. 4 shows the slices of a selection of TFDs at $t_0 = 64$. It indicates that the SCB TFD with smoothing parameters $D = 5$ and $A = 0.13$ is the optimal TFD of the signal $s_2(t)$ at this time instant giving the largest value of $P$. Let us then search for the TFD that best resolves the two chirp components of the signal $s_2(t)$ over the entire time interval $[1,128]$. Table IV contains the optimization process and indicates that the KCS based TFDs outperform the other quadratic time-frequency distributions. Furthermore, it shows that the signal $s_2(t)$ is best presented in the $t-f$ plane using the CB TFD with parameters $D = 2$ and $A = 0.11$ since it has the largest value of $P$.

C. Example 3: Effect of Additive noise

In order to check the behavior of TFDs in the case of noisy multi-component signals, let us search for the optimal TFD of the two-component signal $s_2(t)$ considered in example 2, embedded in additive white Gaussian noise, with a signal-to-noise ratio of 10 dB. The test signal, denoted by $s_3(t)$, is analyzed in the $t-f$ domain using a selection of quadratic TFDs. The time-frequency plots of the optimized TFDs under the constraints of Boashash-Sucic’s criterion are shown in Fig. 5. Here again, from visual inspection, we can see that the KCS based TFDs perform much better that the other considered TFDs since they generate the most appealing plots. Table V records the numerical results of the optimization procedure over the entire time interval $[1,128]$ and reveals that the optim-
Fig. 2. Optimized TFDs over the entire time interval $[1, 128]$ of the signal of example 1 composed of two crossing chirps with frequency ranges $f = 0.1 - 0.2$ Hz and $f = 0.2 - 0.1$ Hz, respectively. (a) WVD, (b) Spectrogram (Hanning, $L = 85$), (c) BJD, (d) CWD ($\sigma = 0.45$), (e) ZAMD ($\alpha = 0.8$), (f) CB TFD ($D = M/B = 4$, $A = \epsilon^C = 1.44$), (g) SCB TFD ($D = 5$, $A = 2.1$) and (h) PCB TFD ($\lambda = \beta, \gamma = 2$).
Fig. 3. Optimized TFDs over the entire time interval [1, 128] of the signal of example 2 composed of two parallel LFs with frequency ranges spreading from 0.15 to 0.25 Hz and 0.2 to 0.3 Hz, respectively. (a) WVD, (b) Spectrogram (Hanning, L = 85), (c) BJD, (d) CWD (σ = 0.45), (e) ZAMD (α = 0.8), (f) CB TFD (D = 2, A = 0.11), (g) SCB TFD (D = 5, A = 0.487) and (h) PCB TFD (λ = 1.5, γ = 1).
TABLE III
PARAMETERS AND THE NORMALIZED INSTANTANEOUS RESOLUTION PERFORMANCE MEASURE $P_i$ OF THE DIFFERENT TFDs FOR THE TIME INSTANT $t_0 = 64$ RELATED TO EXAMPLE 2. THE FIRST SIX MEASUREMENTS ARE ADOPTED FROM [7].

<table>
<thead>
<tr>
<th>TFD (optimal parameter)</th>
<th>$A_M(64)$</th>
<th>$A_X(64)$</th>
<th>$B_i(64)$</th>
<th>$\Delta_f_i(64)$</th>
<th>$S(64)$</th>
<th>$P_i(64)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>0.9153</td>
<td>0.3365</td>
<td>1</td>
<td>0.0130</td>
<td>0.0574</td>
<td>0.7735</td>
</tr>
<tr>
<td>Spectrogram (Hanning, $L = 35$)</td>
<td>0.9119</td>
<td>0.0087</td>
<td>0.5527</td>
<td>0.0266</td>
<td>0.0501</td>
<td>0.4691</td>
</tr>
<tr>
<td>BJD</td>
<td>0.9320</td>
<td>0.1222</td>
<td>0.3579</td>
<td>0.0219</td>
<td>0.0388</td>
<td>0.5532</td>
</tr>
<tr>
<td>CWD ($\sigma = 2$)</td>
<td>0.9355</td>
<td>0.0178</td>
<td>0.4445</td>
<td>0.0238</td>
<td>0.0491</td>
<td>0.5172</td>
</tr>
<tr>
<td>ZAMD ($\alpha = 2$)</td>
<td>0.9416</td>
<td>0.4847</td>
<td>0.4796</td>
<td>0.0214</td>
<td>0.0420</td>
<td>0.4905</td>
</tr>
<tr>
<td>Modified B ($\beta = 0.01$)</td>
<td>0.9676</td>
<td>0.0099</td>
<td>0.0983</td>
<td>0.0185</td>
<td>0.0526</td>
<td>0.5957</td>
</tr>
<tr>
<td>CB TFD ($D = M/B = 2, A = 0.48$)</td>
<td>0.9941</td>
<td>0.0314</td>
<td>0.0179</td>
<td>0.0159</td>
<td>0.0556</td>
<td>0.7143</td>
</tr>
<tr>
<td>SCB TFD ($D = 5, A = 0.13$)</td>
<td>0.9868</td>
<td>0.0183</td>
<td>0.0323</td>
<td>0.0159</td>
<td>0.0556</td>
<td>0.7143</td>
</tr>
</tbody>
</table>

Fig. 4. Normalized slices of TFDs at $t_0=64$ of the signal $s_2(t)$. (a) WVD, (b) Spectrogram (Hanning, $L = 35$), (c) BJD, (d) CWD ($\sigma = 2$), (e) ZAMD ($\alpha = 2$), (f) CB TFD ($D = 2, A = 0.48$) and (g) SCB TFD ($D = 5, A = 0.13$). The first five plots are adopted from [7] and compare the TFDs (dashed) against the Modified B distribution ($\beta = 0.01$) (solid).

TABLE IV
OPTIMIZATION RESULTS FOR A SELECTION OF TFDs OF THE SIGNAL OF EXAMPLE 2.

<table>
<thead>
<tr>
<th>TFD</th>
<th>Optimal kernel Parameters</th>
<th>$P^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>$N/A$</td>
<td>0.6449</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>Hanning, $L = 74$</td>
<td>0.8232</td>
</tr>
<tr>
<td>BJD</td>
<td>$N/A$</td>
<td>0.6860</td>
</tr>
<tr>
<td>CWD</td>
<td>$\sigma = 1.2$</td>
<td>0.7228</td>
</tr>
<tr>
<td>ZAMD</td>
<td>$\alpha = 0.8$</td>
<td>0.6856</td>
</tr>
<tr>
<td>CB TFD</td>
<td>$D = 2, A = 0.11$</td>
<td>0.8449</td>
</tr>
<tr>
<td>SCB TFD</td>
<td>$D = 2, A = 0.1B_7$</td>
<td>0.8414</td>
</tr>
<tr>
<td>PCB TFD</td>
<td>$\lambda = 1.5, \gamma = 1$</td>
<td>0.8409</td>
</tr>
</tbody>
</table>

TABLE V
OPTIMIZATION RESULTS FOR A SELECTION OF TFDs OF THE SIGNAL OF EXAMPLE 3 (ROBUSTNESS TO NOISE TEST).

<table>
<thead>
<tr>
<th>TFD</th>
<th>Optimal kernel Parameters</th>
<th>$P^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>$N/A$</td>
<td>0.6442</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>Bartlett, $L = 71$</td>
<td>0.8224</td>
</tr>
<tr>
<td>BJD</td>
<td>$N/A$</td>
<td>0.6764</td>
</tr>
<tr>
<td>CWD</td>
<td>$\sigma = 0.9$</td>
<td>0.7173</td>
</tr>
<tr>
<td>ZAMD</td>
<td>$\alpha = 0.56$</td>
<td>0.6422</td>
</tr>
<tr>
<td>CB TFD</td>
<td>$D = 2.5, A = 0.11$</td>
<td>0.8443</td>
</tr>
<tr>
<td>SCB TFD</td>
<td>$D = 5, A = 0.26$</td>
<td>0.8435</td>
</tr>
<tr>
<td>PCB TFD</td>
<td>$\lambda = 2, \gamma = 1$</td>
<td>0.8363</td>
</tr>
</tbody>
</table>
Fig. 5. Optimized TFDs over the full duration $T = 128$ of the signal of example 3 composed of two parallel LFM$s$ with frequency ranges spreading from 0.15 to 0.25 Hz and 0.2 to 0.3 Hz, respectively; embedded in 10 dB AWGN. (a) WVD, (b) Spectrogram (Bartlett, $L = 71$), (c) BJD, (d) CWD ($\sigma = 0.9$), (e) ZAMD ($\alpha = 0.56$), (f) CB TFD ($D = 2.5, A = 0.11$), (g) SCB TFD ($D = 3, A = 0.28$) and (h) PCB TFD ($\lambda = 2, \gamma = 1$).
mal TFD of the noisy signal \( s_3(t) \) is the CB TFD with smoothing parameters \( D = 2.5 \) and \( A = 0.11 \) since it possesses the largest value of \( P \).

**D. Example 4: Sum of 2 sinusoidal FM signals and 2 chirp signals.**

In this example, we consider a synthetic signal \( s_4(t) \) consisting of two intersecting sinusoidal FMs and two non-parallel, non-intersecting chirps. The nonlinear components consist of an increasing and decreasing sinusoidal frequency modulated signals at \( f_0 = 0, f(t_0) = 0.35 \) Hz, having both a period \( T = 128 \) sec with smallest and highest frequencies equal to 0.25 Hz and 0.45 Hz respectively. The two chirps occupy the frequency ranges \( f = [0.16 - 0.19] \) Hz and \( f = [0.07 - 0.1] \) Hz respectively. The smallest frequency separation between the linear and nonlinear components is within the range 0.18 – 0.25 Hz near 97 sec and it is low enough and is just avoiding intersection. The purpose here is to confirm again the effectiveness of the KCS based kernels in detecting closely spaced components in the case of mixtures of linear and nonlinear nonstationary signals. Fig. 6 shows the superiority of the KCS based kernels and the spectrogram over the other quadratic time-frequency distributions in resolving the four closely spaced components as well as in reducing the cross-terms. In this example, the Boashash-Sucic’s procedure is applied twice in order to measure the parameters for each of the pairs of consecutive components with equal amplitudes of each TFD time slice. The optimizing TFD’s parameters are chosen so that they produce the greatest value of the Boashash-Sucic’s overall performance measure for both the two linear chirps \((P^{(1)})\) and the two sinusoidal FMs \((P^{(2)})\), the resulting \( P \) to maximize is equal to \((P^{(1)} + P^{(2)})/2\). Table VI presents the numerical results of the optimization procedure and indicates that the CB TFD with parameters \( A = 1.2; D = 3 \) is the optimal TFD for representing \( s_4(t) \) since it produces the largest value of \( P \).

**TABLE VI \[\text{OPTIMIZATION RESULTS OF EXAMPLE 4.}\]**

<table>
<thead>
<tr>
<th>TFD</th>
<th>Optimal kernel Parameters</th>
<th>( P^{(1)} )</th>
<th>( P^{(2)} )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>WVD</td>
<td>( N/A )</td>
<td>0.6428</td>
<td>0.6089</td>
<td>0.6258</td>
</tr>
<tr>
<td>Spectrogram</td>
<td>Hanning, ( L = 45 )</td>
<td>0.8741</td>
<td>0.8644</td>
<td>0.8692</td>
</tr>
<tr>
<td>BJD</td>
<td>( N/A )</td>
<td>0.8069</td>
<td>0.7623</td>
<td>0.7246</td>
</tr>
<tr>
<td>CWD</td>
<td>( \sigma = 0.45 )</td>
<td>0.7602</td>
<td>0.7687</td>
<td>0.7644</td>
</tr>
<tr>
<td>ZAMD</td>
<td>( \theta = 0.5 )</td>
<td>0.7352</td>
<td>0.7381</td>
<td>0.7366</td>
</tr>
<tr>
<td>CB TFD</td>
<td>( D = 1.2, A = 3 )</td>
<td>0.8780</td>
<td>0.8786</td>
<td>0.8783</td>
</tr>
<tr>
<td>SCB TFD</td>
<td>( D = 3, A = 4 )</td>
<td>0.8701</td>
<td>0.8802</td>
<td>0.8751</td>
</tr>
<tr>
<td>PCB TFD</td>
<td>( \lambda = 0.2, \tau = 4 )</td>
<td>0.8815</td>
<td>0.8701</td>
<td>0.8757</td>
</tr>
</tbody>
</table>

**VIII. CONCLUSIONS**

Several time-frequency experimental tests were made to analyze linear and nonlinear FM laws with very closely spaced multi-components signals and noise effects. These tests showed that the KCS based TFDs outperform other well-known classical TFDs in terms of crossterms reduction while still achieving the best time-frequency resolution and then preserving high energy concentration around the components’ instantaneous frequencies. The comparisons made are not based only on visual measure of goodness of TFD plots by looking for the most appealing one but are quantified using the Boashash-Sucic’s objective criterion that implies a deep inspection of each time slice. KCS based TFDs give in all studied cases the largest performance measure value compared to the most known and powerful time-frequency representations. In addition, they reveal the most information about the time-varying test signals in the \( t-f \) plane in terms of detection of the components’ number, extraction of the IF laws from the TFD’s peaks, estimation of signal components bandwidths and evaluation of sidelobe and cross-term amplitudes. The later are the best eliminated using KCS kernels thanks to their compact support nature and the flexibility in tuning the kernel width and amplitude in order to reach their optimization. Note that controlling the kernel amplitude is more flexible using the CB and SCB kernels compared with the PCB kernel that uses, by definition, an integer tuning parameter.

The combination of these results, together with the method and system implementation proposed in [6], opens the way for further promising development in high-performing DSP systems for practical measurement of nonstationary signals’ energy. Future work will also attempt a more detailed and comprehensive comparison with other recently proposed high-resolution TFDs such as the MBD and BD [24] so as to guide the user in terms of how to select a specific TFD for a particular application; such considerations could not be included in this paper due to space limitations.

**VIII. APPENDICES**

**A. Appendix A: Real-valued property**

A time-frequency distribution is real if

\[
TFD_s(t, f) = \Re\{TFD_s(t, f)\} \forall t, f
\]

By calculating the complex conjugate of (1) and making the change of integration variables \( \tau' = -\tau \) and \( \eta' = -\eta \) we get

\[
TFD_s(t, f) = \int \int_{-\infty}^{\infty} e^{-j2\pi \eta(s-t)} \phi^*(\eta, \tau)y^*(s, \tau) d\eta' d\tau'
\]

Thus, a real quadratic distribution is obtained if the corresponding kernel satisfies

\[
\phi(\eta, \tau) = \phi^*(-\eta, -\tau) \quad \forall \eta, \tau \in \mathbb{R}
\]  

Since the CB kernel is an even real function with respect to both \( \eta \) and \( \tau \), then

\[
\phi_{CB}(-\eta, -\tau) = \phi_{CB}(-\eta, -\tau) = \phi_{CB}(\eta, \tau)
\]

Hence, the CB TFD is always real-valued.
Fig. 6. Optimized TFDs over the time duration $[1, 128]$ of the signal $s_4(t)$ composed of two non-parallel, non-intersecting chirps and two intersecting sinusoidal FMs. (a) WVD, (b) Spectrogram (Hanning, $L = 45$), (c) BJD, (d) CWD ($\sigma = 0.45$), (e) ZAMD ($\alpha = 0.5$), (f) CB TFD ($D = 1.2, A = 3$), (g) SCB TFD ($D = 3, A = 4$) and (h) PCB TFD ($\lambda = 0.2, \gamma = 4$).
B. Appendix B: Marginal properties

A time-frequency distribution \( TFD_x(t, f) \) of \( x(t) \) obeys the marginal properties if it reduces to the spectrum and instantaneous power by integrating over \( t \) and \( f \) respectively, i.e.

\[
\begin{align*}
\int_{-\infty}^{+\infty} TFD_x(t, f)dt &= |X(f)|^2 \\
\int_{-\infty}^{+\infty} TFD_x(t, f)df &= |x(t)|^2
\end{align*}
\]

By integrating (1) over \( t \) we obtain

\[
I(f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j2\pi\eta(s-t)}\phi(\eta, \tau)y(s, \tau) e^{-j2\pi f\tau}d\eta d\sigma d\tau dt
\]  

\[
= \int \int \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j2\pi\eta\eta}d\eta \phi(\eta, \tau) y(s, \tau) e^{-j2\pi f\tau}d\eta d\tau
\]  

Since \( \int_{-\infty}^{+\infty} e^{+j2\pi\eta\eta}d\eta = \delta(\eta) \) and \( \delta(\eta)e^{-j2\pi\eta\eta}\phi(\eta, \tau) = \delta(\eta)\phi(0, \tau)\delta(\eta) \) it yields

\[
I(f) = \int \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(\eta) d\eta \phi(0, \tau) y(s, \tau) e^{-j2\pi f\tau}d\eta d\tau
\]

In the case of (the requirement for time marginal property)

\[
\phi(0, \tau) = 1, \forall \tau
\]

(23) becomes

\[
I(f) = \int \int_{-\infty}^{+\infty} x(s + \tau/2)x^*(s - \tau/2)e^{-j2\pi f\tau}d\tau d\eta
\]

Recall that the Wigner-Ville distribution is defined as

\[
WV_x(t, f) = \int_{-\infty}^{+\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j2\pi f\tau}d\tau
\]

Hence, the integral (24) can be rewritten as follows

\[
I(f) = \int \int_{-\infty}^{+\infty} x(s + \tau/2)x^*(s - \tau/2)e^{-j2\pi f\tau}d\tau d\eta
\]

\[
= \int_{-\infty}^{+\infty} WV_x(s, f)ds
\]

\[
I(f) = |X(f)|^2
\]

since the WVD verifies the marginal properties. From the development above, we conclude that in order to a TFD of the quadratic class preserves the marginal property with respect to time, the kernel must satisfy condition (23), which is not the case of the CB kernel. By integrating (1) over \( f \) we obtain

\[
I(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j2\pi\eta(s-t)}\phi(\eta, \tau) y(s, \tau) e^{-j2\pi f\tau}d\eta d\sigma d\tau
\]

\[
= \int \int \int_{-\infty}^{+\infty} e^{-j2\pi f\tau}d\eta d\sigma d\tau
\]

Since \( \int_{-\infty}^{+\infty} e^{-j2\pi f\tau}d\eta = \delta(\tau) \) and \( \delta(\tau)\phi(\eta, \tau)y(s, \tau) = \phi(0, \eta) |x(s)|^2 \delta(\tau) \) it yields

\[
I(t) = \int \int \int_{-\infty}^{+\infty} \delta(\tau) d\tau e^{-j2\pi\eta(s-t)}\phi(0, \eta)
\]

\[
= \int \int e^{-j2\pi\eta(s-t)}\phi(0, \eta)|x(s)|^2 d\eta d\tau
\]

(27)

In the case of (the requirement for frequency marginal property)

\[
\phi(\eta, 0) = 1, \forall \eta
\]

(28)

it results

\[
I(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j2\pi\eta(s-t)}d\eta |x(s)|^2 ds
\]

(29)

From the delta function properties we have:

\[
\int_{-\infty}^{+\infty} e^{-j2\pi\eta(s-t)}d\eta = \delta(s - t), \text{ thus the integral (29) can be rewritten as follows}
\]

\[
I(t) = \int_{-\infty}^{+\infty} \delta(s - t)|x(s)|^2 ds
\]

\[
= \int_{-\infty}^{+\infty} |x(s)|^2 ds
\]

\[
= |x(t)|^2 \int_{-\infty}^{+\infty} \delta(s')ds' \quad (s' = s - t)
\]

(30)

Consequently, the CB kernel does not satisfy the marginal property with respect to frequency as well since condition (28) is not verified.

C. Appendix C: Energy conservation

A given TFD of \( x \) conserves energy if, by integrating it over time and frequency, we obtain the energy of \( x \)

\[
E_x = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} TFD_x(t, f)dt df = \int_{-\infty}^{+\infty} I(f)df
\]

(31)

Referring to (22) the right-side integral in (31) is given as follows

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi(0, \tau) y(s, \tau) e^{-j2\pi f\tau} d\eta d\sigma d\tau
\]

\[
= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j2\pi f\tau}d\eta d\sigma d\tau
\]

\[
\delta(\tau)
\]
Since $\delta(\tau)\phi(0, \tau)y(s, \tau) = \phi(0, 0)\delta(\tau)\phi(0, 0)y(s, 0) = \phi(0, 0)x(s)\phi(x(s))\delta(\tau)$ it yields
\[
\int_{-\infty}^{+\infty} I(f)df = \phi(0, 0)\int_{-\infty}^{+\infty} \delta(\tau)d\tau \int_{-\infty}^{+\infty} |x(s)|^2ds
\]
\[
= \phi(0, 0) \int_{-\infty}^{+\infty} |x(s)|^2ds
\]
(32)

Hence, if one wants to preserve the energy conservation characteristic, the kernel must satisfy the condition
\[
\phi(0, 0) = 1
\]
(33)

which is the case of the CB kernel.

D. Appendix D: Dilation covariance
A given TFD preserves dilations if
\[
z(t) = \sqrt{k}x(kt); k > 0 \Rightarrow TFD_z(t, f) = TFD_z(kt, \frac{f}{k})
\]
(34)

Referring to (2), we have
\[
TFD_z(kt, \frac{f}{k}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J(s - kt, \tau)y(s, \tau)e^{-j2\pi(\frac{f}{k})\tau}dsd\tau
\]
and
\[
TFD_z(t, f) = k \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J(s - t, \tau)z(s + \tau/2)z^*(s - \tau/2)e^{-j2\pi\tau/2}(ks - kt/2)dsd\tau
\]
(35)
\[
= k \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J(s - t, \tau)x(ks + \tau/2)x^*(ks - \tau/2)e^{-j2\pi\tau/2}dsd\tau
\]
(36)

Let: $s' = ks$ and $\tau' = k\tau$. Then
\[
TFD_z(t, f) = \frac{1}{k} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} J(s/k - t, \tau/k)y(s', \tau')e^{-j2\pi\tau'k'}ds'd\tau'
\]
(37)

From (35) and (36), the dilation property is satisfied if
\[
J(s/k - t, \tau/k) = k J(s - kt, \tau)
\]
a condition that the CB TFD does not verify.

E. Appendix E: Perfect localization on linear chirp signals
This property is achieved if the following condition holds
\[
x(t) = e^{j2\pi(f_0 + \beta t)t} \Rightarrow TFD_z(t, f) = \delta(f - (f_0 + \beta t))
\]
(38)

It is obvious that condition (38) only holds for the Wigner-Ville distribution since it is the only case where we get a sum of complex exponentials (the kernel $\phi_{WV}(\eta, \tau) = 1, \forall \eta, \tau$)

\[
WV_z(t, f) = \int_{-\infty}^{+\infty} e^{j2\pi(f_0 + \beta t)/2}((t + \tau/2)\sqrt{t + \tau/2})e^{-j2\pi\tau/2}d\tau
\]
\[
= \int_{-\infty}^{+\infty} e^{-j2\pi(f - (f_0 + \beta t)/2)\tau/2}d\tau
\]
(39)

REFERENCES

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