On $\sigma$-uniform density and ideal convergent sequences of fuzzy real numbers

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Abstract. An ideal $I$ is a family of subsets of positive integers $\mathbb{N}$ which is closed under taking finite unions and subsets of its elements. In this paper we have introduced ideal convergent sequences of fuzzy real numbers using $\sigma$-uniform density. Furthermore, inclusion between $I_\sigma$-convergence and invariant convergence also $I_\sigma$-convergence and $[V_\sigma^*]$-convergence were given.

Keywords: Ideal, $I$-convergence, $\sigma$-uniform density, fuzzy numbers, strongly $\sigma$-convergence

1. Introduction

The notion of $I$-convergence ($I$ denotes the ideal of subsets of $\mathbb{N}$) was initially introduced by Kostyrko et al. [15] as a generalization of statistical convergence (see [6, 28]). More applications of ideals we refer to ([4, 5, 9–13, 16, 20, 21, 25, 27, 29–31]).

A family of sets $I \subseteq \mathcal{P}(\mathbb{N})$ (power sets of $\mathbb{N}$) is called an ideal if and only if for each $A, B \in I$ we have $A \cup B \in I$ and for each $A \in I$ and each $B \subseteq A$, we have $B \in I$. A non-empty family of sets $I \subseteq \mathcal{P}(\mathbb{N})$ is a filter on $\mathbb{N}$ if and only if $\emptyset \notin I$ for each $A, B \in I$, we have $A \cap B \in I$ and each $A \in I$ and each $A \subseteq B$, we have $B \in I$. An ideal $I$ is called non-trivial ideal if $I \neq \emptyset$ and $\mathbb{N} \notin I$. Clearly $I \subseteq \mathcal{P}(\mathbb{N})$ is a non-trivial ideal and only if $I \subseteq \mathcal{P}(\mathbb{N})$ is a filter on $\mathbb{N}$. A non-trivial ideal $I \subseteq \mathcal{P}(\mathbb{N})$ is called admissible if and only if $\{x : x \in \mathbb{N}\} \in I$. A non-trivial ideal $I$ is maximal if there cannot exists any non-trivial ideal $J \neq I$ containing $I$ as a subset. Further details on ideals of $\mathbb{N}$ can be found in Kostyrko et al. [15].

Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $I$-convergent to a real number $\ell$ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon \} \in I$ for every $\varepsilon > 0$ (see [15]). In this case we write $I - \lim x_k = \ell$. If we take $I = I_\sigma = \{A \subseteq \mathbb{N} : A$ is a finite subset$\}$. Then $I_\delta$ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : A(A) = 0\}$ where $A(A)$ denote the asymptotic density of the set $A$. Then $I_\delta$ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the statistical convergence.

In [24], Nuray et al., introduced the $\sigma$-uniform density as follows:

Let $A \subseteq \mathbb{N}$ and

$$t_\sigma := \min_{n} \left| \{ \sigma(n) : \sigma(n) \neq \sigma(n) \} \right|$$

$$S_\sigma := \max_{n} \left| \{ \sigma(n) : \sigma(n) \neq \sigma(n) \} \right|.$$ 

If the following limits exist

$$\overline{V}(A) := \lim_{m \to \infty} \frac{m}{m}, \quad \overline{V}(A) := \lim_{m \to \infty} \frac{m}{m},$$

then they are called a lower and an upper $\sigma$-uniform density of the set $A$, respectively. If $\overline{V}(A) = \overline{V}(A) = \overline{V}(A)$ is called the $\sigma$-uniform density of $A$. 

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If \( r(n) = n + 1 \), then it becomes the uniform density, which is defined in [1].

Denote by \( I_n \) the class of all \( A \subseteq \mathbb{N} \) with \( V(A) = 0 \).

**Definition 1.1.** [24] A sequence \((x_n)\) of real numbers is said to be \( I_n \)-convergent to a real number \( \ell \), if for each \( \varepsilon > 0 \) the set \( \{ k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon \} \) belongs to \( I_n \). The set of all \( I_n \)-convergent sequences will be denoted by \( I_n \).

Throughout the article \( I_n \) is an admissible ideal of \( \mathbb{N} \). If \( r(n) = n + 1 \), then \( I_n \)-convergence coincides with \( I_n \)-convergence which is defined in [1].

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [32] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [18] introduced bounded and convergent sequences of fuzzy numbers and studied their some properties. Later on sequences of fuzzy numbers have been discussed by Diamond and Kloeden [3], Mursaleen and Basarir [19], Mursaleen [22], Nanda [23], Savas [26] and many others.

Let \( C(\mathbb{R}^n) = \{ A \subseteq \mathbb{R}^n : A \) is compact and convex \( \} \). The space \( C(\mathbb{R}^n) \) has a linear structure induced by the operations

\[
A + B = \{ a + b : a \in A, b \in B \}
\]

and

\[
yA = \{ y : a \in A \} \quad \text{for} \quad A, B \in C(\mathbb{R}^n) \quad \text{and} \quad y \in \mathbb{R}.\]

The Hausdorff distance between \( A \) and \( B \) in \( C(\mathbb{R}^n) \) is defined by

\[
\delta_h(A, B) = \max \left\{ \sup_{x \in X} \delta(A, X) \right\}.\]

It is well-known that \((C(\mathbb{R}^n), \delta_h)\) is a complete metric space.

A fuzzy number is a function \( X \) from \( \mathbb{R}^n \) to \([0, 1]\) which is normal, fuzzy convex, upper semicontinuous and the closure of \( X^\alpha = \{ x \in \mathbb{R}^n : X(x) > \alpha \} \) is compact. These properties imply that for each \( 0 < \alpha \leq 1 \), the \( \alpha \)-level set

\[
X^\alpha = \{ x \in \mathbb{R}^n : X(x) > \alpha \}
\]

is a non-empty compact, convex subset of \( \mathbb{R}^n \) with support \( X^0 \).

If \( \mathbb{R}^n \) is replaced by \( \mathbb{R} \), then obviously the set \( C(\mathbb{R}^n) \) is reduced to the set of all closed bounded intervals \( A = \{ a, b \} \) on \( \mathbb{R} \), and also

\[
\delta_h(A, B) = \max(|a - b|, |b - a|) \quad \text{for} \quad A, B \subseteq \mathbb{R}.\]

Let \( L(\mathbb{R}) \) denote the set of all fuzzy numbers. The linear structure of \( L(\mathbb{R}) \) induces the addition \( X + Y \) and the scalar multiplication \( \lambda X \) in terms of \( \alpha \)-level sets, by

\[
[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha
\]

and

\[
[\lambda X]^\alpha = \lambda [X]^\alpha \quad \text{for each} \quad 0 \leq \alpha \leq 1.\]

The set \( \mathbb{R} \) of real numbers can be embedded in \( L(\mathbb{R}) \) if we define \( R \in L(\mathbb{R}) \) by

\[
\tau(t) = \begin{cases} 1, & \text{if} \ t = r; \\ 0, & \text{if} \ t \neq r. \end{cases}
\]

The additive identity and multiplicative identity of \( L(\mathbb{R}) \) are denoted by \( 0 \) and \( T \), respectively.

For \( r \in \mathbb{R} \) and \( X \in L(\mathbb{R}) \), the product \( rX \) is defined as follows:

\[
rX(t) = \begin{cases} X(t/r), & \text{if} \ r \neq 0; \\ 0, & \text{if} \ r = 0. \end{cases}
\]

Define a map \( d : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R} \) by

\[
d(X, Y) = \sup_{0 \leq t \leq 1} \delta_h(X^t, Y^t).\]

For \( X, Y \in L(\mathbb{R}) \) define \( X \leq Y \) if and only if \( X^\alpha \leq Y^\alpha \) for any \( 0 \leq \alpha \leq 1 \). It is known that \((L(\mathbb{R}), d)\) is complete metric space [18].

2. Definitions and preliminaries

A sequence \( X = (X_n) \) of fuzzy numbers is a function \( X \) from the set \( N \) of natural numbers into \( L(\mathbb{R}) \). The fuzzy number \( X_n \) denotes the value of the function at \( k \in N \) [18].

We denote by \( u^\infty \) the set of all sequences \( X = (X_n) \) of fuzzy numbers.

A sequence \( X = (X_n) \) of fuzzy numbers is said to be bounded if the set \( \{ X_n : k \in N \} \) of fuzzy numbers is bounded [18].

We denote by \( b_u^\infty \) the set of all bounded sequences \( X = (X_n) \) of fuzzy numbers.

A sequence \( X = (X_n) \) of fuzzy numbers is said to be convergent to a fuzzy number \( X_0 \) if for every \( \varepsilon > 0 \)
there is a positive integer \( k_0 \) such that \( d(X_k, X_0) < \epsilon \) for \( k > k_0 \) [18].

We denote by \( c^f \) the set of all convergent sequences \( X = (X_k) \) of fuzzy numbers.

It is straightforward to see that \( c^f \subset c^\mathbb{F} \).

Nanda [23] studied the classes of bounded and convergent sequences of fuzzy numbers and showed that these are complete metric spaces.

A metric \( d \) on \( L(\mathbb{R}) \) is said to be translation invariant if \( d(X+Y, Z+W) = d(X, Y) \) for \( X, Y, Z, W \in L(\mathbb{R}) \) (see [19]).

**Definition 2.4.** [16] A sequence of fuzzy numbers.

**Definition 2.5.** [15] An admissible ideal \( I \subset P(\mathbb{N}) \) is said to satisfy the condition (AP) if for every countable family of mutually disjoints sets \( \{A_1, A_2, \ldots \} \) belonging to \( I \) there exists a countable family of sets \( \{B_1, B_2, \ldots \} \) such that \( A_1 \Delta B_j \) is a finite set for \( j \in \mathbb{N} \) and \( B = \bigcup_{j=1}^{\infty} B_j \in I \).

### 3. Main results

Now we give the definition of strongly \( \sigma \)-convergence and \( L_p \)-convergence of fuzzy numbers as follows.

**Definition 2.1.** [25] A bounded sequence \( X = (X_k) \) of fuzzy numbers is said to be strongly Cesàro summable to a fuzzy number \( X_0 \) if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(X_k, X_0) = 0.
\]

**Definition 2.2.** [26] A bounded sequence \( X = (X_k) \) of fuzzy numbers is said to be almost convergent to a fuzzy number \( X_0 \) if and only if

\[
\lim_{n \to \infty} \frac{1}{m} \sum_{k=1}^{n} X_k = X_0 \quad \text{uniformly in } k.
\]

\( c^f \) denotes the set of all almost convergent sequences of fuzzy numbers.

**Definition 2.3.** [26] A bounded sequence \( X = (X_k) \) of fuzzy numbers is said to be strongly almost convergent to a fuzzy number \( X_0 \) if and only if

\[
\lim_{n \to \infty} \frac{1}{m} \sum_{k=1}^{n} d(X_k, X_0) = 0 \quad \text{uniformly in } k.
\]

\( c^\sigma \) denotes the set of all almost convergent sequences of fuzzy numbers.

**Definition 2.4.** [16] A sequence \( X = (X_k) \) of fuzzy numbers is said to be \( I \)-convergent to a fuzzy number \( X_0 \) if for each \( \epsilon > 0 \) the set

\[
A = \{ k \in \mathbb{N} : d(X_k, X_0) \geq \epsilon \} \in I.
\]
Theorem 3.1. (a) If \( L_1 = \lim X_k = X_0 \) and \( L_2 = \lim Y_k = Y_0 \), then \( L_1 = \lim (X_k + Y_k) = X_0 + Y_0 \).
(b) If \( \alpha \) is a constant and \( L_1 = \lim X_k = X_0 \) then \( L_1 - \lim aX_k = aX_0 \).

Theorem 3.2. Suppose that \( X = (X_k) \in \ell_p \), if \( X \) is \( L_2 \)-convergent to \( \ell \), then \( X \) is invariant convergent to \( \ell \).

Proof. Let \( k, m \in \mathbb{N} \) be arbitrary and \( \varepsilon > 0 \). We estimate
\[
T(k, m) = \left( \frac{X_{m,k} + X_{m,k+1} + \cdots + X_{m,k+m}}{m} \right).
\]
We have
\[
T(k, m) \leq T(1)(k, m) + T(2)(k, m),
\]
where
\[
T(1)(k, m) = \frac{1}{m} \sum_{\ell \in K(c)} d(X_{\ell,k}, X_0),
\]
\[
T(2)(k, m) = \frac{1}{m} \sum_{\ell \in K(e)} d(X_{\ell,k}, X_0)
\]
and
\[
K(c) = \{ 1 \leq i \leq m : d(X_{\ell,k}, X_0) \geq \varepsilon \}.
\]
We have \( T(2)(k, m) < \varepsilon \), for every \( k = 1, 2, 3, \ldots \). The boundedness of \( (X_k) \) implies that there exists \( M > 0 \) such that \( d(X_{\ell,k}, X_0) \leq M (i = 1, 2, 3, \ldots ; k = 1, 2, 3, \ldots) \), then this implies that
\[
T(1)(k, m) \leq \sup_{\ell \in K(c)} d(X_{\ell,k}, X_0) \cdot \frac{1}{m} \left( \left\{ 1 \leq i \leq m : d(X_{\ell,k}, X_0) \geq \varepsilon \right\} \right)
\]
\[
\leq M \max_{\ell \in K(c)} \left\{ 1 \leq i \leq m : d(X_{\ell,k}, X_0) \geq \varepsilon \right\} \cdot \frac{1}{m}
\]
\[
= M \frac{S_c}{m},
\]
where \( S_c \) is the sequence
\[
X_k = \begin{cases} \ell, & \text{if } k \text{ is even} \\ \sigma, & \text{if } k \text{ is odd} \end{cases}
\]
\[
\lim_{m \to \infty} M \frac{S_c}{m} = 0.
\]
When \( \sigma(k) = k + 1 \), this sequence is invariant convergent to \( \frac{1}{2} \) but is not \( L_2 \)-convergent.

Theorem 3.3. (a) If \( 0 < p < \infty \) and \( X_k \to X_0([V_0])_p \), then \( X = (X_k) \) is \( L_2 \)-convergent to \( X_0([V_0])_p \).
(b) If \( X = (X_k) \in \ell_p \) and \( L_2 \)-converges to \( X_0([V_0])_p \).

Proof. (a) Let \( X_k \to X_0([V_0])_p \), \( 0 < p < \infty \). Suppose \( \varepsilon > 0 \). Then for every \( k \in \mathbb{N} \), we have
\[
\frac{1}{m} \sum_{\ell \in K(c)} \left| d(X_{\ell,k}, X_0) \right|^p \leq \varepsilon \left\{ 1 \leq i \leq m : d(X_{\ell,k}, X_0) \geq \varepsilon \right\}
\]
\[
\geq \varepsilon \left\{ 1 \leq i \leq m : d(X_{\ell,k}, X_0) \geq \varepsilon \right\}
\]
\[
= \varepsilon \frac{S_c}{m},
\]
for every \( k = 1, 2, 3, \ldots \). This implies that \( \lim_{m \to \infty} \frac{S_c}{m} = 0 \) and so \( L_2 \)-lim \( X_k = X_0 \).
Theorem 3.5. The sequential method $I_{\sigma}$ is regular.

Proof. Proof of the theorem is straightforward, thus omitted. $\square$

Theorem 3.6. The sequential method $I_{\sigma}$ is subsequential.

Proof. Proof of the theorem follows from Theorem 3.5. $\square$

Theorem 3.7. Let $I_{\sigma}$ be an admissible ideal having the property (AB). If $X = (X_{k})$ is $I_{\sigma}$-convergent to $X_{0}$, then $X = (X_{k})$ is $I_{\sigma}$-convergent to $X_{0}$.

Proof. Suppose that $I_{\sigma}$ satisfies the condition (AP). Let $I_{\sigma} - \lim X_{k} = X_{0}$. For $\varepsilon > 0$, the set

$$\{k \in \mathbb{N} : d(X_{k}, X_{0}) \geq \varepsilon\} \in I_{\sigma}.$$

We put

$$K_{1} = \{k \in \mathbb{N} : d(X_{k}, X_{0}) \geq 1\}$$

and

$$K_{m} = \{k \in \mathbb{N} : \frac{1}{m} \leq d(X_{k}, X_{0}) < \frac{1}{m+1}\}$$

for $m \geq 2$, $m \in \mathbb{N}$.

Obviously, $K_{i} \cap K_{j} \neq \emptyset$ for $i \neq j$. By condition (AP) there exists a sequence countable family of sets $\{M_{j}, j \in \mathbb{N}\}$ such that $K_{i} \Delta M_{j}$ are finite sets for $j \in \mathbb{N}$ and $M = \bigcup_{j=1}^{\infty} M_{j} \in I_{\sigma}$. It is sufficient to show that for $k \in B = \mathbb{N} - M$, $\lim_{k \to \infty} X_{k} = X_{0}$.

Let $a > 0$. Choose $m \in \mathbb{N}$ such that $\frac{1}{m^{2}} < a$. Then

$$\{k : d(X_{k}, X_{0}) \geq a\} \subseteq \bigcup_{j=1}^{m+1} K_{j}.$$

Since $K_{j} \Delta M_{j}$, for $j = 1, 2, 3, ... m + 1$ are finite sets, there exists $k_{0} \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{m+1} M_{j}\right) \cap \{k : k > k_{0}\} = \left(\bigcup_{j=1}^{m+1} K_{j}\right) \cap \{k : k > k_{0}\}.$$

(3.2)

If $k > k_{0}$ and $k \notin M_{j}$, then $k \notin \bigcup_{j=1}^{m+1} M_{j}$ and by the relation (3.2), we have $k \notin \bigcup_{j=1}^{m+1} K_{j}$. Then

$$d(X_{k}, X_{0}) < \frac{1}{m+1} < a.$$

Thus, we have $\lim_{k \to \infty} X_{k} = X_{0}$. Hence $X = (X_{k})$ is $I_{\sigma}$-convergent to $X_{0}$. $\square$
4. Applications

Fuzzy numbers finds application in handling vague terms, and therefore they can be suitably used to model real life scenarios involving vague parameters so as to obtain optimal solutions. Fuzzy multi-objective linear programming usually deals with flexible aspiration levels that are indicative of optimality when considering all objectives or goals simultaneously with possible deviation in objectives or constraints. The fuzzy multi-objective linear programming model with nonlinear membership functions for solving a multi objective travelling salesmen problem in order to simultaneously minimize the three parameters cost, distance and time. The primary contribution of a fuzzy mathematical model using nonlinear membership functions, more precisely the exponential functions to ensure an optimal solution in vague, imprecise and uncertain environment.

Fuzzy set theory also finds its applications for modeling, uncertainty and vagueness in various fields of Science and Engineering, e.g. programing [8], nonlinear dynamical systems [14], population dynamics [2], control of chaos [7], quantum physics [17], etc.

5. Conclusions

In this article we have investigated the notion of \( I_\sigma \)-convergence and \( I_\sigma \)-convergence of sequences point of view of fuzzy real numbers. Still there are a lot to be investigated on sequence spaces applying the notion of \( I_\sigma \)-convergence. The workers will apply the techniques used in this article for further investigations on \( I_\sigma \)-convergence.

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References


