Generalized trace formula and asymptotics of the averaged Turan determinant for polynomials orthogonal with a discrete Sobolev inner product

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Abstract

Let \( \mu \) be a finite positive Borel measure supported on \([-1, 1]\) and introduce the discrete Sobolev-type inner product

\[
\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)g^{(i)}(a_k),
\]

where the mass points \( a_k \) belong to \([-1, 1]\), and \( M_{k,i} > 0 (i = 0, 1, \ldots, N_k) \). In this paper, we obtain generalized trace formula and asymptotics of the averaged Turan determinant for the Sobolev-type orthogonal polynomials. Asymptotics of the recurrence coefficients for symmetric Gegenbauer–Sobolev orthogonal polynomials is obtained. Trace formula and asymptotics of Turan’s determinant for Gegenbauer–Sobolev orthogonal polynomials are also given.

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1. Generalized trace formula and asymptotics of the averaged Turan determinant

Let $\mu$ be a finite positive Borel measure supported on the interval $(-1, 1)$ with infinitely many points at the support, and let $a_k (k = 1, 2, \ldots, K)$ be real numbers such that $a_k \in [-1, 1]$. For $f$ and $g$ in $L^2_\mu(-1, 1)$ such that there exist the derivatives in $a_k$, we introduce a discrete Sobolev-type inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) d\mu(x) + \sum_{k=1}^{K} \sum_{i=0}^{N_k} M_{k,i} f^{(i)}(a_k)g^{(i)}(a_k),$$

(1.1)

where $M_{k,i} > 0 (i = 0, 1, \ldots, N_k, k = 1, 2, \ldots, K)$.

As it is well known, this inner product (and corresponding orthogonal systems) is used in some problems of functional analysis, function theory and mathematical physics [7,11,14,18,19].

On the other hand, if we investigate the oscillation of a string loading with masses $M_k$ at the points $a_k$ and use the Fourier method for the corresponding Sturm–Liouville boundary value problem associated with the second-order partial differential equation, then the eigenvectors are orthogonal with respect to the inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) dx + \sum_{k=1}^{N} M_k f(a_k)g(a_k).$$

If we study the oscillation of girder, we get a fourth-order partial differential equation. The corresponding eigenfunctions are orthogonal with respect to the inner product involving derivatives.

This problem also closely related to important type of combinations of manifolds (elastic substructures) of various dimensions. The best-known example, which pertains to the equilibrium theory of plates strengthened by rods, was considered for the first time by S.P. Timoshenko as early as 1915 (he was a famous specialist in elastic theory).

Let $\{\hat{B}_k\} (k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\})$ be the sequence of orthonormal polynomials with respect to the inner product (1.1), i.e.

$$\langle \hat{B}_n, \hat{B}_k \rangle = \delta_{n,k} \quad (n, k \in \mathbb{Z}_+).$$

Let $N_k^*$ be the positive integer number defined by

$$N_k^* = \begin{cases} N_k + 1 & \text{if } N_k \text{ is odd,} \\ N_k + 2 & \text{if } N_k \text{ is even,} \end{cases}$$

and let

$$w_N(x) = \prod_{k=1}^{K} (x - a_k)^{N_k^*},$$

(1.2)

where $N = \sum_{k=1}^{K} N_k^*$.

**Lemma 1.1** (Rocha et al. [31]). The polynomials $\hat{B}_n$ satisfy the recurrence relation:

$$w_N(x)\hat{B}_n(x) = \sum_{j=0}^{N} x_{n+j,j} \hat{B}_{n+j}(x) + \sum_{j=1}^{N} x_{n,j} \hat{B}_{n-j}(x),$$

$$ (n \in \mathbb{Z}_+; \hat{B}_{-j}(x) = 0, j = 1, 2, \ldots; x_{n,s} = 0, n = 0, 1, \ldots, s - 1),$$

(1.3)
furthermore if \( \mu'(x) > 0 \) a.e. then
\[
\lim_{n \to \infty} x_{n,j} = x_j \quad (j = 0, 1, 2, \ldots, N),
\]
where
\[
w_N(x) = x_0 + 2 \sum_{j=1}^{N} x_j T_j(x),
\]
and are given by
\[
x_j = \frac{1}{2^N} \frac{\omega_2(N-j)(0)}{(N-j)!} \quad (j = 0, 1, 2, \ldots, N)
\]
with
\[
\omega_2(N) = \prod_{k=1}^{K} (\xi^2 - 2a_k \xi + 1)^{N_k},
\]
where \( T_s(x) (s = 0, 1, \ldots) \) is the Chebyshev polynomial of the first kind and degree \( s \).

**Lemma 1.2.** Let
\[
\delta = \{\delta_n, \delta_n \in \mathbb{R}^1, \delta_n \neq 0, n \in \mathbb{Z}_+; \delta_{-n} = 0, n = 1, 2, \ldots\}
\]
be an arbitrary sequence. Then the following formula
\[
[w_N(t) - w_N(x)] \sum_{k=0}^{N} \delta_k \hat{B}_k(t) \hat{B}_{k+N}(x)
\]
\[
= \sum_{j=0}^{N} \sum_{k=0}^{n} [\delta_{k-j} x_{k,j} - \delta_k x_{k+N,j}] \hat{B}_k(t) \hat{B}_{k-j+N}(x)
\]
\[
+ \sum_{j=1}^{N} \sum_{k=0}^{n} [\delta_{k+j} x_{k+j,j} - \delta_k x_{k+N+j,j}] \hat{B}_k(t) \hat{B}_{k+j+N}(x)
\]
\[
+ \sum_{j=1}^{n+j} \sum_{k=n+1}^{N} \delta_{k-j} x_{k,j} \hat{B}_k(t) \hat{B}_{k-j+N}(x) - \sum_{j=1}^{n+j} \sum_{k=n+1}^{N} \delta_k x_{k,j} \hat{B}_{k-j}(t) \hat{B}_{k+N}(x)
\]
holds for all \( t, x \) and \( n \in \mathbb{Z}_+ \).

**Proof.** By (1.3) one obtains
\[
[w_N(t) - w_N(x)] \sum_{k=0}^{N} \delta_k \hat{B}_k(t) \hat{B}_{k+N}(x)
\]
\[
= \sum_{j=0}^{N} \sum_{k=0}^{n} \delta_k x_{k+j,j} \hat{B}_{k+j}(t) \hat{B}_{k+N}(x) + \sum_{j=1}^{N} \sum_{k=0}^{n} \delta_k x_{k,j} \hat{B}_{k-j}(t) \hat{B}_{k+N}(x)
\]
\[
- \sum_{j=0}^{n} \sum_{k=0}^{N} \delta_k x_{k+N+j,j} \hat{B}_{k+N+j}(t) \hat{B}_{k+N+j}(x) - \sum_{j=1}^{n} \sum_{k=0}^{N} \delta_k x_{k+N,j} \hat{B}_{k+N}(t) \hat{B}_{k+N-j}(x).
\]
By Abel’s transform and using the initial conditions (1.3), one gets

\[
[w_N(t) - w_N(x)] \sum_{k=0}^{n} \delta_k \hat{B}_k(t) \hat{B}_{k+N}(x)
\]

\[
= \sum_{k=0}^{n} \delta_k [x_{k,0} - x_{k+N,0}] \hat{B}_k(t) \hat{B}_{k+N}(x)
\]

\[
+ \sum_{j=1}^{N} \sum_{k=0}^{n} \delta_k x_{k+j,0} \hat{B}_{k+j}(t) \hat{B}_{k+N}(x) - \sum_{j=1}^{N} \sum_{k=0}^{n} \delta_k x_{k+N,j} \hat{B}_k(t) \hat{B}_{k-j+N}(x)
\]

\[
+ \sum_{j=1}^{N} \sum_{k=0}^{n} \delta_k x_{k-j,0} \hat{B}_{k-j}(t) \hat{B}_{k+N}(x) - \sum_{j=1}^{N} \sum_{k=0}^{n} \delta_k x_{k+N-j,0} \hat{B}_k(t) \hat{B}_{k+N-j}(x)
\]

\[
= \sum_{k=0}^{n} \delta_k [x_{k,0} - x_{k+N,0}] \hat{B}_k(t) \hat{B}_{k+N}(x) + \sum_{j=1}^{N} \sum_{k=0}^{n+j} \delta_k x_{k,j} \hat{B}_k(t) \hat{B}_{k+N-j}(x)
\]

\[
+ \sum_{j=1}^{N} \sum_{k=0}^{n} (\delta_{k-j} x_{k,j} - \delta_k x_{k+N,j}) \hat{B}_k(t) \hat{B}_{k+N-j}(x)
\]

\[
- \sum_{j=1}^{N} \sum_{k=n-j+1}^{n} \delta_k x_{k+j,j} \hat{B}_k(t) \hat{B}_{k+N+j}(x)
\]

\[
+ \sum_{j=1}^{N} \sum_{k=0}^{n} (\delta_{k+j} x_{k+j,j} - \delta_k x_{k+N+j,j}) \hat{B}_k(t) \hat{B}_{k+N+j}(x),
\]

which proves Lemma 1.2. □

We introduce the averaged Turan $\delta$-determinant:

\[
G^{(N)}_n(x; \delta) := \sum_{j=1}^{N} \sum_{k=n+1}^{n+j} x_{k,j} [\delta_{k-j} \hat{B}_k(x) \hat{B}_{k+N-j}(x) - \delta_k \hat{B}_{k-j}(x) \hat{B}_{k+N}(x)].
\]  

(1.9)

Putting $t = x$ in Lemma 1.2, one has

**Corollary 1.3.** For all $t, x$ and $n \in \mathbb{Z}_+$, the following relation

\[
G^{(N)}_n(x; \delta) = \sum_{j=0}^{N} \sum_{k=0}^{n} (\delta_k x_{k+N,j} - \delta_{k-j} x_{k,j}) \hat{B}_k(x) \hat{B}_{k-j+N}(x)
\]

\[
+ \sum_{j=1}^{N} \sum_{k=0}^{n} (\delta_k x_{k+N+j,j} - \delta_{k+j} x_{k+j,j}) \hat{B}_k(x) \hat{B}_{k+j+N}(x).
\]

(1.10)

holds.
Lemma 1.4 (Rocha et al. [31]). If $\mu'(x) > 0$ a.e., then for every function $f$ continuous in $[-1, 1]$ and for every positive $k$, one has

$$
\lim_{n \to \infty} \int_{-1}^{1} f(x) \hat{B}_n(x) \hat{B}_{n+k}(x) \, d\mu(x) = \frac{1}{\pi} \int_{-1}^{1} f(x) T_k(x) \frac{1}{\sqrt{1-x^2}} \, dx \quad (k \in \mathbb{Z}_+). \tag{1.11}
$$

Now we get a weak-type asymptotics of the averaged Turan $\delta$-determinant.

Theorem 1. If $f \in C([-1, 1]), \mu'(x) > 0$ a.e. and for sequence (1.8) relation

$$
\lim_{n \to \infty} \delta_n = \delta \neq 0 \tag{1.12}
$$

holds, then

$$
\lim_{n \to \infty} \int_{-1}^{1} f(x) G_n^{(N)}(x; \delta) \, d\mu(x) = \frac{\delta}{\pi} \int_{-1}^{1} f(x) U_{N-1}(x) w'_N(x) \sqrt{1-x^2} \, dx, \tag{1.13}
$$

where $U_{N-1}(x)$ is the Chebyshev polynomial of the second kind and degree $(N - 1)$, and $w_N(x)$ is defined by (1.2).

Proof. By (1.4), (1.6), (1.7), (1.9), (1.11), (1.12), one obtains

$$
\begin{align*}
&\lim_{n \to \infty} \int_{-1}^{1} f(x) G_n^{(N)}(x; \delta) \, d\mu(x) \\
&= \sum_{j=1}^{N} \lim_{n \to \infty} \sum_{k=n+1}^{n+j} \alpha_{k,j} \int_{-1}^{1} f(x) \hat{B}_k(x) \hat{B}_{k+N-j}(x) \, d\mu(x) \\
&\quad - \sum_{j=1}^{N} \lim_{n \to \infty} \sum_{k=n+1}^{n+j} \alpha_{k,j} \int_{-1}^{1} f(x) \hat{B}_{k-j}(x) \hat{B}_{k+N}(x) \, d\mu(x) \\
&= \frac{\delta}{\pi} \sum_{j=1}^{N} j \alpha_j \int_{-1}^{1} f(x)[T_{N-j}(x) - T_{N+j}(x)] \frac{1}{\sqrt{1-x^2}} \, dx \\
&= \frac{2\delta}{\pi} \sum_{j=1}^{N} j \alpha_j \int_{-1}^{1} f(x) \sqrt{1-x^2} U_{N-1}(x) U_{j-1}(x) \, dx,
\end{align*}
$$

where

$$
U_s(x) = \frac{\sin(s+1) \arccos x}{\sin(\arccos x)} \quad (-1 \leq x \leq 1; s \in \mathbb{Z}_+)
$$

is the Chebyshev polynomial of the second kind and degree $s$. Combining relations

$$
\lim_{n \to \infty} \int_{-1}^{1} f(x) G_n^{(N)}(x; \delta) \, d\mu(x) = \frac{2\delta}{\pi} \int_{-1}^{1} f(x) \sum_{j=1}^{n} j \alpha_j U_{j-1}(x) U_{N-1}(x) \sqrt{1-x^2} \, dx
$$
and

\[ T'_j(x) = jU_{j-1}(x) \quad (j = 1, 2, \ldots) \]

with (1.5), one obtains (1.13). Theorem 1 is proved. □

Let

\[ E_K := (-1, 1) \setminus \bigcup_{k=1}^{K} \{a_k\}. \tag{1.14} \]

Using (1.10) and (1.13) by standard methods [13,22,24–27,35,36], one can obtain the main result.

**Theorem 2.** Assume that sequence (1.8) satisfies (1.12), and the estimate

\[ \sum_{j=0}^{N} \sum_{k=0}^{\infty} |\delta_k - \delta_{k+j}| < \infty \tag{1.15} \]

holds. Suppose the orthonormal polynomial system \( \{\hat{B}_n\} (n \in \mathbb{Z}_+) \) satisfies the following conditions:

(a) The recurrence coefficients (1.3) are of \( N \)-bounded variation, i.e.

\[ \sum_{j=0}^{N} \sum_{k=0}^{\infty} |\alpha_{k,j} - \alpha_{k+N,j}| < \infty. \]

(b) There exists a continuous function \( h(x) \) on \( E_K \) (see (1.14)) such that

\[ |\hat{B}_n(x)| \leq h(x) \quad (x \in E_K). \]

(c) The measure \( \mu \) is absolutely continuous in \( E_K \), and \( \mu'(x) = \rho(x) \) is strictly positive and continuous on \( E_K \). Then the following assertions are valid:

(1) At every \( x \in E_K \) and uniformly on every compact subset of \( E_K \), the asymptotics of the averaged Turan \( \delta \)-determinant

\[ \lim_{n \to \infty} G_n^{(N)}(x; \delta) = \frac{\delta}{\pi} U_{N-1}(x) w'_N(x) \frac{\sqrt{1-x^2}}{\rho(x)} \]

holds; an upper bound for the uniform error of approximation for \( x \) on a closed subset of \( E_K \) is given by

\[ \left| G_n^{(N)}(x; \delta) - \frac{\delta}{\pi} U_{N-1}(x) w'_N(x) \frac{\sqrt{1-x^2}}{\rho(x)} \right| \leq C \sum_{j=0}^{N} \sum_{k=n+1}^{\infty} |\delta_k - \delta_{k+j}| + C \sum_{j=0}^{N} \sum_{k=n+1}^{\infty} |\alpha_{k,j} - \alpha_{k+N,j}|, \]

where \( C \)'s are positive constants independent of \( n \in \mathbb{Z}_+; \)
(2) At every \( x \in E_K \) and uniformly on every compact subset of \( E_K \), the following generalized trace formula
\[
\sum_{j=0}^{N} \sum_{k=0}^{\infty} (\delta_k y_{k+N,j} - \delta_k y_{k,j}) \hat{B}_k(x) \hat{B}_{k+N-j}(x)
\]
\[
+ \sum_{j=1}^{N} \sum_{k=0}^{\infty} (\delta_k y_{k+j+N,j} - \delta_k y_{k+j,j}) \hat{B}_k(x) \hat{B}_{k+N+j}(x)
\]
\[= \frac{\delta}{\pi} U_{N-1}(x) w_N'(x) \frac{\sqrt{1-x^2}}{\rho(x)} \]
holds.

Put \( \delta_k = 1 (k \in \mathbb{Z}_+) \), then we get the following analog of Turan’s determinant [12,13,16,22,24,26,32,35,36]:
\[
G^{(N)}_n(x) = G^{(N)}_n(x, 1) = \sum_{j=1}^{n+j} \sum_{k=n+1}^{\infty} \alpha_{k,j} [\hat{B}_k(x) \hat{B}_{k+N-j}(x) - \hat{B}_{k-j}(x) \hat{B}_{k+N}(x)].
\]

**Corollary 1.5.** Under conditions (a)–(c) of Theorem 2, the following assertions are valid at every \( x \in E_K \) and uniformly on every compact subset of \( E_K \):

(a) \[
\lim_{n \to \infty} G^{(N)}_n(x) = \frac{1}{\pi} U_{N-1}(x) w_N'(x) \frac{\sqrt{1-x^2}}{\rho(x)} ;
\]

(b) \[
\sum_{j=0}^{N} \sum_{k=0}^{\infty} (\alpha_k y_{k+N,j} - \alpha_k y_{k,j}) \hat{B}_k(x) \hat{B}_{k+N-j}(x)
\]
\[+ \sum_{k=1}^{N} \sum_{k=0}^{\infty} (\alpha_{k+N+j,j} - \alpha_{k+j,j}) \hat{B}_k(x) \hat{B}_{k+N+j}(x) = \frac{1}{\pi} U_{N-1}(x) w_N'(x) \frac{\sqrt{1-x^2}}{\rho(x)} .
\]

**Remarks.** (1) This result partially have been announced in [30].

(2) For \( N = 1 \) we obtain a new approach to the proof of the remarkable formulas of papers [8,22].

Let \( \{p_n\} (n \in \mathbb{Z}_+) \) be a sequence of polynomials
\[
p_n(x) = k(p_n)x^n + r(p_n)x^{n-1} + \cdots, \quad k(p_n) > 0 \quad (n \in \mathbb{Z}_+),
\]
such that
\[
\int_{-1}^{1} p_n(x) p_m(x) \, d\mu(x) = \delta_{m,n} \quad (m, n \in \mathbb{Z}_+).
\]
These orthonormal polynomials satisfy a three-term recurrence relation:
\[
x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) \quad (n \in \mathbb{Z}_+; p_{-1}(x) = 0),
\]
where
\[ a_{n+1} = \frac{k(p_n)}{k(p_{n+1})} > 0, \quad b_n = \frac{r(p_n)}{k(p_n)} - \frac{r(p_{n+1})}{k(p_{n+1})} \quad (n \in \mathbb{Z}_+) . \]

We define the Turan determinant as follows:
\[ G_n(x) = p_{n+1}^2(x) - p_n(x) p_{n+2}(x) \quad (n \in \mathbb{Z}_+) . \]

If the recurrence coefficients converge
\[ \lim_{n \to \infty} a_n = \frac{1}{2}, \quad \lim_{n \to \infty} b_n = 0 , \]

and
\[ \sum_{k=0}^{\infty} (|a_{k+1} - a_{k+2}| + |b_k - b_{k+1}|) < \infty , \]

then uniformly on every closed subset of \((-1, 1)\), the following formulas are valid:

(1) \[ \lim_{n \to \infty} G_n(x) = \frac{2}{\pi} \frac{\sqrt{1-x^2}}{\mu'(x)} ; \]

(2) Trace formula:
\[ \sum_{n=0}^{\infty} [a_{n+1}^2 - a_n^2] p_n^2(x) + a_{n+1} (b_n - b_{n+1}) p_{n+1} p_n(x) = \frac{1}{2\pi} \frac{\sqrt{1-x^2}}{\mu'(x)} . \]

(3) One can apply this result to recover the spectral measure from the Jacobi matrix [23,13,36].

“. . . highly effective method for recovering the spectral measure from the Jacobi matrix is the one using Turan determinant” [23, p. 457]. For analogous remark on use of Trace formula see [23, p. 460].

“It is evident that Legendre projection is poor near the end-points, whereas the Sobolev projection displays reasonably good behavior throughout the interval” [15, p. 124].

2. Symmetric Gegenbauer–Sobolev orthogonal polynomials

We consider a nonstandard inner product:
\[
\langle f, g \rangle = \int_{-1}^{1} fg \, d\mu_x + M [f(1)g(1) + f(-1)g(-1)] \\
+ N [f'(1)g'(1) + f'(-1)g'(-1)],
\]

where \( M > 0, N > 0 \) and
\[
d\mu_x(x) = \frac{\Gamma(2\alpha + 2)}{2^{2\alpha+1} \Gamma^2(\alpha + 1)} (1-x^2)^\alpha \, dx \quad (x > -1)
\]
a probability Gegenbauer measure.
Define the Gegenbauer polynomial $R_n^{(2)}(x)$ by the Rodrigues formula

$$R_n^{(2)}(x) = \frac{(-1)^n \Gamma(n + 1)}{2^n \Gamma(n + 2x + 1)} (1 - x^2)^{-x}((1 - x^2)^{n + x})^{(n)} (x > -1).$$

Then

$$\int_{-1}^{1} R_m^{(2)}(x) R_n^{(2)}(x) (1 - x^2)^x \, dx = 0 \quad (m \neq n; m, n \in \mathbb{Z}_+),$$

moreover $R_n^{(2)}(1) = 1(n \in \mathbb{Z}_+)$. 

**Lemma 2.1 (Szegö [33]).** For polynomial coefficients

$$R_n^{(2)}(x) = k(R_n^{(2)}) x^n + l(R_n^{(2)}) x^{n-2} + m(R_n^{(2)}) x^{n-4} + \ldots \quad (2.2)$$

the following formulas are valid:

$$k(R_n^{(2)}) = \frac{\Gamma(n + 1) \Gamma(2n + 2x + 1)}{2^n \Gamma(n + 2x + 1) \Gamma(n + 2x + 3)} \quad (n \in \mathbb{Z}_+) \quad (2.3)$$

$$l(R_n^{(2)}) = -\frac{\Gamma(n + 1) (n - 1) n \Gamma(2n + 2x - 1)}{2^n \Gamma(n + 2x) \Gamma(n + 2x + 1)} \quad (n \geq 2) \quad (2.4)$$

and

$$m(R_n^{(2)}) = \frac{\Gamma(n + 1) \Gamma(2n + 2x - 3)(n - 3) n}{2^{n+1} \Gamma(n + x - 1) \Gamma(n + 2x + 1)} \quad (n \geq 4), \quad (2.5)$$

where $(c)_n$ is the shifted factorial defined by

$$(c)_n := c(c + 1) \ldots (c + n - 1) = \frac{\Gamma(n + c)}{\Gamma(c)} \quad (n = 1, 2, \ldots), \quad (c)_0 = 1.$$ 

**Corollary 2.2.**

\[
\begin{align*}
k(R_n^{(2)})^2 &= \frac{(n + 2x + 1)(n + 2x + 2)}{(2n + 2x + 1)(2n + 2x + 3)} = \frac{1}{4} \left[ 1 + \frac{2x + 1}{n} + O(n^{-2}) \right], \quad (2.6) \\
l(R_{n+2}^{(2)})^2 &= -\frac{n(n - 1)}{2(2n + 2x - 1)} \\
&= -\frac{n}{4} \left[ 1 - \frac{2x + 1}{2} + \frac{4x^2 - 1}{4} + \frac{1}{n^2} + \frac{(1 - 4x^2)(2x - 1)}{8} \frac{1}{n^3} + O(n^{-4}) \right], \quad (2.7) \\
m(R_n^{(2)})^2 &= \frac{1}{8} \frac{(n - 3) n^2}{(2n + 2x - 1)(2n + 2x - 3)} \\
&= \frac{n^2}{32} \left[ 1 - \frac{2(x + 2)}{n} + \frac{3(2x + 1)(2x + 3)}{4} \frac{1}{n^2} - \frac{8x^3 + 12x^2 - 2x - 3}{2} \frac{1}{n^3} + O(n^{-4}) \right]. \quad (2.8)
\end{align*}
\]
Bavinck and Meijer [3,4] have introduced polynomials \( \{ B_n^{(z)}(x) = B_n^{(z)}(x; M, N) \}(n \in \mathbb{Z}_+) \), orthogonal with respect to the inner product (2.1), as

\[
\langle B_m^{(z)}(x; M, N), B_n^{(z)}(x; M, N) \rangle = 0 \quad (m \neq n; m, n \in \mathbb{Z}_+)
\]

and have proved the following representation (see also [10]):

\[
B_n^{(z)}(x) = a_n (1 - x^2)^2 \{ R_n^{(z)}(x) \}^{(4)} + b_n (1 - x^2) \{ R_n^{(z)}(x) \}^{(2)} + c_n R_n^{(z)}(x),
\]

where

\[
a_n = MN \frac{1}{(x + 1)_3(x + 2)} \frac{(n - 1)n(n + x + 2)}{n + 2x + 1} (\delta_n^{(z)})^2
\]

\[
+ N \frac{1}{2(x + 1)_3} n\delta_n^{(z)} \quad (n \in \mathbb{Z}_+),
\]

\[
b_n = -N \frac{1}{2(x + 1)(x + 3)} (n - 2)n(n + 2x + 3)\delta_n^{(z)}
\]

\[
- M \frac{2n}{n + 2x + 1} \delta_n^{(z)} \quad (n \in \mathbb{Z}_+),
\]

\[
c_n = 1 - N \frac{1}{2(x + 1)_3} (n - 2)_3(n + 2x + 2)_2 \delta_n^{(z)}
\]

\[
(n = 3, 4, \ldots; c_0 = c_1 = c_2 = 1),
\]

\[
\delta_n^{(z)} = \frac{\Gamma(n + 2x + 2)}{\Gamma(2x + 3)n!} \quad (n \in \mathbb{Z}_+).
\]

Let \( \{ \hat{B}_n^{(z)}(x) = \hat{B}_n^{(z)}(x; M, N) \}(x \in [-1, 1], n \in \mathbb{Z}_+) \) be the corresponding sequence of orthonormal polynomials (symmetric Gegenbauer–Sobolev orthonormal polynomials):

\[
\frac{\Gamma(2x + 2)}{2^{2x+1} \Gamma^2(x + 1)} \int_{-1}^{1} \hat{B}_m^{(z)}(x) \hat{B}_n^{(z)}(x)(1 - x^2)^x \, dx
\]

\[
+ M[\hat{B}_m^{(z)}(1) \hat{B}_n^{(z)}(1) + \hat{B}_m^{(z)}(-1) \hat{B}_n^{(z)}(-1)]
\]

\[
+N[\{ \hat{B}_m^{(z)} \}'(1)\{ \hat{B}_n^{(z)} \}'(1) + \{ \hat{B}_m^{(z)} \}'(-1)\{ \hat{B}_n^{(z)} \}'(-1)] = \delta_{m,n}
\]

\((m, n \in \mathbb{Z}_+; M > 0, N > 0)\).

Then

\[
\hat{B}_n^{(z)}(x) = \hat{\lambda}_n^{(z)} B_n^{(z)}(x),
\]

where

\[
(\hat{\lambda}_n^{(z)})^2 = (\lambda_n^{(z)})^2 = \frac{\Gamma(2x + 2)}{2^{2x+1} \Gamma^2(x + 1)} \int_{-1}^{1} [B_n^{(z)}(x)]^2 (1 - x^2)^x \, dx
\]

\[
+ 2M[B_n^{(z)}(1)]^2 + 2N[\{ B_n^{(z)} \}'(1)]^2.
\]
Lemma 2.3 (Osilenker [28, 29], (z = 0, M = 0), (N = 0)). The following representation is valid:

\[(\lambda_n^{(z)})^{-2} = (B_n^{(z)}(x; M, N), B_n^{(z)}(x; M, N))\]

\[= \frac{\Gamma(2x + 2)}{2n + 2x + 1 \Gamma(n + 2x + 1)} \frac{n!}{\Gamma(n + 2x + 1)^2} O_n^{(z)}(M, N) O_n^{(z)}(M, N) \quad (n \in \mathbb{Z}_+); \tag{2.15}\]

in addition

\[O_n^{(z)}(M, N) = MN \zeta_n^{(z)} + N \nu_n^{(z)} + M \theta_n^{(z)} + 1 \quad (n \in \mathbb{Z}_+), \tag{2.16}\]

where

(i)

\[\zeta_n^{(z)} = \frac{1}{(x + 1)3(x + 2)\Gamma^2(2x + 3)} \frac{(n - 3)(n - 2)}{\Gamma(n + 2x + 1)\Gamma(n + 2x + 3)} \times \frac{1}{[(n - 2)!]^2}, \tag{2.17}\]

(ii)

\[\nu_n^{(z)} = \frac{\Gamma(n + 2x + 2)}{2(x + 1)3\Gamma(2x + 3)(n - 3)} \frac{[(x + 2)n^2 + (x + 2)(2x - 1)n}{-2(x + 1)(2x + 3)}, \tag{2.18}\]

(iii)

\[\theta_n^{(z)} = \frac{2}{\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 1)}{(n - 2)!}. \tag{2.19}\]

Proof. First, we consider case \(n = 4, 5, \ldots, \). By formula (3.9) from [10], one has

\[(\lambda_n^{(z)})^{-2} = 2M c_n^2 + 2N \left[ \frac{n(n + 2x + 1)}{2(x + 1)} + M \left( \frac{n + 2x + 2}{(x + 1)(x + 2)} \delta_n^{(z)} \right) \right]^2 + \frac{1}{2(x + 1)2n + 2x + 1} \frac{1}{\delta_n^{(z)}} [(n - 3)4(n + 2x + 1)4a_n^2 + n(n - 1)(n + 2x + 1)(n + 2x + 2)b_n^2 - 2(n - 3)4(n + 2x + 1)(n + 2x + 2)a_n b_n + c_n^2 - 2(n - 1)nb_n c_n + 2(n - 3)4a_n c_n], \tag{2.20}\]

where \(a_n, b_n, \) and \(c_n\) are defined by (2.10)–(2.13). Since (see (2.13))

\[\frac{1}{2(x + 1)} \frac{n + 2x + 1}{2n + 2x + 1} \frac{1}{\delta_n^{(z)}} = \frac{\Gamma(2x + 2)}{\Gamma(n + 2x + 1)} \frac{\Gamma(n + 1)}{\Gamma(n + 2x + 1)}, \]

we get

\[(\lambda_n^{(z)})^{-2} = \frac{\Gamma(2x + 2)}{2n + 2x + 1} \frac{\Gamma(n + 1)}{\Gamma(n + 2x + 1)} [N^2[A_n M^2 + B_n M + C_n] + N[D_n M^2 + E_n M + F_n] + K_n M^2 + L_n M + R_n]. \tag{2.21}\]

Our aim is to find all coefficients in (2.21).
Taking into account (2.17) and (2.18), from (2.22) to (2.24), one gets
\[
A_n = \frac{[\delta_n^{(x)}]^4}{(x+1)^3(x+2)^2} \frac{(n-3)_4(n-1)^2n^2(n+2x+2)^2}{n+2x+1}, \tag{2.22}
\]
\[
B_n = \frac{[\delta_n^{(x)}]^3}{2(x+1)^2(x+2)} \frac{1}{n+2x+1} (n-2)(n-1)^2n^2(n+2x+2)^2(n+2x+3)
\times[(x+2)n^4 + 2(x+2)(2x+1)n^3 + (6x^3 + 15x^2 - 14)n^2
+ (4x^4 + 8x^3 - 15x^2 - 41x - 16)n
-2(x+1)(x+2)(2x+1)(2x+3)] \tag{2.23}
\]
and
\[
C_n = \frac{[\delta_n^{(x)}]^2}{4(x+1)^2} (n-2)(n-1)n^2(n+2x+2)(n+2x+3)
\times[(x+2)n^4 + 2(x+2)(2x+1)n^3 + (x+2)(4x^3 + 8x^2 - 5x - 14)n^2
+ 2(x+2)(4x^3 + 16x^2 + 23x + 8)n + 4(x+1)(2x+3)]. \tag{2.24}
\]
Taking into account (2.17) and (2.18), from (2.22) to (2.24), one gets
\[
A_n = \zeta_n^{(x)} \zeta_n^{(x)} n_{n+2},
\]
\[
B_n = \zeta_n^{(x)} v_{n+2}^{(x)} + \zeta_n^{(x)} v_n^{(x)} n_{n+2},
\]
\[
C_n = v_n^{(x)} v_{n+2}^{(x)}.
\]
So expression at $N^2$ is the following:
\[
N^2 \left\{ \zeta_n^{(x)} \zeta_n^{(x)} n_{n+2} M^2 + \left[ \zeta_n^{(x)} v_n^{(x)} n_{n+2} + \zeta_n^{(x)} v_n^{(x)} n_{n+2} \right] M + v_n^{(x)} v_{n+2} \right\}. \tag{2.25}
\]
Find coefficient at $N$. One gets from (2.20) that
\[
D_n = [\delta_n^{(x)}]^3 \left[ \frac{4}{(x+1)(x+2)^2} \frac{(2n+2x+1)(n+2x+2)^2n^2(n-1)^2}{n+2x+1}
+ \frac{4}{(x+1)^3(x+2)} (n-3)(n-2)(n-1)^2n^2(n+2x+2)^2 \right]
\frac{4[\delta_n^{(x)}]^3}{(x+1)^2(x+2)} \frac{(n-1)^2n^2(n+2x+2)^2}{n+2x+1} [n^2 + (2x+1)n + (2x^2 + 7x + 9)].
\]
By definitions (2.17) and (2.19), one has
\[
D_n = \zeta_n^{(x)} \zeta_n^{(x)} n_{n+2} + \zeta_n^{(x)} \zeta_n^{(x)} n_{n+2}. \tag{2.26}
\]
By a similar way
\[
E_n = \frac{2[\delta_n^{(x)}]^2}{(x+1)^2(x+2)} \frac{n(n-1)(n+2x+2)}{n+2x+1} [(x^2 + 4x + 5)n^4
+ 2(2x^3 + 9x^2 + 14x + 5)n^3 + (4x^4 + 16x^3 + 31x^2 + 40x + 31)n^2
+ (-8x^4 + 16x^3 + 26x^2 + 68x + 26)n + 4(x+1)(x+2)(2x+1)(2x+3)],
\]
If we substitute $a_n, b_n, c_n$ (see formulas (2.10)–(2.13)) in (2.20), then we obtain the coefficients in (2.21):
and by (2.17)–(2.19), one obtains
\[ E_n = v_n^{(2)} \theta_{n+2}^{(2)} + v_{n+2}^{(2)} \theta_n^{(2)} + \zeta_n^{(z)} + \zeta_{n+2}^{(z)}. \]  
\[ (2.27) \]
As before, from (2.20), we get the following:
\[ F_n = \frac{\delta_n^{(2)}}{(x + 1)^3} \left[ (x + 2)(x + 3)n(2n + 2x + 1)(n + 2x + 1) \right. 
\- (n - 2)(n - 1)(2n + 2x + 2)(2n + 2x + 3) 
\+ (x + 2)(n - 2)(n + 2x + 3) + (n - 3)_4], 
\]
and by straightforward calculation, we obtain
\[ F_n = \frac{1}{(x + 1)^3} n \delta_n^{(2)} [(x + 2)n^4 + 2(x + 2)(2x + 1)n^3 
\+ (6x^2 + 3x^2 + 34x + 22)n^2 + (2x + 1)(2x^3 + 11x^2 + 25x + 20)n 
\- 4(x + 1)(2x + 3)]. 
\]
Using notation (2.18), one gets
\[ F_n = v_n^{(2)} + v_{n+2}^{(2)}. \]  
\[ (2.28) \]
Taking into account (2.26)–(2.28), we get the following representation for the expression in (2.21) containing \( N \):
\[ N \left\{ \left[ \frac{\delta_n^{(z)}}{\delta_n^{(z)}} \theta_{n+2}^{(z)} + \zeta_n^{(z)} \theta_n^{(z)} \right] M^2 + \left[ v_n^{(z)} \theta_{n+2}^{(z)} \right. 
\- v_{n+2}^{(z)} \theta_n^{(z)} + \zeta_n^{(z)} \theta_n^{(z)} \right] M 
\- \left[ v_n^{(z)} + v_{n+2}^{(z)} \right] M \right\} . \]  
\[ (2.29) \]
We find the remaining coefficients in (2.21). Using (2.19) from (2.20), one obtains
\[ K_n = \frac{4}{\Gamma^2(2x + 3)} \frac{\Gamma(n + 2x + 1)}{\Gamma(n - 1)} \frac{\Gamma(n + 2x + 3)}{\Gamma(n + 1)} = \theta_n^{(z)} \theta_{n+2}^{(z)} 
\]  
and
\[ L_n = \frac{4}{\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 1)}{n + 1} \frac{\Gamma(n + 2x + 3)}{\Gamma(n + 1)} \frac{[n^2 + (2x + 1)n + (x + 1)(2x + 1)]}{n + 1}, \]
\[ R_n = 1. \]
By the last three formulas, we get the following representation for the expression in (2.21), which does not contain \( N \):
\[ \theta_n^{(z)} \theta_{n+2}^{(z)} M^2 + \left[ \theta_n^{(z)} + \theta_{n+2}^{(z)} \right] M + 1. \]  
\[ (2.30) \]
Finally, substituting expressions (2.25), (2.29) and (2.30) in (2.21), one obtains
\[ \left( \lambda_n^{(z)} \right)^{-2} = \frac{\Gamma(2x + 2)}{2n + 2x + 1} \frac{\Gamma(n + 1)}{\Gamma(n + 2x + 1)} \left\{ \left[ \zeta_n^{(z)} \theta_n^{(z)} \right] \right. 
\- v_n^{(z)} \theta_{n+2}^{(z)} \right] M^2 + \left[ \zeta_n^{(z)} \theta_n^{(z)} + \zeta_{n+2}^{(z)} \theta_{n+2}^{(z)} \right] M 
\- v_n^{(z)} \theta_{n+2}^{(z)} + v_{n+2}^{(z)} \theta_{n+2}^{(z)} \right] M 
\- \left[ \zeta_n^{(z)} + \zeta_{n+2}^{(z)} \right] M + v_n^{(z)} + v_{n+2}^{(z)} \right] N 
\+ \theta_n^{(z)} \theta_{n+2}^{(z)} M^2 + \left( \theta_n^{(z)} + \theta_{n+2}^{(z)} \right) M + 1, \]
and this coincides with (2.15) and (2.16). We prove Lemma 2.3 in the case \( n = 4, 5, \ldots \) Now we prove formulas (2.15)–(2.19) in the case, \( n = 0, 1, 2, 3 \). In fact, it follows from (2.16) to (2.19) that

\[
\Omega_0^{(2)}(M, N) = \Omega_1^{(2)}(M, N) = 1; \quad \Omega_2^{(2)}(M, N) = 2M + 1,
\]

\[
\Omega_3^{(2)}(M, N) = 2(2x + 3)(M + N) + 1,
\]

\[
\Omega_4^{(2)}(M, N) = \frac{4(2x + 3)^2(2x + 5)}{\alpha + 1} MN + \frac{2(2x + 5)(2x + 3)^2}{\alpha + 1} N
\]
\[
+ 2(2x + 2)(2x + 3)M + 1,
\]

\[
\Omega_5^{(2)}(M, N) = \frac{4}{3} \frac{(2x + 3)^2(2x + 5)^2(2x + 7)}{\alpha + 1} MN
\]
\[
+ \frac{\Gamma(2x + 7)}{2(2x + 1)\Gamma(2x + 3)} (3x^2 + 15x + 17)N + \frac{1}{3} \frac{\Gamma(2x + 6)}{\Gamma(2x + 3)} M + 1.
\]

On the other hand, it follows from (2.9) to (2.13) that

\[
B_0^{(2)}(x) = 1, \quad B_1^{(2)}(x) = x,
\]

\[
B_2^{(2)}(x) = \frac{1}{2(\alpha + 1)} \left\{ (2x + 3)(2M + 1)x^2 - [2(2x + 3)M + 1] \right\},
\]

\[
B_3^{(2)}(x) = \left[ \frac{(2x + 3)(2x + 5)}{\alpha + 1} M + \frac{(2x + 3)(2x + 5)}{\alpha + 1} N + \frac{2x + 5}{2(\alpha + 1)} \right] x^3
\]
\[
- \left[ \frac{(2x + 3)(2x + 5)}{\alpha + 1} M + \frac{3(2x + 3)(2x + 5)}{\alpha + 1} N + \frac{3}{2(\alpha + 1)} \right] x.
\]

By definition of inner product (2.1), one has

\[
(B_0^{(2)}(x; M, N), B_0^{(2)}(x; M, N)) = 2M + 1 = \Omega_0^{(2)}(M, N)\Omega_2^{(2)}(M, N),
\]

\[
(B_1^{(2)}(x; M, N), B_1^{(2)}(x; M, N)) = 2(M + N) + \frac{1}{2x + 3}
\]
\[
= \frac{1}{2x + 3} \Omega_1^{(2)}(M, N)\Omega_3^{(2)}(M, N),
\]

\[
(B_2^{(2)}(x; M, N), B_2^{(2)}(x; M, N))
\]
\[
= \frac{8(2x + 3)^2}{(\alpha + 1)^2} (M^2N + MN) + \frac{2(2x + 3)^2}{(\alpha + 1)^2} N + \frac{4(\alpha + 2)(2x + 3)}{(\alpha + 1)(2x + 5)} M^2
\]
\[
+ \frac{2(2x^2 + 7x + 7)}{(\alpha + 1)(2x + 5)} M + \frac{1}{(\alpha + 1)(2x + 5)}
\]
\[
= \frac{2\Gamma(2x + 2)}{(2x + 5)\Gamma(2x + 3)} \Omega_2^{(2)}(M, N)\Omega_4^{(2)}(M, N),
\]
and
\[
\langle B_3^{(2)}(x; M, N), B_3^{(2)}(x; M, N) \rangle \\
= \frac{8(2x + 3)^2(2x + 5)^2}{(x + 1)^2} (MN^2 + M^2N) \\
+ \frac{8(2x + 3)(2x + 5)(7x^2 + 36x + 44)}{(x + 1)^2(2x + 7)} MN + \frac{4(x + 2)(2x + 3)(2x + 5)}{(x + 1)(2x + 7)} M^2 \\
+ \frac{6}{(x + 1)^2(2x + 7)} (6x^3 + 45x^2 + 110x + 86)N \\
+ \frac{2(2x^2 + 9x + 13)}{(x + 1)(2x + 7)} M + \frac{3}{6\Gamma(2x + 2)} \\
(2x + 7)\Gamma(2x + 5) \Omega_3^{(2)}(M, N)\Omega_3^{(2)}(M, N).
\]

Lemma 2.3 is completely proved. \(\square\)

**Lemma 2.4.** For coefficients of polynomial \(B_n^{(2)}(x) = B_n^{(2)}(x; M, N)(n \in \mathbb{Z}_+)\)
\[
B_n^{(2)}(x) = k(B_n^{(2)})x^n + l(B_n^{(2)})x^{n-2} + m(B_n^{(2)})x^{n-4} + \cdots
\]
the following representations are valid:

(i)
\[
k(B_n^{(2)}) = \Omega_n^{(2)}(M, N)k(R_n^{(2)}) \quad (n \in \mathbb{Z}_+),
\]

(ii)
\[
l(B_n^{(2)}) = \left\{ \begin{array}{l}
\Omega_n^{(2)}(M, N) + MN \frac{8}{(x + 1)\Gamma(2x + 3)} (n - 3)(n - 2) \\
x(n + 2x + 1)\Gamma(n + 2x + 3) \\
\times n!(n - 2)!
\end{array} \right. + N \frac{2(n - 2)(n + 2x - 1)}{(x + 1)\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 2)}{(n - 1)!} \\
+ M \frac{4}{\Gamma(2x + 2)} \frac{\Gamma(n + 2x + 1)}{n!} \right\} l(R_n^{(2)}) \quad (n \geq 2),
\]

and

(iii)
\[
m(B_n^{(2)}) = \left\{ \begin{array}{l}
\Omega_n^{(2)}(M, N) + MN \frac{16}{(x + 1)\Gamma(2x + 3)} \\
\times (n + 2x + 1)\Gamma(n + 2x + 3) \\
\times [n^2 - 5n + 2(x + 6)] (n - 2)!
\end{array} \right. \\
+ N \frac{4\Gamma(n + 2x + 2)}{(x + 1)\Gamma(2x + 3)(n - 1)!} [n^2 + (2x - 3)n - 2(2x - 3)] \\
+ M \frac{8}{\Gamma(2x + 2)} \frac{\Gamma(n + 2x + 1)}{n!} \right\} m(R_n^{(2)}) \quad (n \geq 4),
\]

where \(k(R_n^{(2)}), l(R_n^{(2)}), m(R_n^{(2)}), \Omega_n^{(2)}(M, N)\) are defined in (2.3)–(2.5) and (2.16)–(2.19).
Proof. We substitute (2.2) and (2.31) in (2.9)
\[ k(B_n^{(x)}) x^n + l(B_n^{(x)}) x^{n-2} + m(B_n^{(x)}) x^{n-4} + \cdots \]
\[ = a_n (x^4 - 2x^2 + 1) [(n-3)_4 k(R_n^{(x)}) x^{n-4} + (n-5)_4 l(R_n^{(x)}) x^{n-6} \]
\[ + (n-7)_4 m(R_n^{(x)}) x^{n-8} + \cdots ] \]
\[ + b_n (1 - x^2) [(n-1)n k(R_n^{(x)}) x^{n-2} + (n-3)(n-2) l(R_n^{(x)}) x^{n-4} \]
\[ + (n-5)(n-4) m(R_n^{(x)}) x^{n-6} + \cdots ] \]
\[ + c_n [k(R_n^{(x)}) x^n + l(R_n^{(x)}) x^{n-2} + m(R_n^{(x)}) x^{n-4} + \cdots ]. \] (2.35)

The comparison of the coefficients of \( x^n \) in the above relation yields
\[ k(B_n^{(x)}) = [(n-3)_4 a_n - (n-1)n b_n + c_n] k(R_n^{(x)}) \]
and substituting \( a_n, b_n, c_n \) for (2.10)–(2.13), we get
\[ (n-3)_4 a_n - (n-1)n b_n + c_n = M N \frac{1}{(x+1)_3(x+2)} \frac{(n-3)_4 \Gamma(n+2x+1) \Gamma(n+2x+3)}{(n-2)! n!} \]
\[ + N \frac{1}{2(x+1)_3 \Gamma(2x+3)} \frac{(n-3)_4 \Gamma(n+2x+2)}{(n-1)!} \]
\[ + N \frac{1}{2(x+1)(x+3) \Gamma(2x+3)} \frac{(n-1)n(n-2)(n+2x+3) \Gamma(n+2x+2)}{(n-1)!} \]
\[ - N \frac{1}{2(x+1)_3 \Gamma(2x+3)} \frac{\Gamma(n+2x+4)}{(n-3)!} \]
\[ + M \frac{2}{\Gamma(2x+3)} \frac{(n-1)n \Gamma(n+2x+1)}{n!} + 1, \]
It follows from (2.16) to (2.19) that
\[ (n-3)_4 a_n - (n-1)n b_n + c_n = Q_n^{(x)}(M, N), \] (2.36)
and we get relation (2.32).

By (2.3) and (2.4) it is easily to see that
\[ k(R_n^{(x)}) = - \frac{2(2n+2x-1)}{(n-1)n} l(R_n^{(x)}). \]

Thus, comparing the coefficients at \( x^{n-2} (n \geq 2) \) in relation (2.35), one obtains
\[ l(B_n^{(x)}) = a_n [(n-5)_4 l(R_n^{(x)}) - 2(n-3)_4 k(R_n^{(x)})] \]
\[ + b_n [(n-1)n k(R_n^{(x)}) - (n-3)(n-2) l(R_n^{(x)})] + c_n l(R_n^{(x)}) \]
\[ = l(R_n^{(x)}) [(n-5)_4 + 4(n-3)(n-2)(2n+2x-1)] a_n \]
\[ - [ (n-3)(n-2) + 2(2n+2x-1) ] b_n + c_n \]
\[ = [[(n-3)_4 a_n - (n-1)n b_n + c_n] + 8(x+2)(n-3)(n-2)a_n \]
\[ - 4(x+1)b_n] l(R_n^{(x)}). \]
By (2.36), we get
\[
I(B_n^{(x)}) = [\Omega_n^{(x)}(M, N) + 8(x+2)(n-3)(n-2)a_n - 4(x+1)b_n]l(R_n^{(x)}).
\]  
(2.37)

A consequence of (2.10), (2.11) and (2.13) is the following:
\[
8(x+2)(n-3)(n-2)a_n - 4(x+1)b_n
= MN (n-3)(n-2) \frac{\Gamma(n+2x+1)\Gamma(n+2x+3)}{(x+1)^3\Gamma^2(2x+3)} (n-2)!n! \frac{2(2x+1)(n-2)(n+2x-1)}{(x+1)^3\Gamma(2x+3)} (n-1)! \frac{8(x+1)\Gamma(n+2x+1)}{n!}.
\]

Substituting this expression in (2.37) we obtain (2.33).

For finding the coefficient \(m(B_n^{(x)})\), we compare coefficients at \(x^{n-4}(n \geq 4)\) in (2.35):
\[
m(B_n^{(x)}) = (n-7)_4a_n m(R_n^{(x)}) - 2(n-5)_4a_n l(R_n^{(x)}) + (n-3)_4a_n k(R_n^{(x)})
\]
\[
+ (n-3)(n-2)b_n l(R_n^{(x)}) - (n-5)(n-4)b_n m(R_n^{(x)}) + c_n m(R_n^{(x)}).
\]

Note by definition \((n-7)_4 = 0\) \((n = 4, 5, 6, 7)\) and \((n-5)_4 = 0\) \((n = 4, 5)\).

From (2.3) to (2.5), one has
\[
\frac{k(R_n^{(x)})}{m(R_n^{(x)})} = \frac{8(2n+2x-1)(2n+2x-3)}{(n-3)_4},
\]
\[
\frac{l(R_n^{(x)})}{m(R_n^{(x)})} = \frac{4(2n+2x-3)}{(n-3)(n-2)} \quad (n \geq 4).
\]

Consequently,
\[
m(B_n^{(x)}) = \{(n-7)_4 + 8(2n+2x-3)(n^2 - 7n + 2x + 19)]a_n
\]
\[
- [n^2 - n + 8(x+1)]b_n + c_n \} m(R_n^{(x)}).
\]  
(2.38)

We calculate the expression in brackets. As above, by (2.10)–(2.13) and (2.36), one gets
\[
(n-7)_4 + 8(2n+2x-3)(n^2 - 7n + 2x + 19)a_n - [n^2 - n + 8(x+1)]b_n + c_n
\]
\[
= [(n-3)_4a_n - (n-1)n b_n + c_n] + 16(x+2)[n^2 - 5n + 2(x+6)]a_n - 8(x+1)b_n
\]
\[
= \Omega_n^{(x)}(M, N) + 16MN(x+2) \frac{n^2 - 5n + 2(x+6)}{(x+1)_3(x+2)\Gamma^2(2x+3)}
\]
\[
\times \frac{\Gamma(n+2x+1)\Gamma(n+2x+3)}{(n-2)!n!} \frac{1}{2(x+1)_3\Gamma(2x+3)} \frac{n^2 - 5n + 2(x+6)}{(n-2)(n+2x+3)\Gamma(n+2x+2)} (n-1)! \frac{8(x+1)\Gamma(n+2x+1)}{n!}.
\]
From the last relation, we deduce
\[
(n - 7)^4 + 8(2n + 2x - 3)(n^2 - 7n + 2x + 19)a_n - [n^2 - n + 8(x + 1)]b_n + c_n
\]
\[
= \Omega_n^{(2)}(M, N) + MN \frac{16}{(x + 1)^3 \Gamma^2(2x + 3)} [n^2 - 5n + 2(x + 6)]
\]
\[
\times \frac{\Gamma(n + 2x + 1)\Gamma(n + 2x + 3)}{(n - 2)!n!} + N \frac{4}{(x + 1)\Gamma(2x + 3)}
\]
\[
\times [n^2 + (2x - 3)n - 2(2x - 3)] \frac{\Gamma(n + 2x + 2)}{(n - 1)!}
\]
\[
+ M \frac{16(x + 1)}{\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 1)}{n!},
\]
Substituting this relation in (2.38) one obtains (2.34). Lemma 2.4 is completely proved. □

In the next section we use the following assertion.

Lemma 2.5 (Abramowitz and Stegun [1], Copson [6], Tricomi and Erdelyi [34]). For \( n \) large enough, the following asymptotic equality:
\[
\frac{\Gamma(n + z)}{\Gamma(n + \beta)} = n^{z-\beta} \left[ 1 + \frac{(z - \beta)(z + \beta - 1)}{2} \frac{1}{n}
\right.
\]
\[
\left. + \frac{1}{24} (z - \beta)(z - \beta - 1) \frac{3(z + \beta - 1)^2 - z + \beta - 1}{n^2}
\right]
\]
\[
+ O \left( \frac{1}{n^3} \right)
\]
holds. (2.39)

Using representation (2.9)–(2.13) and Lemmas 2.3 and 2.5 one can get another proof of the following proposition.

Lemma 2.6 (Marcellan and Osilenker [21], Foulquie Moreno et al. [10]). (case \( M > 0, N > 0 \)). The following properties for symmetric Gegenbauer–Sobolev polynomials hold:

(i) \( \hat{\theta}^{(n)}(n) \sim n^{-3x-\frac{15}{2}} \),

(ii) \( |\hat{B}_n^{(x)}(x)| \leq C_x (1 - x^2)^{-\frac{3}{2} - \frac{1}{4}}, x \geq -\frac{1}{2} \),

\( |\hat{B}_n^{(x)}(x)| \leq C_x \left( -1 < x < -\frac{1}{2} \right) (n \in \mathbb{Z}_+, -1 < x < 1), \)

(iii) \( \max_{-1 \leq x \leq 1} |\hat{B}_n^{(x)}(x)| \leq C_x (n + 1)^\frac{1}{2} (n \in \mathbb{Z}_+) \),

(iv) \( |\hat{B}_n^{(x)}(1)| \leq n^{-x-\frac{3}{2}}, |[\hat{B}_n^{(x)}]'(1)| \leq n^{-x-\frac{7}{2}} \),
where the constants $C_x$, are independent of $x \in (-1, 1)$, and $n \in \mathbb{Z}_+$ (in formula ii) and $n \in \mathbb{Z}_+$ (in formula iii).

**Remark.** It should be noted that the Gegenbauer–Sobolev orthonormal polynomials $\hat{B}_n^{(2)}(x)$ have some properties other than the corresponding Gegenbauer orthonormal polynomials $R_n^{(2)}(x)$ (see, for example [2–5,9,10,15,17,18,20,21,28–31]). We mention the following properties only.

1. The inner product (2.1) cannot be obtained by a weight function since $\langle x, x \rangle \neq (1, x^2)$.

   This implies that well-known results for classical orthogonal polynomials, depending on the existence of a weight function do not longer hold.

   The well-known three-term recurrence relation for orthogonal polynomials does not hold in this case, but has to be replaced by a seven-term relation [5], in addition, this order is minimal [9].

   For $n$ large enough, there exists exactly one pair of real zeros $-\rho_n$ and $\rho_n$ of $\hat{B}_n^{(2)}(x)$ lying outside $(-1, 1)$ [2,5].

2. Polynomials $\hat{B}_n^{(2)}(x)$ are eigenfunctions of a class of linear differential operators, usually of infinite order. In the case that $\alpha$ is a nonnegative integer, this class contains a differential operator of finite order. This order is $4\alpha + 10$ [17,3]).

   We recall that polynomials $\hat{R}_n^{(2)}(x)$ are eigenfunctions of differential operator of second order [33].

3. As well-known [33]

   $$\hat{R}_n^{(2)}(\pm 1) \asymp n^{\frac{1}{2} + \frac{1}{2}}, \{\hat{R}_n^{2}\}'(\pm 1) \asymp n^{2 + \frac{1}{2}}.$$  

   These asymptotics are different from (iv).

### 3. Asymptotics of the averaged Turan determinant and generalized trace formula for symmetric Gegenbauer–Sobolev orthogonal polynomials

As known [5], symmetric Gegenbauer–Sobolev orthonormal polynomials satisfy a nine-term recurrence relation (note that a degree $N$ in (1.2) is even)

\begin{align*}
(x^2 - 1)^2 \hat{B}_n^{(2)}(x) &= \varepsilon_n^{(2)} \hat{B}_{n+4}^{(2)}(x) + \beta_n^{(2)} \hat{B}_{n+2}^{(2)}(x) + 2 \gamma_n^{(2)} \hat{B}_n^{(2)}(x) + \beta_n^{(2)} \hat{B}_{n-2}^{(2)}(x) \\
&+ \varepsilon_n^{(2)} \hat{B}_{n-4}^{(2)}(x) (n \in \mathbb{Z}_+; \hat{B}_{-s}^{(2)}(x) = 0, s = 1, 2, \ldots; \varepsilon_n^{(2)} = 0) \tag{3.1}
\end{align*}

Our main aim is to calculate the recurrence coefficients and to prove that they are bounded variation.

Let

\begin{align*}
\hat{B}_n^{(2)}(x) &= k(\hat{B}_n^{(2)})x^n + l(\hat{B}_n^{(2)})x^{n-2} + m(\hat{B}_n^{(2)})x^{n-4} + \cdots. \tag{3.2}
\end{align*}

By (2.14) and (2.31) one gets

\begin{align*}
k(\hat{B}_n^{(2)}) &= \lambda_n^{(n)} k(B_n^{(2)}), l(\hat{B}_n^{(2)}) = \lambda_n^{(n)} l(B_n^{(2)}), m(\hat{B}_n^{(2)}) = \lambda_n^{(n)} m(B_n^{(2)}), \tag{3.3}
\end{align*}

where $\lambda_n^{(n)}$ are defined in Lemma 2.3.
Comparing coefficients at $x^{n+4}$, $x^{n+2}$, $x^n$ on both sides of (3.1) and using (3.2), (3.3), we obtain the following.

**Lemma 3.1.** For the recursion coefficients of the symmetric band matrix associated with symmetric Gegenbauer–Sobolev orthonormal polynomials $\{\hat{B}_n^{(2)}\} (n \in \mathbb{Z}_+)$, the following representations are valid:

(i) $$\tilde{\epsilon}_{n+4}^{(2)} = \frac{\gamma_n^{(n)}}{\tilde{\lambda}_{n+4}^{(n+4)}} \frac{k(B_n^{(2)})}{k(B_{n+4}^{(2)})}, \quad (3.4)$$

(ii) $$\tilde{\rho}_{n+2}^{(2)} = \frac{\gamma_n^{(n+2)}}{\tilde{\lambda}_{n+2}^{(n+2)}} \frac{k(B_n^{(2)})}{k(B_{n+2}^{(2)})} \left[ \frac{l(B_n^{(2)})}{k(B_n^{(2)})} - \frac{l(B_{n+4}^{(2)})}{k(B_{n+4}^{(2)})} - 2 \right], \quad (3.5)$$

(iii) $$\tilde{\gamma}_n^{(2)} = \frac{m(B_n^{(2)})}{k(B_n^{(2)})} - \frac{m(B_{n+4}^{(2)})}{k(B_{n+4}^{(2)})} - 2 \frac{l(B_n^{(2)})}{k(B_n^{(2)})} + 1 - \frac{l(B_{n+4}^{(2)})}{k(B_{n+4}^{(2)})} \left[ \frac{l(B_n^{(2)})}{k(B_n^{(2)})} - \frac{l(B_{n+4}^{(2)})}{k(B_{n+4}^{(2)})} - 2 \right]. \quad (3.6)$$

**Lemma 3.2.** For the sequence $\Omega_n^{(2)}(M, N)$, defined by formulas (2.16)–(2.19), the following relations hold:

(i) $$\frac{\Omega_n^{(2)}(M, N)}{\Omega_{n+2}^{(2)}(M, N)} = 1 - \frac{8(z+2)}{n} + O \left( \frac{1}{n^2} \right), \quad (3.7)$$

(ii) $$\frac{\Omega_{n+2}^{(2)}(M, N)}{\Omega_n^{(2)}(M, N)} = 1 + \frac{8(z+2)}{n} + O \left( \frac{1}{n^2} \right), \quad (3.8)$$

(iii) $$\frac{\Omega_n^{(2)}(M, N)}{\Omega_{n+4}^{(2)}(M, N)} = 1 - \frac{16(z+2)}{n} + O \left( \frac{1}{n^2} \right), \quad (3.9)$$

(iv) $$\frac{\Omega_{n+4}^{(2)}(M, N)}{\Omega_n^{(2)}(M, N)} = 1 + \frac{16(z+2)}{n} + O \left( \frac{1}{n^2} \right) \quad (3.10)$$

hold.
We prove equality (3.8) only because the other ones can be deduced from (3.8). Taking into account (2.16), we find
\[
\frac{\Omega_{n+2}(M, N)}{\Omega_{n}^{(2)}(M, N)} = 1 + \frac{MN[\nu_{n+2} - \nu_{n}] + N[\nu_{n+2} - \nu_{n}] + M[\theta_{n+2} - \theta_{n}]}{MN\nu_{n}^{(2)} + N\nu_{n} + M\theta_{n} + 1}.
\] (3.11)

It is not difficult to see that from (2.17) to (2.19), we have
\[
\nu_{n+2} - \nu_{n} = \frac{(n - 1)n}{(x + 1)3(x + 2)\Gamma^2(2x + 3)}\frac{\Gamma(n + 2x + 1)\Gamma(n + 2x + 3)}{(n!)^2} \times [8(x + 2)n^3 + 12(2x^2 + 5x + 2)n^2 + 4(8x^3 + 30x^2 + 35x + 14)n + 4(x + 1)(x + 2)(2x + 1)(2x + 3)],
\]
and
\[
\theta_{n+2} - \theta_{n} = \frac{4(x + 1)}{\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 1)}{2n + 2x + 1} \frac{2n + 2x + 1}{n!} (2n + 2x + 1).
\]

Putting the last three relations in (3.11) and using (2.39) we obtain (3.8). This completes the proof of Lemma 3.2.

Lemma 3.3. For the ratio of the normalized multipliers, the following relations
\[
\frac{\lambda_{n}^{(n)}}{\lambda_{n+2}^{(n+2)}} = 1 + \frac{6x + 15}{n} + O \left( \frac{1}{n^2} \right), \quad \text{(3.12)}
\]
\[
\frac{\lambda_{n}^{(4n)}}{\lambda_{n+4}^{(n+4)}} = 1 + \frac{2(6x + 15)}{n} + O \left( \frac{1}{n^2} \right) \quad \text{(3.13)}
\]
hold.

Proof. We prove relation (3.12). By (2.15), one has
\[
\frac{\lambda_{n}^{(n)}}{\lambda_{n+2}^{(n+2)}} = \frac{\sqrt{2n + 2x + 1}}{2n + 2x + 5} \frac{(n + 1)(n + 2)}{(n + 2x + 1)(n + 2x + 2)} \sqrt{\frac{\Omega_{n+4}^{(2)}(M, N)}{\Omega_{n}^{(2)}(M, N)}} \quad \text{(n \in \mathbb{Z}^+)}.
\]
Since
\[
\sqrt{1 + x} = 1 + \frac{1}{2} x + O(x^2) (x \to 0), \quad \text{(3.14)}
\]
we see that
\[ \sqrt{\frac{2n + 2z + 1}{2n + 2z + 5}} = 1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right) \]
and by (3.10), (3.14)
\[ \sqrt{\frac{(n + 1)(n + 2)}{(n + 2z + 1)(n + 2z + 2)}} = 1 - \frac{2z}{n} + O\left(\frac{1}{n^2}\right) \]
we deduce (3.12). Relation (3.13) follows from (3.12). Lemma 3.3 is completely proved. □

**Lemma 3.4.** For the ratio of the leading coefficients of polynomial \( B_n^{(z)}(x) \) (see (2.31)), the following relations are valid:

(i) \[
\frac{l(B_n^{(z)})}{k(B_n^{(z)})} = \frac{l(R_n^{(z)})}{k(R_n^{(z)})} \left[ 1 + \frac{8(z + 2)}{(n - 1)n + T_n^{(z)}(M, N)} \right],
\]
where
\[
|T_n^{(z)}(M, N)| \leq C \frac{1}{n^{2z+4}},
\]
\[
|T_n^{(z)}(M, N) - T_{n+1}^{(z)}(M, N)| \leq C \frac{1}{n^{2z+5}}.
\]

(ii) \[
\frac{m(B_n^{(z)})}{k(B_n^{(z)})} = \frac{m(R_n^{(z)})}{k(R_n^{(z)})} \left\{ 1 + \frac{16(z + 2)[n^2 - 5n + 2(z + 6)]}{(n - 3)_4} + W_n^{(z)}(M, N) \right\}
\]
where
\[
|W_n^{(z)}(M, N)| \leq C \frac{1}{n^{2z+4}}, \quad |W_n^{(z)}(M, N) - W_{n+1}^{(z)}(M, N)| \leq C \frac{1}{n^{2z+5}}.
\]

The constants in (3.16) and (3.18) are independent of \( n \in \mathbb{Z}_+ \).

**Proof.** It follows from (2.32) and (2.33) that
\[
\frac{l(B_n^{(z)})}{k(B_n^{(z)})} = \frac{l(R_n^{(z)})}{k(R_n^{(z)})} \left[ 1 + \frac{\tilde{l}(B_n^{(z)})}{O_n^{(z)}(M, N)} \right],
\]
where
\[
\tilde{l}(B_n^{(z)}) = \frac{8MN(n - 3)(n - 2) \Gamma(n + 2z + 1) \Gamma(n + 2z + 3)}{(z + 1)_3 \Gamma^2(2z + 3) \Gamma(n - 2)!n!} + N \frac{2(n - 2)(n + 2z - 1) \Gamma(n + 2z + 2)}{(z + 1) \Gamma(2z + 3) \Gamma(n - 1)!} + \frac{4M}{\Gamma(2z + 2) n!} \Gamma(n + 2z + 1).
\]
and function $Q_{n}^{(2)}(M, N)$ is defined by (2.16)–(2.19). It is not difficult to see that

$$\frac{\tilde{I}(B_{n}^{(2)})}{\Omega_{n}^{(2)}(M, N)} = \frac{8(x + 2)}{(n - 1)n} + T_{n}^{(2)}(M, N),$$

where

$$T_{n}^{(2)}(M, N) = \frac{S_{n}^{(2)}(M, N)}{(n - 1)n\Omega_{n}^{(2)}(M, N)}$$

with

$$S_{n}^{(2)}(M, N) = \frac{2N(n - 2)(n - 1)}{(x + 1)(x + 3)\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 2)}{(n - 1)!} \times [(x + 1)n^2 + (2x^2 + x - 1)n + 4(x + 1)(2x + 3)]$$

$$- \frac{8M(x + 3)}{\Gamma(2x + 3)} \frac{\Gamma(n + 2x + 1)}{(n - 2)!} + 8(x + 2).$$

Then,

$$T_{n}^{(2)}(M, N) - T_{n+1}^{(2)}(M, N) = \frac{(n - 1)S_{n+1}^{(2)}(M, N)\Omega_{n}^{(2)}(M, N) - (n + 1)S_{n}^{(2)}(M, N)\Omega_{n+1}^{(2)}(M, N)}{(n - 1)^3\Omega_{n}^{(2)}(M, N)\Omega_{n+1}^{(2)}(M, N)}.$$

Using representation of $Q_{n}^{(2)}(M, N)$, $S_{n}^{(2)}(M, N)$ and relation (2.39), we get estimates (3.16). Taking into account (2.32) and (2.34), we obtain

$$\frac{m(B_{n}^{(2)})}{k(B_{n}^{(2)})} = \left[1 + \frac{\tilde{m}(B_{n}^{(2)})}{\Omega_{n}^{(2)}(M, N)}\right] \frac{m(R_{n}^{(2)})}{k(R_{n}^{(2)})},$$

where

$$\tilde{m}(B_{n}^{(2)}) = \frac{16MN}{(x + 1)\Gamma^2(2x + 3)} \frac{\Gamma(n + 2x + 1)\Gamma(n + 2x + 3)}{(n - 2)!n!} [n^2 - 5n + 2(x + 6)]$$

$$+ \frac{4N}{(x + 1)\Gamma(2x + 3)} [n^2 + (2x - 3)n - 2(2x - 3)] \frac{\Gamma(n + 2x + 2)}{(n - 1)!}$$

$$+ \frac{8M}{\Gamma(2x + 2)} \frac{\Gamma(n + 2x + 1)}{n!}.$$

Hence,

$$\frac{\tilde{m}(B_{n}^{(2)})}{\Omega_{n}^{(2)}(M, N)} = \frac{16(x + 2)[n^2 - 5n + 2(x + 6)]}{(n - 3)_4} + W_{n}^{(2)}(M, N),$$

where

$$W_{n}^{(2)}(M, N) = \frac{U_{n}^{(2)}(M, N)}{(n - 3)_4\Omega_{n}^{(2)}(M, N)}.$$
with
\[
U_n^{(2)}(M, N) = -\frac{4N(n-1)(n-2)}{(x+1)(x+3)\Gamma(2x+3)} \frac{\Gamma(n+2x+2)}{(n-1)!} [(x+1)n^4 + 2x(x-1)n^3 - (10x^2 + 3x + 25)n^2 + (8x^3 + 52x^2 + 116x + 156)n - (16x^3 + 112x^2 - 300x + 252)].
\]

Thus we have
\[
W_n^{(2)}(M, N) - W_{n+1}^{(2)}(M, N)
= \frac{(n+1)U_n^{(2)}(M, N)\Omega_{n+1}^{(2)}(M, N) - (n-3)U_{n+1}^{(2)}(M, N)\Omega_n^{(2)}(M, N)}{(n-3)\Omega_n^{(2)}(M, N)\Omega_{n+1}^{(2)}(M, N)},
\]
and application of definition of \(\Omega_n^{(2)}(M, N), U_n^{(2)}(M, N)\) and formula (2.39) yields estimates (3.18).

Lemma 3.4 is proved. □

Our main result of this section is the following

**Theorem 3.** For the recurrence coefficients \(\varepsilon_n^{(2)}, \beta_n^{(2)}, \gamma_n^{(2)}\) (see (3.1)) the following assertions are valid:

(i) \[
\lim_{n \to \infty} \varepsilon_n^{(2)} = \frac{1}{16}, \quad \lim_{n \to \infty} \beta_n^{(2)} = \frac{1}{4}, \quad \lim_{n \to \infty} \gamma_n^{(2)} = \frac{3}{8};
\]

(ii) The sequences \(\{\varepsilon_n^{(2)}\}, \{\beta_n^{(2)}\}, \{\gamma_n^{(2)}\}\) \((n \in \mathbb{Z}_+)\) are bounded variation
\[
\sum_{n=0}^{\infty} (|\Delta\varepsilon_n^{(2)}| + |\Delta\beta_n^{(2)}| + |\Delta\gamma_n^{(2)}|) < \infty.
\]

**Proof.** By (2.32)
\[
\frac{k(B_n^{(2)})}{k(B_{n+4}^{(2)})} = \frac{\Omega_n^{(2)}(M, N)}{\Omega_{n+4}^{(2)}(M, N)} \frac{k(R_n^{(2)})}{k(R_{n+4}^{(2)})} (n \in \mathbb{Z}_+).
\]
If we combine this with (2.6), (3.4), (3.9) and (3.13), then we get
\[
\varepsilon_{n+4}^{(2)} = \frac{1}{16} \left[ 1 + O \left( \frac{1}{n^2} \right) \right],
\]
and we prove our result for the sequence \(\{\varepsilon_n^{(2)}\}(n \in \mathbb{Z}_+)\).

From relations (2.7), (3.15) and (3.16), we obtain the following:
\[
\frac{l(B_n^{(2)})}{k(B_n^{(2)})} - \frac{l(B_{n+4}^{(2)})}{k(B_{n+4}^{(2)})} - 2 = -1 - \frac{4x^2 + 32x + 63}{4n^2} + O \left( \frac{1}{n^3} \right)
+ E_n^{(2)}(M, N),
\]
(3.19)
with
\[ |E_n^{(x)}(M, N)| \leq C_x \frac{1}{n^{2x+4}}, \tag{3.20} \]
where the constant \( C_x > 0 \) is independent of \( n \in \mathbb{Z}_+ \). Taking into account (2.6), (2.32) and (3.7), we obtain
\[
\frac{k(B_n^{(x)})}{k(B_{n+2}^{(x)})} = \frac{\Omega_n^{(x)}(M, N)}{\Omega_{n+2}^{(x)}(M, N)} \frac{k(R_n^{(x)})}{k(R_{n+2}^{(x)})} = \frac{1}{4} \left[ 1 - \frac{6x + 15}{n} + O \left( \frac{1}{n^2} \right) \right].
\]
Now, by relations (3.5) and (3.12), one gets
\[
\beta_{n+2}^{(x)} = -\frac{1}{4} \left[ 1 - \frac{6x + 5}{n} + O \left( \frac{1}{n^2} \right) \right] + \tilde{E}_n^{(x)}(M, N),
\]
where
\[ |\tilde{E}_n^{(x)}(M, N)| \leq C \frac{1}{n^{2x+4}}, \]
and we obtain the assertions of Theorem 3 for the sequence \( \{\beta_n^{(x)}\}(n \in \mathbb{Z}_+) \).

From (2.7), (3.15) and (3.16) we deduce the following:
\[
\frac{l(B_n^{(x)})}{k(B_{n+2}^{(x)})} = -\frac{1}{4} \left[ 1 + \frac{3 - 2x}{2n} + \frac{4x^2 + 32x + 63}{4n^2} + O \left( \frac{1}{n^3} \right) \right] + \tilde{E}_n^{(x)}(M, N),
\]
where
\[ |\tilde{E}_n^{(x)}(M, N)| \leq C \frac{1}{n^{2x+4}}, \]
and the constant \( C_x > 0 \) is independent of \( n \in \mathbb{Z}_+ \). Then, using (3.19), (3.20) and last two relations, one obtains
\[
\frac{1}{4} \left[ 1 + \frac{3 - 2x}{2n} + \frac{4x^2 + 32x + 63}{2n^2} + O \left( \frac{1}{n^3} \right) \right] + F_n^{(x)}(M, N),
\]
where
\[ |F_n^{(x)}(M, N)| \leq C \frac{1}{n^{2x+4}}, \]
and the constant \( C_x > 0 \) is independent of \( n \in \mathbb{Z}_+ \). Combining this with (3.15) and (3.16), we get
\[
1 - \frac{l(B_n^{(x)})}{k(B_{n+2}^{(x)})} \left[ \frac{l(B_n^{(x)})}{k(B_{n+4}^{(x)})} - B_n^{(x)} \right] - \frac{1}{k(B_{n+2}^{(x)})} \left[ \frac{B_n^{(x)}}{k(B_{n+4}^{(x)})} - 2 \right] - 2 \frac{l(B_n^{(x)})}{k(B_{n+2}^{(x)})} = \frac{n}{4} - \frac{2x - 3}{8} + O \left( \frac{1}{n^2} \right) + \tilde{F}_n^{(x)}(M, N), \tag{3.21}
\]
Theorem 4. At every point $x \in (-1, 1)$ and uniformly on every closed subset of $(-1, 1)$ the following statements are valid:

1. If for the sequence (1.8), conditions (1.12) and (1.15) are satisfied, then the following asymptotics

$$
\lim_{n \to \infty} G_n^{(2)}(x; \delta) = \frac{\delta}{\pi} \frac{2^{2x+5} \Gamma(x+1)}{\Gamma(2x+2)} x^2 (1 - 2x^2)(1 - x^2)^{-2} - x^{2-x}
$$

holds.

2. The generalized trace formula

$$
\sum_{n=0}^{\infty} [4(\varepsilon_{n+4}^2 - \varepsilon_n^2) + 2(\beta_{n+2}^2 - \beta_n^2)] [\tilde{B}^{(2)}_n(x)]^2
$$

with the constant $C_x > 0$ independent of $n \in \mathbb{Z}_+$. On the other hand, using (2.8), (3.17) and (3.18), we obtain

$$
m(B^{(2)}_n) - m(B^{(2)}_{n+4}) = \frac{n^2}{32} \left[ -8 + \frac{8x}{n^2} - \frac{32x}{n^2} + O \left(\frac{1}{n^4}\right) \right] + G_n^{(2)}(M, N),
$$

where

$$
|G_n^{(2)}(M, N)| \leq C_x \frac{1}{n^{2x+3}},
$$

with the constant $C_x > 0$ independent of $n \in \mathbb{Z}_+$.

By (3.6), summing the last relations and (3.21), (3.22), we obtain

$$
\gamma_n^{(2)} = \frac{3}{8} - \frac{x}{n} + O \left(\frac{1}{n^2}\right) + \tilde{G}_n^{(2)}(M, N), |\tilde{G}_n^{(2)}(M, N)| \leq C_x \frac{1}{n^{2x+3}},
$$

with the constant $C_x > 0$ independent of $n \in \mathbb{Z}_+$.

Thus we have the assertion of Theorem 3 for the sequence $\gamma_n^{(2)}$. Theorem 3 is completely proved.

Let us introduce the averaged Turan $\delta$-determinant for the system of symmetric Gegenbauer–Sobolev orthonormal polynomials

$$
G_n^{(2)}(x; \delta) = \sum_{k=n+1}^{n+4} \varepsilon_k [\delta_{k-4} [\tilde{B}_k^{(2)}(x)]^2 - \delta_k [\tilde{B}_{k-4}^{(2)}(x) [\tilde{B}_{k+4}^{(2)}(x)]]
$$

and

$$
+ \sum_{k=n+1}^{n+2} \beta_k [\delta_{k-2} [\tilde{B}_k^{(2)}(x) [\tilde{B}_{k+2}^{(2)}(x)] - \delta_k [\tilde{B}_{k-2}^{(2)}(x) [\tilde{B}_{k+4}^{(2)}(x)].
$$

Combining Theorem 2 with Lemma 2.6 and Theorem 3, we obtain the following statement.

Theorem 4. At every point $x \in (-1, 1)$ and uniformly on every closed subset of $(-1, 1)$ the following statements are valid:

1. If for the sequence (1.8), conditions (1.12) and (1.15) are satisfied, then the following asymptotics

$$
\lim_{n \to \infty} G_n^{(2)}(x; \delta) = \frac{\delta}{\pi} \frac{2^{2x+5} \Gamma(x+1)}{\Gamma(2x+2)} x^2 (1 - 2x^2)(1 - x^2)^{-2} - x^{2-x}
$$

holds.

2. The generalized trace formula

$$
\sum_{n=0}^{\infty} [4(\varepsilon_{n+4}^2 - \varepsilon_n^2) + 2(\beta_{n+2}^2 - \beta_n^2)] [\tilde{B}^{(2)}_n(x)]^2
$$

with the constant $C_x > 0$ independent of $n \in \mathbb{Z}_+$.
\[
+4 \sum_{n=0}^{\infty} \varepsilon_{n+4}(\varepsilon_{n+4} - \varepsilon_n) \widehat{B}_{n+4}^{(2)}(x) \widehat{B}_n^{(2)}(x) \\
+2 \sum_{n=0}^{\infty} (\varepsilon_{n+4} \beta_{n+6} - \varepsilon_{n+6} \beta_{n+2}) \widehat{B}_{n+6}^{(2)}(x) \widehat{B}_n^{(2)}(x) \\
= \frac{2^{2z+4}}{\pi \Gamma(2z+2)} \chi^2(1 - \chi^2)^{\frac{5}{2} - z}.
\]

holds.

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