On controllability of diagonal systems with one-dimensional input space

Birgit Jacob∗ Jonathan R. Partington†

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Abstract

This paper deals with diagonal systems on a Hilbert space with a one-dimensional input space and a (possibly unbounded) input operator. A priori it is not assumed that the input operator is admissible. Necessary and sufficient conditions for different notions of controllability such as null controllability and exact controllability are presented. These conditions, which are given in terms of the eigenvalues of the diagonal operator and in terms of the input operator, are linked with the theory of interpolation in Hardy spaces.

1 Introduction

Controllability is an important property of a distributed parameter system, which has been extensively studied in the literature, see for example [3] and [1]. In this paper, we study controllability of systems having a Riesz basis of eigenvectors and a one-dimensional input space. This class might seem restrictive, but it is fairly general nevertheless, because many semigroups considered in the literature have a Riesz basis of eigenvectors, and because a practically implemented system will have a finite-dimensional input space. It has been noted in the literature that exact controllability rarely holds if the input space is one-dimensional, and therefore we study weaker notions such as null-controllability and approximate controllability as well. For results concerning exact controllability we refer the reader to Jacob and Zwart [2], Rebarber and Weiss [7], and the references therein. On a Hilbert space $H$ we consider the following system

$$
\dot{x}(t) = Ax(t) + bu(t), \quad x(0) = x_0, \quad t \geq 0.
$$

∗Fachbereich Mathematik, Universität Dortmund, D-44221 Dortmund, Germany, birgit.jacob@math.uni-dortmund.de
†School of Mathematics, University of Leeds, Leeds LS2 9JT, U.K., J.R.Partington@leeds.ac.uk
We assume that $A$ is the infinitesimal generator of an exponentially stable $C_0$-semigroup $(T(t))_{t \geq 0}$ on $H$ which possesses a sequence of normalized eigenvectors $\{\phi_n\}_{n \in \mathbb{N}}$ forming a Riesz basis for $H$, with associated eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$, that is,

$$A\phi_n = \lambda_n \phi_n, \quad n \in \mathbb{N}.$$ 

Since $(T(t))_{t \geq 0}$ is assumed to be exponentially stable we have $\sup_{n \in \mathbb{N}} \Re \lambda_n < 0$. Let $\psi_n$ be an eigenvector of $A^*$ corresponding to the eigenvalue $\lambda_n$. Without loss of generality we can assume that $\langle \phi_n, \psi_n \rangle = 1$. Then the sequence $\{\psi_n\}_{n \in \mathbb{N}}$ forms a Riesz basis of $H$ and every $x \in H$ can be written as

$$x = \sum_{n \in \mathbb{N}} \langle x, \psi_n \rangle \phi_n = \sum_{n \in \mathbb{N}} \langle x, \phi_n \rangle \psi_n.$$ 

By $H_\alpha, \alpha \in \mathbb{R}$, we denote the interpolation spaces

$$H_\alpha = \left\{ \sum_{n=1}^{\infty} x_n \phi_n | \{x_n|\lambda_n^\alpha\}_{n \in \mathbb{N}} \in \ell^2 \right\},$$

equipped with the scalar product

$$\langle x, y \rangle := \sum_{n \in \mathbb{N}} \langle x, \psi_n \rangle \langle y, \psi_n \rangle |\lambda_n|^{2\alpha}.$$ 

Thus the spaces $H_\alpha$ are Hilbert spaces with $H_0 = H$ and $H_1 = D(A)$. In the following let $\alpha \geq 0$, $b \in H_{-\alpha}$ and $u \in L^2(0, \infty)$. Thus $b$ can be represented by a sequence $\{b_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ with $\{b_n|\lambda_n^{-\alpha}\}_{n \in \mathbb{N}} \in \ell^2$, that is $b_n := \langle b, \psi_n \rangle$. For more information on the spaces $H_{-\alpha}$ see for example [8]. One important feature of this interpolation spaces $H_{-\alpha}$ is that the semigroup $(T(t))_{t \geq 0}$ can be extended to a $C_0$-semigroup on $H_{-\alpha}$, which we denote by $(T_{-\alpha}(t))_{t \geq 0}$, and the generator of this extended semigroup, denoted by $A_{-\alpha}$ is an extension of $A$. By a solution of (1) we mean the so-called mild solution

$$x(t) := T(t)x_0 + \int_0^t T_{-\alpha}(t-s)bu(s) \, ds, \quad t \geq 0.$$ 

Thus the solution is a continuous $H_{-\alpha}$-valued function. For $t_1 \geq 0$ we introduce the operators $B_{t_1} \in \mathcal{L}(L^2(0, \infty), H_{-\alpha})$ and $B_\infty \in \mathcal{L}(L^2(0, \infty), H_{-\alpha})$ by

$$B_{t_1}u := \int_0^{t_1} T_{-\alpha}(t_1-s)bu(s) \, ds,$$

and

$$B_\infty u := \int_0^\infty T_{-\alpha}(s)bu(s) \, ds,$$

respectively. We shall discuss the following controllability concepts.
Definition 1.1 We say that system (1) is
1. null-controllable on $[0, t_1]$, if $\text{Im} \ T(t_1) \subset \text{Im} \ B_{t_1}$;
2. exactly controllable on $[0, t_1]$, if $H \subset \text{Im} \ B_{t_1}$;
3. null-controllable on $[0, \infty)$, if $\text{Im} \ T(t_1) \subset \text{Im} \ B_{\infty}$ for some $t_1 \geq 0$;
4. exactly controllable on $[0, \infty)$, if $H \subset \text{Im} \ B_{\infty}$.

Every exactly controllable system is null-controllable. If system (1) is null-controllable (resp. exactly controllable) on $[0, t_0]$, then it is null-controllable (resp. exactly controllable) on $[0, t_1] \text{ and } [0, \infty)$ for every $t_1 > t_0$. Further, if system (1) is null-controllable (resp. exactly controllable) on $[0, \infty)$, then it is null-controllable (resp. exactly controllable) on some interval $[0, t_0]$. Based on these observation we only study these notion on an infinite interval and thus write exactly (resp. null-) controllable instead of exactly (resp. null-) controllable on $[0, \infty)$.

2 Criteria for controllability

In this section we develop equivalent condition for the controllability notion introduced in the previous section. We have the following two main results. The proofs of these theorems are given at the end of this section.

Theorem 2.1 The following statements are equivalent:

1. System (1) is null-controllable.
2. There exists a constant $A > 0$ such that for all $h > 0$ and all $\omega \in \mathbb{R}$:
   \[
   \sum_{\lambda_n \in R(\omega, h)} \frac{|\text{Re} \lambda_n|^2 |b_n|^2 e^{2t_1 |\text{Re} \lambda_n|} \prod_{k \neq n} \frac{\lambda_n + \lambda_k}{\lambda_k - \lambda_n}^2}{|b_n|^2 e^{2t_1 |\text{Re} \lambda_n|}} \leq Ah,
   \]
   where $R(\omega, h) := \{ s \in \mathbb{C}_+ \mid \text{Re} s < h, \omega - h < \text{Im} s < \omega + h \}$.

The proof of this theorem will be given at the end of this section. In the following examples we study the condition (3).

Example 2.2 For $\beta > 0$ we consider the sequence $\{\lambda_n\}_{n \in \mathbb{N}} := \{-n^\beta\}_{n \in \mathbb{N}}$. Then for $k \gg n$ we have
   \[
   \left| \frac{\lambda_n + \lambda_k}{\lambda_k - \lambda_n} \right|^2 \approx 1 + 4 \left( \frac{n}{k} \right)^\beta,
   \]
   and thus for each $n$ the infinite product converges to a finite number only in the case $\beta > 1$. That is, the system is not null-controllable if $\beta \in (0, 1]$.
Example 2.3 It is not necessary for \( \{-\lambda_n\}_{n \in \mathbb{N}} \) to be a Carleson sequence in \( \mathbb{C}_+ \), which would mean that

\[
\prod_{k \neq n} \left| \frac{\lambda_n + \lambda_k}{\lambda_k - \lambda_n} \right|^2
\]

was uniformly bounded in \( n \), although we do need the infinite product to converge for each \( n \). However, this implies that \( \{-\lambda_n\}_{n \in \mathbb{N}} \) needs to be a Blaschke sequence.

In the case of \( \{\lambda_n\}_{n \in \mathbb{N}} = \{-2^n\}_{n \in \mathbb{N}} \), the product is uniformly bounded in \( n \) and any sequence \( \{b_n\} \) such that

\[
\frac{|\lambda_n|^2}{|b_n|^2} e^{2t_1\lambda_n} = O(2^n) \quad \text{as} \quad n \to \infty,
\]

including all sequences bounded away from zero, will provide null-controllability.

Theorem 2.4 The following statements are equivalent:

1. System (1) is exactly controllable.

2. There exists a constant \( A > 0 \) such that for all \( h > 0 \) and all \( \omega \in \mathbb{R} \):

\[
\sum_{-\lambda_n \in R(\omega, h)} \frac{|\text{Re} \lambda_n|^2}{|b_n|^2} \prod_{k \neq n} \left| \frac{\lambda_n + \lambda_k}{\lambda_k - \lambda_n} \right|^2 \leq Ah,
\]

where \( R(\omega, h) := \{ s \in \mathbb{C}_+ \mid \text{Re} s < h, \omega - h < \text{Im} s < \omega + h \} \).

The proofs of Theorems 2.1 and 2.4 are based on interpolation problems on \( \mathbb{C}_+ \). The following theorem is a half-plane version of a result given by McPhail [5].

Theorem 2.5 Let \( \{s_n\}_n \) be a sequence in \( \mathbb{C}_+ \) and \( \{b_n\}_n \) a sequence of nonzero complex numbers. Then it is possible for every \( \{x_n\}_n \) in \( \ell^2 \) to solve the equation \( F(s_n) = b_n x_n \) for \( F \in H^2(\mathbb{C}_+) \), if and only if

\[
\nu = \sum_{n=1}^{\infty} \frac{(\text{Re} s_n)^2 |b_n|^2}{\epsilon_n^2} \delta_{s_n}
\]

is a Carleson measure on \( \mathbb{C}_+ \), where

\[
\epsilon_n = \prod_{k \neq n} \left| \frac{s_n - s_k}{s_n + s_k} \right|.
\]

Equivalently, let \( \{w_n\}_n \) be a non-negative sequence. Then it is possible for all sequences \( \{a_n\}_n \) satisfying \( \sum |a_n w_n|^2 < \infty \) to find \( F \in H^2(\mathbb{C}_+) \) with \( F(z_n) = a_n \) for all \( n \), if and only if the measure

\[
\nu = \sum_{n=1}^{\infty} \frac{(\text{Re} s_n)^2}{w_n^2 \epsilon_n^2} \delta_{s_n},
\]

is a Carleson measure on \( \mathbb{C}_+ \).
where $\delta_s$ is the (Dirac) point mass at $s$, is a Carleson measure on $\mathbb{C}_+$, i.e., if and only if there is a finite constant $K$ such that

$$\sum_{s_n \in R(I)} \frac{(\text{Re } s_n)^2}{w_n^2 \epsilon_n^2} \leq K|I|,$$

for all intervals $I \subset \mathbb{R}$, where $R(I)$ is the rectangle

$$R(I) = \{x + iy : y \in I, 0 < x < |I|\}.$$

**Proof of Theorem 2.1** Using the fact that

$$B_\infty f = \sum_{n=1}^{\infty} b_n \int_0^\infty e^{\lambda_n t} f(t) \, dt \phi_n = \sum_{n=1}^{\infty} b_n \hat{f}(-\lambda_n) \phi_n,$$

(4)

where $\hat{f}$ denotes the Laplace transform of $f$, we get that null-controllability of system (1) is equivalent to a positive answer to the following interpolation problem: Does there exist for every sequence $\{x_n\}_n \in \ell^2$ a function $g \in H^2(\mathbb{C}_+)$ such that

$$b_n g(-\lambda_n) = e^{\lambda_n t_1} x_n?$$

By Theorem 2.5 this interpolation problem is equivalent to Part 2.

**Proof of Theorem 2.4** As in the proof of Theorem 2.1 we get that exact controllability of system (1) is equivalent to a positive answer to: Does there exist for every sequence $\{x_n\}_n \in \ell^2$ a function $g \in H^2(\mathbb{C}_+)$ such that

$$b_n g(-\lambda_n) = x_n?$$

By Theorem 2.5 this interpolation problem is equivalent to Part 2 of Theorem 2.4.

**References**


