On the geodetic number of median graphs

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Abstract

A set of vertices $S$ in a graph is called geodetic if every vertex of this graph lies on some shortest path between two vertices from $S$. In this paper, minimum geodetic sets in median graphs are studied with respect to the operation of peripheral expansion. Along the way geodetic sets of median prisms are considered and median graphs that possess a geodetic set of size two are characterized. © 2007 Elsevier B.V. All rights reserved.

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1. Introduction

The metric in graphs derived from the shortest-path distance between pairs of vertices yields various metric concepts in graphs, such as intervals, convexity and boundary sets. While for the first two concepts there is basically one possible natural definition (interval $I(u, v)$ being the set of vertices that lie on a shortest path between vertices $u$ and $v$, and a set $S$ of vertices being convex if $I(x, y) \subseteq S$ for all $x, y \in S$), there are several different ways of defining peripheral or boundary vertices in graphs [6,14]. One of these is presented through the concept of geodetic sets from [3,13] that was further studied in [4,7,8].

A set $S$ of vertices of a graph $G = (V(G), E(G))$ is called a geodetic set in $G$ if its geodetic closure (that is the union of intervals between all pairs of vertices from $S$) equals $V(G)$. The size of a minimum (with respect to cardinality) geodetic set in a graph $G$ is called the geodetic number of $G$ and is denoted by $g(G)$. The problem of determining the geodetic number of a graph is known to be NP-hard [13] in general. On the other hand, bounds and some exact results are known for several classes of graphs. In particular, graphs with $g(G) = 2$ have been characterized in [3].

One of the most important classes of graphs derived from metric structures are median graphs. They are defined as graphs in which for every triple of vertices $u, v, w \in V(G)$ the intersection $I(u, v) \cap I(u, w) \cap I(v, w)$ consists of precisely one vertex (which is called the median of the triple $u, v, w$). Median graphs have been rediscovered several times, and a rich structure theory has been developed, cf. the survey [16]. It is natural to study metric properties such as geodetic number in a “metrically defined” class such as median graphs (in [5], for instance, another such class of distance-hereditary graphs was studied).

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In the next section we present basic preliminary results about median graphs, in particular the operation of peripheral expansion, by which larger median graphs are obtained from smaller ones. We also present some straightforward observations about geodetic sets in median graphs. In Section 3 we prove our main results. We show that the geodetic number of a median graph \( G \), obtained by the peripheral expansion from a graph \( H \) along a convex subgraph \( P \), lies between \( g(H) \) and \( g(H) + g(P) \) with both bounds being sharp. In addition we characterize the case when \( g(G) = g(H) \), and as a by-product median prisms \( G \square K_2 \) are characterized which have the same geodetic number as \( G \). In Section 4 we prove three characterizations of median graphs with geodetic number 2, and conclude the paper with an open problem.

2. Preliminaries and basic observations

Throughout the paper we consider finite, undirected graphs. One of the most useful characterizations of median graphs due to Mulder is based on a special expansion procedure \[18,19\]. We will use its variation that involves so-called peripheral subgraphs of a median graph \[20\], see also \[1\].

Let \( H \) be a connected graph and \( P \) a convex subgraph, meaning the subgraph, induced by a convex subset \( V(P) \) of \( V(H) \). Then the peripheral expansion of \( H \) along \( P \) is the graph \( G \) obtained as follows. Take the disjoint union of a copy of \( H \) and a copy of \( P \). Join each vertex \( u \) in the copy of \( P \) with the vertex that corresponds to \( u \) in the copy of \( H \) (actually in the subgraph \( P \) of \( H \)). We say that the resulting graph \( G \) is obtained by a (peripheral) expansion from \( H \) along \( P \), and denote this operation in symbols by \( G = \text{pe}(H, P) \). We also say that we expand \( P \) in \( H \) to obtain \( G \). We will denote by \( H \) also the subgraph of \( G \) that corresponds (and is isomorphic to) \( H \), and by \( P' \) the subgraph of \( G \) induced by \( V(G) \setminus V(H) \).

It is easy to prove that expanding a convex subgraph of a median graph yields again a median graph. It is more surprising that the converse is also true, as proved by Mulder in \[20\]:

**Theorem 1.** A graph \( G \) is a median graph if and only if it can be obtained from \( K_1 \) by a sequence of peripheral expansions.

The Cartesian product \( G \square H \) of graphs \( G \) and \( H \) is the graph with the vertex set \( V(G) \times V(H) \) in which vertices \((a, x)\) and \((b, y)\) are adjacent whenever \( ab \in E(G) \) and \( x = y \), or \( a = b \) and \( xy \in E(H) \). The Cartesian product is associative and commutative with \( K_1 \) as its unit. The Cartesian product of \( n \) copies of \( K_2 \) is called the hypercube \( Q_n \). Hypercubes are important examples of median graphs. It is easy to see that \( g(Q_n) = 2 \) for all \( n \) \[3\].

The distance \( d_G(u, v) \) between vertices \( u \) and \( v \) in a graph \( G \) is defined as the length of a shortest path between \( u \) and \( v \). We say that a subgraph \( H \) of a graph \( G \) is isometrically embedded in \( G \) if \( d_H(u, v) = d_G(u, v) \) for every \( u, v \in V(H) \). Two edges \( e = xy \) and \( f = uv \) of a graph \( G \) are in the Djoković–Winkler \[10,21\] relation \( \Theta_G \), \( \Theta \) for short, if \( d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u) \). Winkler \[21\] proved that bipartite graphs in which the relation \( \Theta \) is transitive are precisely the graphs that are isometrically embeddable into hypercubes (often called partial cubes), and it is well-known that median graphs have this property. Hence \( \Theta \) is an equivalence relation on median graphs. Note that by the peripheral expansion exactly one new \( \Theta \)-class appears.

For an edge \( ab \) of a connected, bipartite graph \( G \), let

\[
W_{ab} = \{ w \in V(G) \mid d_G(a, w) < d_G(b, w) \},
\]

\[
U_{ab} = \{ w \in W_{ab} \mid w \text{ has a neighbor in } W_{ba} \}.
\]

In median graphs sets \( W_{ab} \) and \( U_{ab} \) are convex, hence they induce median graphs \[15\]. A subgraph \( P \) of a median graph \( G \) is called a peripheral subgraph if it is induced by some \( W_{ab} \) which is at the same time equal to \( U_{ab} \). Hence a peripheral subgraph also corresponds to an expanded copy of \( P \) in a graph \( \text{pe}(H, P) \).

Notice that trees are median graphs in which peripheral subgraphs are precisely the one-vertex subgraphs induced by each leaf. Moreover, the minimum geodetic set of a tree consists of all leaves \[8\]. Hence, it is natural to ask what is the relation between geodetic sets and peripheral subgraphs in median graphs. Here is the first observation.

**Lemma 2.** If \( S \) is a geodetic set of a median graph \( G \) then every peripheral subgraph of \( G \) contains a vertex in \( S \).
Proof. Let $G$ be a median graph and $H$ a peripheral subgraph. Then $V(H) = U_{ab} = W_{ab}$ for some edge $ab \in E(G)$. Let $S$ be a geodetic set of $G$. Suppose $H$ does not contain any vertex of $S$. Then every vertex from $S$ lies in $W_{ba}$ which is convex in $G$. Thus the geodetic closure of $S$ lies in $W_{ba}$ and so $S$ is not a geodetic set in $G$, a contradiction. □

However, unlike in trees, it may happen that in a median graph any minimum geodetic set contains vertices that are not in a peripheral subgraph. Consider the graph from Fig. 1. Peripheral subgraphs are induced by the vertices of degree 1; however, these vertices do not form a geodetic set (note that the black vertex is not on any shortest path between vertices of degree 1).

3. Geodetic sets and peripheral expansion

We will first prove a somewhat general result about so-called gated subsets in a graph, and then apply it to median graphs. Gated sets were first introduced in [11].

A subset $S$ of the vertex set $V(G)$ is called gated if for every $x \in V(G)$ there exists a vertex $u \in S$ such that $u \in I(x, v)$ for all $v \in S$. Given $x \in V(G)$, such a vertex $u \in S$ is always unique, and we denote it by $p_S(x)$. Clearly, if $x \in S$, $p_S(x) = x$. A subgraph $H$ of $G$ is called gated if it is induced by a gated set in $G$.

Lemma 3. Let $H$ be a gated subgraph of $G$. If $S$ is a geodetic set of $G$ then $p(S) = \{x \in V(H) | x = p_H(y), \text{for } y \in S\}$ is a geodetic set of $H$. In particular, $g(G) \geq g(H)$.

Proof. Let $S$ be a geodetic set of $G$, $t \in V(H)$ and $t \in I(y, u)$ for some $y, u \in S$. We claim that $t \in I(x, v)$ where $x = p_H(y)$ and $v = p_H(u)$.

Suppose $t \notin I(x, v)$. Since $t \in I(y, u)$, we have $d(y, u) = d(y, t) + d(t, u)$ and $d(x, t) + d(u, v) + d(v, t) > d(y, x) + d(x, v) + d(u, v) = d(y, v) + d(v, u)$ which is a contradiction to the triangle inequality. □

Note that gated and convex sets coincide in median graphs [2,9]. Hence we derive the following observation.

Corollary 4. If $H$ is a convex subgraph of a median graph $G$, then $g(G) \geq g(H)$.

In particular, this yields the following relation between geodetic numbers of a median graph and its peripheral expansion.

Lemma 5. Let $G$ be a median graph such that $G = pe(H, P)$. Then $g(G) \geq g(H)$.

Consider a median graph $H$ with a convex subgraph $P$ and the peripheral expansion $G = pe(H, P)$. In view of the above result it is natural to ask when $g(G) = g(H)$. We will characterize such cases by using the structure of minimum geodetic sets in $H$. 
We call a $u, v$-path of length $d(u, v)$ a $u, v$-geodesic. Let $S$ and $T$ be two disjoint subsets of $V(G)$. Then a $u, v$-geodesic is called a geodesic on $S$ if $u, v \in S$ and a $u, v$-geodesic is called a geodesic between $S$ and $T$ if $u \in S$ and $v \in T$.

**Theorem 6.** Let $H$ be a median graph, $P$ a convex subgraph and $G = pe(H, P)$. Then $g(H) = g(G)$ if and only if there exists a minimum geodetic set $S$ of $H$ with partition $S = S_1 \cup S_2$ such that $S_1, S_2 \neq \emptyset$, $S_2 \subseteq V(P)$ and for every $v \in V(P)$:

- $v$ lies on some geodesic between $S_1$ and $S_2$, or
- $v$ lies on some geodesic on $S_1$ and on some geodesic on $S_2$.

**Proof.** Let $G$ be a nontrivial median graph, and suppose $G$ can be obtained by a peripheral expansion from a median graph $H$ along a convex subgraph $P$. Denote by $P'$ the copy of $P$ in $G$. Let $S = S_1 \cup S_2$ be a partition of a minimum geodetic set $S$ of $H$ satisfying the conditions of the theorem. Denote by $S'$ the set of vertices in $P'$ corresponding to the vertices of $S$. Let $S' = S_1 \cup S_2$. We claim that $S'$ is a geodetic set of $G$. If $v \in V(H)$ lies on an $x, y$-geodesic with $x \in S_1$ and $y \in S_2$, then $v$ clearly lies on an $x, y'$-geodesic where $y'$ is the vertex in $S_2$ corresponding to $y$. In particular, if $v \in V(P)$, then $v' \in V(P')$, that corresponds to $v$, also lies on an $x, y'$-geodesic. On the other hand, if $v \in V(P)$ lies on some geodesic on $S_1$ and on some geodesic on $S_2$, then evidently $v' \in V(P')$ lies on some geodesic on $S_2$. Thus $S'$ is a geodetic set of $G$ and so $g(G) \leq \max\{|S'| = |S| = g(H)\}$. By Lemma 5 the desired equality follows.

For the converse assume $g(G) = g(H)$ where $G$ is a median graph obtained by the peripheral expansion from $H$ along $P$ (by $H$ we also denote the corresponding convex subgraph of $G$). Let $S'$ be a minimum geodetic set of $G$. Denote by $P'$ the expanded part of $P$ in $G$, and note that $S' \cap V(P') \neq \emptyset$ by Lemma 2. Denote by $S_2$ the set of vertices from $P$ that correspond to vertices of $S' \cap V(P')$, and let $S_2 = S' \cap V(H)$. Since $S_1 \cup S_2 = S$ equals $p(S')$ in $H$, by Lemma 3 we infer that $S$ is a geodetic set of $H$. By the assumption that $g(G) = g(H)$ we also derive that $S$ is a minimum geodetic set and that $S_1 \cap S_2 = \emptyset$. It is also clear that $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. Consider now an arbitrary vertex $v \in V(P)$, and suppose that $v$ does not lie on any geodesic between $S_1$ and $S_2$. Suppose it does not lie on both. If $v$ does not lie on any geodesic on $S_2$ then $v' \in V(P')$ would not lie on any geodesic on $S' \cap V(P')$ but also not between $S' \cap V(H)$ and $S' \cap V(P').$This would imply that $v'$ does not lie on any geodesic on $S'$ in $G$ which is a contradiction. Similarly, if $v$ does not lie on any geodesic on $S_1$ in $H$ then also $v$ would not lie on any geodesic on $S_1$ in $G$. In addition, $v$ would not lie on any geodesic between $S' \cap V(H)$ and $S' \cap V(P')$. This is a contradiction since $v$ would not lie on any geodesic on $S'$ in $G$. The proof is complete. \(\square\)

**Corollary 7.** Let $G$ be a nontrivial median graph. Then $g(G \Box K_2) = g(G)$ if and only if there exists a minimum geodetic set $S$ of $G$ with partition $S = S_1 \cup S_2$ such that $S_1, S_2 \neq \emptyset$, and for every $v \in V(G)$:

- $v$ lies on some geodesic between $S_1$ and $S_2$ or
- $v$ lies on some geodesic on $S_1$ and on some geodesic on $S_2$.

In fact, in Theorem 6 we did not use that $G$ is a median graph. Hence by a more general definition of peripheral expansion the theorem could easily be extended. In particular, the corollary holds for arbitrary prisms. A similar result was proved in [17].

The question that now immediately comes forward is, how large can be the difference between $g(pe(H, P))$ and $g(H)$. We answer this by introducing the following family of median graphs.

For $k \geq 2$ we define a squared daisy $D_k$ as follows: let $c$ be a vertex with neighbors $p_1, p_2, \ldots, p_{2k}$ and for every pair of vertices $p_{2i-1}, p_{2i}$, $i \in \{1, 2, \ldots, k\}$ let there be a vertex $q_i$ adjacent to $p_{2i-1}$ and $p_{2i}$, $i \in \{1, 2, \ldots, k\}$. For $k \geq 2$ we define $A_k$ as the graph obtained from $D_k$ by adding the set of vertices $\{r_1, r_2, \ldots, r_{2k}\}$ and such that $r_ip_i \in E(A_k)$ for every $i \in \{1, 2, \ldots, 2k\}$. Note that $A_k$ contains a copy of $D_k$ as a convex subgraph. (See also Fig. 2 where $A_3$ is depicted, and vertices of $D_3$ are dark.) Hence the operation of the peripheral expansion of $A_k$ along the convex subgraph $D_k$ makes sense.

**Theorem 8.** For $k \geq 2$ let $B_k = pe(A_k, D_k)$. Then $g(B_k) = g(A_k) + g(D_k)$.
Proof. Observe that \( L = \{r_1, r_2, \ldots, r_{2k}\} \) is a minimum geodetic set of \( A_k \). Denote by \( c', p'_i, o'_i \) the vertices of \( B_k \) that are obtained by the peripheral expansion and are adjacent to \( c, p_i, o_i \), respectively.

Every vertex of \( D_k \) still belongs to an interval in \( A_k \) between two vertices of \( L \). On the other hand \( p'_{2i-1} \) and \( p'_{2i} \) belong to \( I(r_{2i-1}, o'_i) \) and \( I(r_{2i}, o'_i) \), respectively, and \( c' \in I(o'_i, o'_j) \) for \( i \neq j \). Thus \( L \cup \{o'_1, o'_2, \ldots, o'_k\} \) is a geodetic set of \( B_k \) and \( g(B_k) \leq |L| + k \).

To prove that \( g(B_k) \geq |L| + k \) it suffices to show that every geodetic set of \( B_k \) contains at least one vertex from \( \{p'_{2i-1}, p'_{2i}, o'_i, o'_i\} \) for every \( i \in \{1, 2, \ldots, k\} \). Note that any geodetic set of \( B_k \) must contain \( L \) by Lemma 2.

Suppose that there is a geodetic set \( Z \) of a graph \( B_k \) and an index \( i \) such that \( p'_{2i-1}, p'_{2i}, o'_i \notin Z \). Consider the vertex \( o'_i \). Since \( Z \) is a geodetic set there exist \( x, y \in Z \) such that \( o'_i \in I(x, y) \). Vertices \( x \) and \( y \) cannot be both in \( A_k \) because of the convexity of \( A_k \) in \( B_k \). Since \( p'_{2i-1}, p'_{2i}, o'_i \notin Z \) we derive that one of the vertices \( x, y \) must be in \( A_k \), and let \( x \in A_k \) (without loss of generality). It is then easy to see that \( x = o_i \) and so \( o_i \in Z \). \( \square \)

Corollary 9. For every \( k \in \mathbb{N} \) there exists a median graph \( G \) which can be obtained by a peripheral expansion from a median graph \( H \) such that \( g(G) = g(H) + k \).

Proof. For \( k = 1 \) we obtain the smallest example by taking \( H = P_3 \) and expanding it along the vertex of degree 2 in \( H \). Then \( G = K_{1,3} \), \( g(H) = 2 \) and \( g(G) = 3 \).

For \( k \geq 2 \) we get from Theorem 8 that \( g(B_k) = g(A_k) + k \), where \( B_k = \text{pe}(A_k, D_k) \). \( \square \)

Results of this section are summarized in the following theorem.

Theorem 10. Let \( H \) be a median graph, \( P \) a convex subgraph, and \( G = \text{pe}(H, P) \). Then \( g(H) \leq g(G) \leq g(H) + g(P) \), and both bounds are sharp.

Proof. Lower bound follows from Lemma 5. For the upper bound note that the union of geodetic sets of \( H \) and \( P' \) forms a geodetic set of \( G \). That both bounds are sharp follows from Theorems 6 and 8. \( \square \)

4. Median graphs with geodetic number 2

In this section we prove three characterizations of median graphs with geodetic number 2. The second and the third one also hold in the more general case of partial cubes. We start with a characterization of median graphs \( G \) with \( g(G) = 2 \) using the peripheral expansion procedure.
Theorem 11. Let $G$ be a median graph, different from $K_1$. Then $g(G) = 2$ if and only if $G$ can be obtained by a sequence of peripheral expansions from $K_1$ such that in each peripheral expansion step the convex subgraph $P$ with respect to which the expansion is performed has a nonempty intersection with some minimum geodetic set.

Proof. Since $G$ is a median graph it can be obtained by a sequence of peripheral expansions by Theorem 1. In this way we obtain a sequence of median graphs $G_1 = K_2, G_2, \ldots, G_k = G$ where $G_{i+1} = \text{pe}(G_i, P_i)$ and $P_i$ is a peripheral subgraph of $G_i$ for $i \in \{1, 2, \ldots, k - 1\}$. Suppose $g(G) = g(G_k) = 2$. Then Lemma 4 implies $g(G_{k-1}) = 2$, since the geodetic number of every nontrivial graph is at least 2. Thus $g(G) = g(G_{k-1})$ and $P_{k-1}$ has a nonempty intersection with some minimum geodetic set of $G_{k-1}$ by Theorem 6. By repeating the same argument the result follows.

For the converse we use the induction on the number of peripheral expansion steps. Suppose $G = \text{pe}(H, P)$. By induction hypothesis $g(H) = 2$. Since $P$ has a nonempty intersection with some minimum geodetic set of $H$, say $\{a, b\}$, we have a partition $S = S_1 \cup S_2 = \{a\} \cup \{b\}$ which obviously fulfills the conditions of Theorem 6. Thus $g(H) = g(G)$ and $g(G) = 2$. □

For the following two characterizations of partial cubes with geodetic number 2 we will introduce two types of antipodal vertices in partial cubes. While the first type pertains with the metric nature of partial cubes, the second one also has a geometric flavor. From the main theorem it will follow that both concepts in fact coincide.

Let $G$ be a graph isometrically embeddable into a hypercube. Vertices $u$ and $v$ are called antipodal in the partial cube $G$ if their distance is equal to the number of $\Theta$-classes of $G$.

By a result of Eppstein [12] every partial cube can be isometrically embedded into an $n$-dimensional grid $\mathbb{Z}^n$. Denote by $p_i$ a projection from $\mathbb{Z}^n$ to $\mathbb{Z}_i$ where $\mathbb{Z}_i$ is the $i$th copy of $\mathbb{Z}$ in $\mathbb{Z}^n$. A projection $p_i$ of vertices of a graph $G$, isometrically embedded into $\mathbb{Z}^n$, is a discrete interval $[a_i, b_i] \subset \mathbb{Z}_i$ which is the set of all positive integers between $a_i$ and $b_i$. By representing this interval as the path $p_i$ of length $d_i$ we can interpret the isometric embedding into $\mathbb{Z}^n$ as an isometric embedding into the Cartesian product of paths $\pi_1, \pi_2, \ldots, \pi_n$. We say that vertices $u$ and $v$ of a graph $G$ which is isometrically embedded in $\mathbb{Z}^n$ are antipodal with respect to $\mathbb{Z}^n$ if $d_G(u,v) = d_1 + d_2 + \ldots + d_n$.

The following lemma asserts that every $\Theta$-class corresponds to exactly one unit interval of one copy of $\mathbb{Z}$ in $\mathbb{Z}^n$.

Lemma 12. Let $G$ be a partial cube graph, isometrically embedded into $\mathbb{Z}^n$, and let $ab, xy \in E(G)$. Then, $ab \Theta xy$ if and only if there exists $i$ such that $p_i(b) \neq p_i(a) \neq p_i(y) \neq p_i(x)$.

Proof. Suppose first that $ab$ and $xy$ are edges of a graph $G$ which are in relation $\Theta$. Since $G$ is isometrically embedded into $\mathbb{Z}^n$ there exists a unique index $i$ such that $p_i(a) = p_i(b)$ and $p_j(a) = p_j(b)$ for all $j \neq i$. We may assume without loss of generality that $p_i(a) + 1 = p_i(b)$, and that $x \in W_{ab}$ (which implies $y \in W_{ba}$ since $ab \Theta xy$). Since $G$ is isometric, the distances between vertices in $G$ correspond to the distances, derived from coordinates of their embedding. Thus, since $x \in W_{ab}$, we have $p_i(x) \leq p_i(a)$. By an analogous argument $p_i(y) \leq p_i(b)$. Since $|p_i(x) - p_i(y)| = 1$ we derive $p_i(x) = p_i(a)$ and $p_i(y) = p_i(b)$.

For the converse, assume that $ab$ and $xy$ are edges, such that there is an index $i$ with $p_i(x) = p_i(a) = p_i(b) = p_i(y)$. Then clearly $p_j(x) = p_j(y)$ and $p_j(a) = p_j(b)$ for all $j \neq i$. In the following equalities we denote by $v_i$ the $i$th coordinate $p_i(v)$ of a vertex $v$:

$$d(a, y) = \sum_{j=1}^{n} d(a_j, y_j) = d(a_i, y_i) + \sum_{j=1, j \neq i}^{n} d(a_j, y_j)$$

$$= d(a_i, b_i) + \sum_{j=1, j \neq i}^{n} d(b_j, y_j) = 1 + \sum_{j=1}^{n} d(b_j, y_j)$$

$$= 1 + d(b, y),$$

where equality in the middle row follows from $d(b_j, y_j) = 0$. By arguing in the same way we derive also $d(b, x) = d(a, x) + 1$ hence

$$d(a, y) + d(b, x) = 2 + d(b, y) + d(a, x),$$

and so $ab \Theta xy$. □
Lemma 13. (Imrich and Klavžar [15]). Suppose that a walk \( \pi \) connects the endvertices of an edge \( e \) but does not contain it. Then \( \pi \) contains an edge \( f \) with \( e \sim f \).

Lemma 14. (Imrich and Klavžar [15]). Let \( P \) be a path in a median graph \( G \). Then \( P \) is a geodesic if and only if all edges in \( P \) lie in pairwise distinct \( \Theta \)-classes.

Theorem 15. Let \( G \) be a partial cube, different from \( K_1 \). Then the following assertions are equivalent:

(i) \( g(G) = 2 \);
(ii) there exist vertices \( a \) and \( b \) that are antipodal in the partial cube \( G \);
(iii) \( G \) can be isometrically embedded into \( \mathbb{Z}^n \) in such a way that there exist antipodal vertices \( a \) and \( b \) with respect to \( \mathbb{Z}^n \).

Proof. (i)\(\rightarrow\)(ii): Suppose \( G \) is a partial cube with \( g(G) = 2 \) and let \( S = \{a, b\} \) be a minimum geodetic set. First, we claim that for each \( \Theta \)-class \( F \), there is an \( a, b \)-geodesic \( II \) that contains an edge in \( F \).

Let \( u \) and \( v \) be adjacent vertices and \( uv \in F \). Since \( \{a, b\} \) is a geodetic set, both \( u \) and \( v \) must lie on a shortest path between \( a \) and \( b \). If they lie on the same \( a, b \)-geodesic then clearly also the edge \( uv \) lies on it. Now consider the case when \( u \) and \( v \) lie on different \( a, b \)-geodesics. Let \( Q \) and \( R \) be the parts of these geodesics from \( a \) to \( u \) and from \( a \) to \( v \), respectively. Then by Lemma 13 either \( Q \) or \( R \) contains an edge of the \( \Theta \)-class \( F \) and the claim is proved.

If \( G \) contains only one \( a, b \)-geodesic, then by the claim this geodesic needs to contain an edge in each \( \Theta \)-class. Now, assume that an edge \( uv \) of arbitrary \( \Theta \)-class \( F \) lies on an \( a, b \)-geodesic \( II \) and that there is another \( a, b \)-geodesic \( II' \). Then \( II \) without the edge \( uv \) and together with \( II' \) is a shortest path connecting vertices \( u \) and \( v \). Thus by Lemma 13 this walk contains an edge \( e \) with \( e \sim uv \). Since \( II \) is a geodesic, \( e \) cannot lie in \( II \) by Lemma 14, thus \( e \) is contained in \( II' \).

We derive that any \( a, b \)-geodesic contains an edge from each \( \Theta \)-class and so \( a \) and \( b \) are antipodal in the partial cube \( G \).

(ii)\(\rightarrow\)(iii): Since \( G \) is a partial cube, we derive from Eppstein’s theorem that \( G \) can be isometrically embedded into the Cartesian product of paths \( \pi_1, \pi_2, \ldots, \pi_n \), where \( d_i \) is the length of a path \( \pi_i \). Let \( a \) and \( b \) be antipodal vertices in the partial cube \( G \), that is \( d_G(a, b) = d_1 + d_2 + \cdots + d_n \), implying \( a \) and \( b \) are antipodal with respect to \( \mathbb{Z}^n \).

(iii)\(\rightarrow\)(i): Suppose that a nontrivial partial cube \( G \) can be isometrically embedded into \( \mathbb{Z}^n \) in such a way that there exist vertices \( a \) and \( b \) that are antipodal with respect to \( \mathbb{Z}^n \). Let \( B = \pi_1 \square \pi_2 \square \cdots \square \pi_n \). Obviously, every vertex of \( B \) and thus every vertex of \( G \) lies on some \( a, b \)-geodesic in \( B \). Since \( G \) is an isometric subgraph of \( B \), for every \( x \in V(G) \) we have \( d_G(a, b) = d_B(a, b) = d_B(a, x) + d_B(x, b) = d_G(a, x) + d_G(x, b) \). This implies the existence of a shortest path between \( a \) and \( b \) in \( G \) going through \( x \). Thus \( \{a, b\} \) is a minimum geodetic set and the proof is complete.

A geometric interpretation of characterization (iii) above is that a partial cube \( G \) has geodetic set \( \{a, b\} \) if and only if \( G \) can be isometrically embedded into \( \mathbb{Z}^n \) in such a way that the box in \( \mathbb{Z}^n \) obtained as the convex closure of vertices \( a \) and \( b \) contains entire \( G \).

5. Open problem

Let \( G \) be a graph. The eccentricity of a vertex \( u \in V(G) \) is defined as \( \text{ecc}(u) = \max\{d(u, v) : v \in V(G)\} \). In [5] contour vertices of a graph \( G \) were defined as vertices \( v \) for which \( \text{ecc}(v) \geq \text{ecc}(x) \) for all neighbors \( x \) of \( v \). The set of all contour vertices of \( G \) is called the contour set of \( G \) and is denoted by \( Ct(G) \). It was proved that contour vertices of a distance-hereditary graph form a geodetic set (which is, of course, not necessarily minimum) [5]. The same holds for chordal graphs and several other classes of perfect graphs, yet it is open for bipartite graphs [6]. We believe this is true at least for median graphs.

Conjecture 1. For a median graph \( G \), the set \( Ct(G) \) is a geodetic set of \( G \).

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References