ANALYZING AHP-MATRICES BY ROBUST PARTIAL LEAST SQUARES REGRESSION

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Abstract. The Analytic Hierarchy Process (AHP) [8] is a powerful process to help people to express priorities and make the best decision when both qualitative and quantitative aspects of a decision need to be considered. In this paper, in order to eliminate the influence of outliers, we use an approach based on Robust Partial Least Squares (R-PLS)[12] regression for the computation of the values for the weights of a comparison matrix. A simulation study to compare the results with other methods for computing the weights proposed to analyze comparison matrix.

Keywords: Analytic Hierarchy Process, Robust Regression, Simulation Study.

1 Introduction

The Analytic Hierarchy Process (AHP) is a systematic procedure for representing the elements of a multicriteria decision maker (MCDM) problem, hierarchically. A decision problem is broken by means of AHP into smaller parts and then decision makers lead through a series of pairwise comparison judgements to express the relative intensity of the impact of the elements in the hierarchy. The AHP procedure shows how to use judgements and experience to analyze a complex decision problem by combining its qualitative and quantitative aspects into a single framework and generating a set of priorities for alternative courses of action. The process has inherent flexibilities in structuring a problem and in taking different judgements from people. These judgements are converted into numbers. Detailed exposition of the AHP is found in Saaty [8]. Saaty and Vargas [9], illustrate applications of the AHP in various real-life systems. A fundamental problem of decision theory is to derive weights for a set of activities according to the importance. The object of this approach is to use the weights, which we call priorities, to allocate a resource among the activities or, if precise weights cannot be obtained, to rank the most important activities. The AHP falls into the broad category of mathematical and behavioural science interests. The dominance matrices play a central role in the AHP [10]. The aim of our work is to investigate the consequence of changes in the judgements through perturbations on the entire set of
judgements. This type of approach leads to the criterion of consistency. Thus, obtaining solutions in our method is not a statistical procedure. However, if we want to compare any solution with the criterion of consistency, we refer to statistical reasoning and perturbations over the entire matrix of judgements. Different approaches are developed in order to evaluate the weights, using statistical procedures. Tucker [13] presents a method for the determination of parameters of a functional relation by factor analysis. Saaty and Vargas [9] have studied the eigenvalue approach with the least squares and logarithmic least squares methods. With inconsistency, just the eigenvalue method is applied. Laininen and Hmlinen [6], starting from the consideration that the presence of outliers can have a significant influence on the weight estimates given by the eigenvector method and the logarithmic least squares regression, propose a robust version of logarithmic least squares regression in the case early mentioned.

2 Methods of Estimating and Analysis of Priority Vectors

Methods of estimating the vector of priorities were introduced along with clustering analysis to make the process more efficient and consistent. In the AHP the weight ratios are asked for all pairs of attributes. Usually the weights are derived by the principal eigenvector (EV) of the comparison matrix. Other estimation methods, such as the logarithmic least squares method, can be used to derive the weights as well.

In Saaty [8] is described how the eigenvector associated with the principal eigenvalue of $A$ can be obtained as:

$$\lim_{k \to \infty} A^k e = C w$$

Where $e$ is the column vector unity, $e^T$ its transpose, and $C$ a positive constant. This result allows us to approximate $\lambda_{\text{max}}$ and $w$ as accurately as desired but within computational capabilities.

In situations in which accuracy is not the most important factor, the vector of priorities can be approximated by one of three methods: average of normalized columns (ANC), normalization of row averages (NRA) and normalization of the geometric mean of rows (NGM) [9].

Let $\hat{w}_i$ be the priority estimate of the $i$th activity. According to the first method we have:

$$\hat{w}_i(\text{ANC}) = \frac{1}{n} \sum_{j=1}^{n} \frac{a_{ij}}{\sum_{k=1}^{n} a_{kj}}.$$ 

The second method yields:
\[ \hat{w}_i(NRA) = \frac{\sum_{j=1}^{n} a_{ij}}{\sum_{i,j=1}^{n} a_{ij}}, \]

and

\[ \hat{w}_i(NGM) = \frac{\left( \prod_{j=1}^{n} a_{ij} \right)^{1/n}}{\sum_{k=1}^{n} \left( \prod_{j=1}^{k} a_{kj} \right)^{1/n}}. \]

A drawback of this method is that there is no practical statistical theory behind it.

If the matrix \( A \) is consistent, the three methods reproduce the original vector of priorities. However, in the inconsistent case ANC is more accurate than the other two methods. The NGM method provides a good estimate of the priorities if accuracy is not of extreme importance.

Other ways of approximating the vector of priorities are discussed in [8].

3 Methods Based on Regression Approaches

The standard method to calculate the weights from an AHP-matrix is to take the eigenvector corresponding to the largest eigenvalue of the matrix, standardizing the sum of the component equal to 1. This is not-statistical approach. A statistical approach is needed if there are random fluctuations in the ratios of the relative importance. All the human decisions are affected by random variations [3]. The use of Regression Methods for analyzing AHP-matrix is not new [2] [1]. The regression techniques resolves a classical statistical problem, to estimate the linear relationship between two sets of variables, \( X_{n,p} \) (explicative variables) and \( y_{n,1} \) (dependent variable) where \( n \) is the number of statistical units and \( q \) the number of the explanatory variables. The technique which is largely used to solve this problem is the multivariate regression model

\[ y = a + Xb + e \]

where \( a(n,1) \) is the intercept term, \( b(p,1) \) the gradient and \( e(n,1) \) the error term. The parameters are generally computed using least squares criteria. This techniques is well note as Least Squares Regression (LS). Laininen, R.P. Hämäläinen [6] instead of use of LS regression methods use a variant Logarithmic Regression (LOG) that is lesser sensitive in presence of value that are different from corresponding consistent value. In the statistical theory these values are called outliers. Such technique, Robust Logarithm Regression (RLOG) is more stable under random occurrences of outliers compared to EV method.
3.1 Robust PLS Regression Based on Simple Least Median Regression

When $X'X$ does not exist (or is unstable) therefore the classical least squares regression model cannot be applied. The solution for overcoming this problem is offered by Partial Least Squares (PLS) Regression [14]. Garthwaite [4] proposed an alternative approach of the PLS based on simple linear regression showing that linear combinations of the explanatory variables can be formed sequentially and related to $y$ variable by ordinary least squares regression. Since PLS regression is very sensitive to the presence of outliers in the data, different algorithms of robust version have been proposed. Rousseeuw [7] proposed the Least Median of Squares (LMS) estimator which is obtained from the following optimization set up

$$\text{Minimize } \left( \text{median}_i e_i^2 \right)$$

where $e_i$ is the residual of observation $i$.

Simonetti et al. [12], propose a robust version of PLS regression (R-PLS), based on least median square regression [7]. This median square regression is substituted for the least square regression. The R-PLS regression is lesser sensitive to presence of outliers that is in AHP context when the comparisons matrix is non consistent.

4 Inconsistency

A foundamental property of the pairwise comparison matrix is the consistency. If the decision maker is perfectly consistent in making estimates, the matrix $A$ will satisfy the following consistency condition:

$$a_{ij} = a_{ik}a_{kj},$$

for each $i,j,k=1,2,\ldots n$ [8].

In the case of perfectly consistency, the reciprocal matrix $A$ has unit rank since every row is a constant multiple of the first row.

If $A$ is consistent then all its eigenvalues except one are zero. Since the sum of the eigenvalues of a matrix is equal to its trace,

$$a_{ij} = a_{ik}a_{kj}$$

then $n$ is the only non zero eigenvalue of $A$.

In the inconsistent case, which is the most common, the pairwise comparisons are not perfect, that is, the entry $a_{ij}$ might deviate from the ratio of the real membership values $w_i$ and $w_j$.

When the entries $a_{ij}$ change slightly then the eigenvalues change in a similar fashion. The maximum eigenvalue is close to $n$, greater than $n$, while the remaining eigenvalues are close to zero.
In order to find the priority vector of $A$ one should find the eigenvector corresponding to the maximum eigenvalue.

### 4.1 Measures of Consistency

Since small changes in the entries $a_{ij}$ imply a small change in $\lambda_{max}$, the deviation of the greatest eigenvalue of $A$ from the dimension of this matrix, $n$, represents a deviation from consistency. The following expression

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

called Consistency Index, is proposed by Saaty as a measure of the consistency of comparisons. If the coefficient is not within certain acceptable limits, then the comparisons have to be repeated until an acceptable consistency level is reached [8].

Another measure of consistency is the Consistency Ratio:

$$CR = \frac{CI}{RI}$$

The Random Index ($R.I.$) is the average value of $C.I.$ derived from a sample of size 500 of randomly generated reciprocal matrices with entries from the set $[1/9, 1/8, \ldots, 1, 2, \ldots, 8, 9]$.

The ratio $CR$ should be about 10 percent or less for acceptable overall consistency. Otherwise, the quality of the judgmental data should be improved, perhaps by revising the manner in which questions are posed to make the pairwise comparisons.

Another measure of consistency is the so called Consistency Measure ($CM$), defined as

$$CM = \frac{2}{n(n-1)} \sum_{i>j} \frac{\tau_{ij} - \bar{\tau}_{ij}}{(1 + \tau_{ij})(1 + \bar{\tau}_{ij})}$$

where $\tau_{ij} = \max a_{ik}a_{kj}$, $k / [1, \ldots, n]$ stands for the extended bound of the comparison matrix element $a_{ij}$ and $\bar{\tau}_{ij}$ is the inverse of $\tau_{ij}$. This measure ranges from 0 to 1 and its value increases as the inconsistency of the comparison matrix increases [11].

### 5 Influence of the Outliers on the Estimates of the Weights

The AHP is a method for formalizing decision making where there are limited number of choices but each has a number of attributes and it is difficult to formalize some of those attributes.
The AHP has been used in a large number of applications to provide some structure on a decision making process. Note that the system is somewhat ad-hoc (for example a scale 1-9 range) and there are number of “hidden assumption”.

Furthermore, it is possible to manipulate the rankings to get a preferred outcome (by “lying”). In particular, the method can be influenced by random errors and the values of the ratios of the weights may be exceptionally different from the corresponding consistent value: in this case the statement is called an outlier.

This kind of an outlier has powerful influence on the values of the estimated weights given by the standard eigenvector method and the least square regression method. It has been demonstrated that the robust regression technique is a method which is lesser sensitive to outliers [6] than the eigenvector method.

6 Example: Kangas Data

Consider the following consistent (CR=0) 4x4 AHP-matrix from a real case study [5].

Table 1. A consistent AHP-matrix.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/5</td>
<td>1/7</td>
<td>1/9</td>
<td></td>
</tr>
<tr>
<td>1/5</td>
<td>1</td>
<td>7/5</td>
<td>9/5</td>
<td></td>
</tr>
<tr>
<td>1/7</td>
<td>5/7</td>
<td>1</td>
<td>9/7</td>
<td></td>
</tr>
<tr>
<td>1/9</td>
<td>9/9</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

For such table, all methods considered in this paper, give the same results: 0.6878, 0.1376, 0.0983, 0.0764. Laininen, R.P. Häämäinen [6] call these the correct weights (CW). Let us suppose that the judgment in comparison of the entities number three and four is not the consistent relation 9/7 but it is 7/9 as shown in [6].

Table 2. An inconsistent AHP-matrix.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1/5</td>
<td>1/7</td>
<td>1/9</td>
<td></td>
</tr>
<tr>
<td>1/5</td>
<td>1</td>
<td>7/5</td>
<td>9/5</td>
<td></td>
</tr>
<tr>
<td>1/7</td>
<td>5/7</td>
<td>1</td>
<td>7/9</td>
<td></td>
</tr>
<tr>
<td>1/9</td>
<td>9/9</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

The CR for the data in table 2 is CR=0.12. The results of the estimate with the five methods considered here are shown in table 3.
We can note that the RLOG and R-PLS methods give the same results and are exactly the same of the correct weights.

7 A simulation Study

From the previous example seems that the RLOG and R-PLS regression are the more stable methods in presence of outliers, reaching the same value of the CW. In this paragraph, we compare the two methods using a simulation study. Let consider the data of Table 1. Then we generate from this matrix, adding random errors, new matrices with fixed range of CR (0.00−|0.02; 0.02−|0.06; 0.06−|0.1). We have considered CR value lesser or equal than 0.1 because if CR is greater than 0.10 then a re-examination of the pairwise judgments is recommended until a C.R. less than or equal to 0.10 is achieved. For each range we generate 200 matrices. We show, by introducing a simulation study, how the estimates of the weights calculated by the R-PLS regression is lesser sensitive to the occurrence of the outliers than the RLOG. For each range of CR we compute the mean and the standard deviation of the 200 samples, as shown in tables 4, 5, 6.

Table 4. Comparison of estimated weights between RLOG and R-PLS for simulated matrices with a 0 < CR ≤ 0.02.

<table>
<thead>
<tr>
<th>0 &lt; CR ≤ 0.02</th>
<th>Mean</th>
<th>St. Dev.</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>RLOG</td>
<td>0.691</td>
<td>0.084</td>
<td>R-PLS</td>
<td>0.688</td>
</tr>
<tr>
<td>RLOG</td>
<td>0.128</td>
<td>0.044</td>
<td>R-PLS</td>
<td>0.133</td>
</tr>
<tr>
<td>RLOG</td>
<td>0.080</td>
<td>0.041</td>
<td>R-PLS</td>
<td>0.099</td>
</tr>
<tr>
<td>RLOG</td>
<td>0.099</td>
<td>0.023</td>
<td>R-PLS</td>
<td>0.078</td>
</tr>
</tbody>
</table>

From the data in tables 4, 5, 6, is very clear that the mean of the estimate computed with R-PLS is more closed to the CW than RLOG with a very low variance.
Table 5. Comparison of estimated weights between RLOG and R-PLS for simulated matrices with a $0.02 < CR \leq 0.06$.

<table>
<thead>
<tr>
<th></th>
<th>Mean RLOG</th>
<th>St. Dev. RLOG</th>
<th>Mean R-PLS</th>
<th>St. Dev. R-PLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.699</td>
<td>0.091</td>
<td>0.698</td>
<td>0.055</td>
</tr>
<tr>
<td></td>
<td>0.110</td>
<td>0.085</td>
<td>0.135</td>
<td>0.029</td>
</tr>
<tr>
<td></td>
<td>0.078</td>
<td>0.062</td>
<td>0.094</td>
<td>0.063</td>
</tr>
<tr>
<td></td>
<td>0.111</td>
<td>0.041</td>
<td>0.071</td>
<td>0.022</td>
</tr>
</tbody>
</table>

Table 6. Comparison of estimated weights between RLOG and R-PLS for simulated matrices with a $0.06 < CR \leq 0.1$.

<table>
<thead>
<tr>
<th></th>
<th>Mean RLOG</th>
<th>St. Dev. RLOG</th>
<th>Mean R-PLS</th>
<th>St. Dev. R-PLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>0.699</td>
<td>0.091</td>
<td>0.698</td>
<td>0.055</td>
</tr>
<tr>
<td></td>
<td>0.110</td>
<td>0.085</td>
<td>0.135</td>
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<td>0.022</td>
</tr>
</tbody>
</table>

8 Conclusions

The use of statistical theory is needed in the estimate of the weights for AHP-matrices if there are random fluctuations in the ratios of the relative importance. We have shown, using a simulation study, that R-PLS regression is a more robust technique compared to the other regression methods applied in AHP context. Further investigations on the influence of the outliers on the estimates, considering different range of coefficient ratio index, are in progress.

References