Lifting Structures for Quincunx FIR Filter-Banks With Quadrantal or Diagonal Symmetry

Bhushan D. Patil, Pushkar G. Patwardhan, and Vikram M. Gadre

Abstract—Lifting structures, also referred to as ladder structures, have been used for the design of linear phase (centro-symmetric) 2-D Quincunx filter-banks. In this letter, our aim is to use lifting structures for the construction of Quincunx filter-banks with the filters having the higher order symmetries viz. quadrantal or diagonal symmetries. We derive conditions on the predict and update steps of the lifting structure so that the filters in the filter-bank have quadrantal or diagonal symmetry. Using these conditions, we construct example structures for the predict and update steps which yield Quincunx filter-banks with quadrantly or diagonally symmetric filters.

Index Terms—lifting, Quincunx filter-banks, two-dimensional filter-banks.

I. INTRODUCTION

VARIOUS approaches for the design of 2-D linear phase Quincunx filter-banks have been considered in the literature. Design using cascade structures has been considered in [1], whereas [2]–[8] use lifting structures. Design approaches using the method of generalized McClellan transformation have been presented in [9] and [10]. All these approaches impose the condition of linear phase in the filters of the Quincunx filter-bank which implies that the filter impulse response has centro-symmetry. As observed in [12] and [13], higher order symmetries are possible in the 2-D impulse response (2-D signals in general). Quadrantal and diagonal symmetries are some of the more common symmetries. These symmetries are also particularly useful in the context of Quincunx filter-banks, since the desired ideal passbands of the filters in a Quincunx filter-bank have quadrantal and/or diagonal symmetry. For the case of 2-D Quincunx filter-banks, filters having these symmetries are required to “preserve” signal symmetries in 2-D signal extension schemes [14], [15]. One of the requirements for preserving signal symmetries, as discussed in the above references, is to design Quincunx filter-banks with the filters having quadrantal or diagonal symmetries.

Cascade structures which yield Quincunx filter-banks with the filters having quadrantal or diagonal symmetry have been proposed in [15]. In particular, [15] derives a set of conditions on the polyphase matrix of the Quincunx filter-bank such that the filters in the filter-bank have quadrantal or diagonal symmetry. In this letter, our aim is to consider lifting structures for the design of Quincunx filter-bank with quadrantal or diagonal symmetry. Using the results of [15], we derive a set of conditions on the “lifting steps” (the predict and update steps) which guarantee that the final filters have the desired symmetry.

The rest of this letter is organized as follows: In Section I-A, we review the lifting structure for the Quincunx filter-bank, and in Section I-B, we briefly review the characterization of quadrantal and diagonal symmetries. In Section II-A, we consider quadrantly symmetric Quincunx filter-banks, and we derive conditions on the lifting steps for the filter-bank to have quadrantal symmetry. In Section II-B, we present conditions on the lifting steps for diagonally symmetric Quincunx filter-banks. We present design examples in Section III.

Notation: Boldfaced lowercase letters are used to represent vectors, and boldfaced uppercase letters are used for matrices. $A^T$ denotes the transpose of $A$, $A^{-1}$ denote the inverse of $A$, and $\det(A)$ denotes the determinantal of $A$. A vector $z = [z_0 \ z_1]^T$ raised to a matrix power $A = [A_{00} \ A_{10}; A_{01} \ A_{11}]$ is defined as a vector $z^A$ is a vector whose $i$th entry is $z_0^{A_{0i}} z_1^{A_{1i}}$, where $i = 0, 1$. We use $Q$ to refer to the specific Quincunx sampling matrix $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

A. Review of Quincunx Filter-Bank and the Lifting Structure

Fig. 1 shows the lifting structure for the Quincunx filter-bank. Here, $E(z)$ and $R(z)$ are the analysis and synthesis polyphase matrices of a “base” perfect reconstruction (PR) Quincunx filter-bank. The predict and update lifting steps can be used to increase the order of the polyphase matrix (and thus of the filters) while maintaining PR. The predict and update steps involve adding the following factors to the analysis polyphase matrix:

$E'(z) = \begin{bmatrix} 1 & U(z) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} E(z)$.

The corresponding synthesis matrix is given by
\[ R'(z) = E^{-1}(z) \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -U(z) \end{bmatrix}. \]

We note that multiple predict and/or update steps could be added, possibly with different \( P(z) \) and \( U(z) \) functions, while maintaining PR of the filter-bank. \( E(z) = R(z) = I \) also referred to as the “lazy wavelet,” is a typical choice as the “base filter-bank” before applying the lifting steps. However, in general, any PR filter-bank could be used as the starting point.

In this letter, the problem that we address is as follows: Given a PR Quincunx filter-bank with quadrantly or diagonally symmetric filters, what are the conditions on the predict and update factors, \( P(z) \) and \( U(z) \), respectively, so that the symmetry of the filters in the filter-bank is maintained?

**B. Review of Quadrantal and Diagonal Symmetry Characterization** [14], [15]

A quadrantly symmetric 2-D sequence, with the center of symmetry \( c = [c_1 \ c_2]^T \), can be characterized as [14], [15]:
\[ X(z) = \gamma_1 z^{-2A_1 c} X(z) T_3 = \gamma_2 z^{-2A_2 c} X(z) T_4 = \gamma_3 z^{-2A_3 c} X(z) T_4 = \gamma_4 z^{-2A_4 c} X(z) T_4 \]
where
\[ T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma_3 = \pm 1, \quad \gamma_4 = \pm 1 \]
and
\[ A_i = (I - T_1) \text{ for } i = 1, 2, \ldots, 4. \]

A diagonally symmetric 2-D signal, with center of symmetry \( c = [c_1 \ c_2]^T \), can be characterized as follows [14], [15]:
\[ X(z) = \gamma_1 z^{-2A_1 c} X(z) T_3 = \gamma_4 z^{-2A_4 c} X(z) T_4 = \gamma_3 z^{-2A_3 c} X(z) T_4 = \gamma_4 z^{-2A_4 c} X(z) T_4 \]
where
\[ T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_3 = \pm 1, \quad \gamma_4 = \pm 1 \]
and
\[ A_i = (I - T_1) \text{ for } i = 3, 4. \]

**II. QUADRANTALLY SYMMETRIC QUINCUNCX FILTER-BANKS**

In this section, we derive conditions on the lifting steps so that the filters in the Quincunx filter-bank have quadrantal or diagonal symmetry. We consider quadrantal symmetry in Section II-A and diagonal symmetry in Section II-B.

**A. Conditions for Quadrantly Symmetric Quincunx Filter-Banks**

From [15, Proposition-1], the conditions on the polyphase analysis for a quadrantly symmetric Quincunx filter-bank are as follows.

**Proposition-1: [15]:** Let the analysis polyphase matrix be \( E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} \). Then, \( E_{00}(z), E_{01}(z), E_{10}(z), E_{11}(z) \) should have diagonal symmetry with symmetry parameters \( \gamma_s = 1 \) (i.e., with symmetry, and not anti-symmetry), and should have symmetry as \( d_0, d_0 \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}^T, d_1, \) and \( d_1 \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}^T \) respectively, and satisfying the following constraints:

1. \( \det(E(z)) = z^r \), where \( r \) is an arbitrary integer vector, and
2. \( d_0 + d_1 = r = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}^T. \)

Now, our aim is as follows: Suppose we have a polyphase matrix \( E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} \) satisfying the conditions of Proposition-1. We now wish to increase the order by using a lifting step (either a predict or an update step). The question we want to address is: What are the conditions on the lifting steps such that the new polyphase matrix obtained after the lifting step also satisfies Proposition-1?

First consider the predict step. The new polyphase matrix obtained after the predict step is:
\[ E'(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} E(z). \] 

Note that the predict-function \( P(z) \) is a 2-D transfer function. From (1), it is evident that
\[ \det(E'(z)) = \det(E(z)) = z^r. \]

Further, we have from (1)
\[ E'(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}'(z) & E_{11}'(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) - P(z) E_{00}(z) & E_{11}(z) - P(z) E_{01}(z) \end{bmatrix}. \]

Now, first of all, we would like \( E_{10}'(z) \) and \( E_{11}'(z) \) to be both diagonally symmetric. This can be achieved as follows.

Let \( P(z) \) be diagonally symmetric. We will denote the center of symmetry (COS) of \( P(z) \) as \( \text{COS}(P(z)) \). From (3), we have \( \text{COS}(E_{10}'(z)) = \text{COS}(E_{10}(z) - P(z) E_{00}(z)) \). We would like to have \( \text{COS}(E_{10}'(z)) = \text{COS}(E_{10}(z)) \). We can achieve this if we can ensure that
\[ \text{COS}(P(z) \cdot E_{00}(z)) = \text{COS}(E_{10}(z)) \]
But \( \text{COS}(P(z) \cdot E_{00}(z)) = \text{COS}(E_{00}(z)) + \text{COS}(P(z)) \).

Thus, we require that
\[ \text{COS}(E_{00}(z)) + \text{COS}(P(z)) = \text{COS}(E_{10}(z)) \]
but, since the above equation satisfies Proposition-1, we have \( \text{COS}(E_{00}(z)) = d_0 \) and \( \text{COS}(E_{10}(z)) = d_1 \).

Thus, we choose
\[ \text{COS}(P(z)) = d_1 - d_0. \]

With this, we have
\[ \text{COS}(E_{10}'(z)) = \text{COS}(E_{10}(z)). \]

It can be verified that with the choice of \( P(z) \) as in (4), we also have
\[ \text{COS}(E_{11}'(z)) = \text{COS}(E_{11}(z)). \]

From (3)–(5), it follows that \( E(z) \) also satisfies Proposition-1.

**Lemma-1:** Given a polyphase matrix \( E(z) \) satisfying Proposition-1, the new polyphase matrix \( E'(z) \) obtained by the predict lifting step as \( E'(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} E(z) \) will also satisfy
Proposition-1 if the predict function \( P(z) \) is chosen to be diagonally symmetric with center of symmetry as \( \text{COS}(P(z)) = d_1 - d_0 \).

Now consider the update step: The new polyphase matrix obtained after the predict step is

\[
E'(z) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} E(z). \tag{7}
\]

Again, the update function, \( U(z) \), is a 2-D transfer function.

From (7), it is evident that

\[
\text{det}(E'(z)) = \text{det}(E(z)) = z^r. \tag{8}
\]

Further, we have from (7)

\[
E'(z) = \begin{bmatrix}
E_{0,0}^r(z) & E_{0,1}^r(z) \\
E_{1,0}^r(z) & E_{1,1}^r(z)
\end{bmatrix} = \begin{bmatrix}
E_{0,0}(z) + U(z)E_{1,0}(z) & E_{0,1}(z) + U(z)E_{1,1}(z) \\
E_{1,0}(z) & E_{1,1}(z)
\end{bmatrix}. \tag{9}
\]

We would like \( E_{0,0}^r(z) \) and \( E_{0,1}^r(z) \) to be both diagonally symmetric. Following similar arguments as done for the predict step, we can achieve this by having

\[
\text{COS}(U(z) \cdot E_{1,0}(z)) = \text{COS}(E_{0,0}(z))
\Rightarrow \text{COS}(U(z)) = d_0 - d_1. \tag{10}
\]

It can be verified that, with (10), we have

\[
\text{COS}(E_{0,0}^r(z)) = \text{COS}(E_{0,0}(z))
\]
and

\[
\text{COS}(E_{0,1}^r(z)) = \text{COS}(E_{0,1}(z)). \tag{11}
\]

Thus, from (8) and (11), it follows that \( E(z) \) also satisfies Proposition-1. We can summarize this as follows.

**Lemma-2:** Given a polyphase matrix \( E(z) \) satisfying proposition-1, the new polyphase matrix \( E'(z) \) obtained by the update step is

\[
E'(z) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} E(z)
\]

Proposition-1 if the update function \( U(z) \) is chosen to be diagonally symmetric with center of symmetry as \( \text{COS}(U(z)) = d_0 - d_1 \).

**B. Conditions for Diagonally Symmetric Quincunx Filter-Banks**

In this section, we present conditions on the lifting steps for Quincunx filter-banks with diagonal symmetry. For diagonally symmetric Quincunx filter-banks, the analysis polyphase matrix should satisfy [15, Proposition-2].

**Lemma-3:** Given a polyphase matrix \( E(z) \) satisfying [15, Proposition-2], the new polyphase matrix \( E'(z) \) obtained by the predict lifting or update steps as \( E'(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E(z) \), or

\[
E''(z) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} U(z) \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} E(z),
\]

will also satisfy [15, Proposition-2] if the lifting steps satisfy the following conditions.

a) The predict function \( P(z) \) is quadrantly symmetric with center of symmetry as \( \text{COS}(P(z)) = d_1 - d_0 \).

b) The update function \( U(z) \) is quadrantly symmetric with center of symmetry as \( \text{COS}(U(z)) = d_0 - d_1 \).

**III. DESIGN EXAMPLES**

We now present examples of lifting steps which satisfy the conditions derived in Section II and which can be used to construct Quincunx filter-banks with quadrantal or diagonal symmetry.

**Example-1:** In this example, we construct lifting steps for a quadrantly symmetric Quincunx filter-bank. As the “base filter-bank,” we start with the PR quadrantly symmetric Quincunx filter-bank from [15]. We then give examples of lifting predict and update steps which can be used to increase the order of the polyphase matrix, while maintaining PR and symmetry. From [15, Section III-A], we have the following polyphase matrix:

\[
E_{0,0}(z) = a + h \left( z_1^{-1}z_2 + z_1z_2^{-1} \right) + g \left( z_1z_2 + z_1^{-1}z_2^{-1} \right)
\]

\[
E_{0,1}(z) = c(z_1 + z_2) + b(1 + z_1z_2)
\]

\[
E_{1,0}(z) = d \left( 1 + z_1^{-1}z_2^{-1} \right) + e \left( z_1^{-1} + z_2^{-1} \right), \quad E_{1,1}(z) = f.
\]

Here \( a, b, c, d, e, f, g \), and \( h \) are parameters subject to some constraints [15]. For this polyphase matrix we have

\[
d_0 = \begin{bmatrix} 0 \\
0
\end{bmatrix}, \quad d_1 = -\begin{bmatrix} 1/2 \\
1/2
\end{bmatrix}, \quad r = \begin{bmatrix} 0 \\
0
\end{bmatrix}.
\]

We note that the above \( E(z) \) includes the identity matrix as a special case with \( a = h = 1 \) and \( b = c = d = e = f = g = 0 \). Thus, from Lemma-1, we should choose a diagonally symmetric \( P(z) \) such that

\[
\text{COS}(P(z)) = -\begin{bmatrix} 1/2 \\
1/2
\end{bmatrix} = d_1 - d_0.
\]

A possible choice of \( P(z) \) is

\[
P(z) = u \left( 1 + z_1^{-1}z_2^{-1} \right) + v \left( z_1^{-1} + z_2^{-1} \right)
\]

where \( u \) and \( v \) are parameters.

With this, the new polyphase matrix is given by

\[
E'(z) = \begin{bmatrix}
1 & 0 \\
-\text{P}(z) & 1
\end{bmatrix} E(z) = \begin{bmatrix}
E_{00}(z) & E_{01}(z) \\
E_{10}^r(z) & E_{11}^r(z)
\end{bmatrix}
\]

where

\[
E_{00}^r(z) = (d - au - gv) \left( 1 + z_1^{-1}z_2^{-1} \right) + (e - av - hw) \left( z_1^{-1} + z_2^{-1} \right) - \left( ub \right) \left( z_1^{-1}z_2 + z_1z_2^{-1} + z_1^{-1} + z_2^{-1} \right) - \left( gu \right) \left( z_1 + z_2 + z_1^{-1}z_2^{-1} + z_1^{-1} + z_2^{-1} \right)
\]

\[
E_{10}^r(z) = \left( f - 2ub - 2cv \right) \left( z_1^{-1}z_2 + z_1z_2^{-1} \right) - \left( wc + bv \right) \left( z_1 + z_2 + z_1^{-1} + z_2^{-1} \right) - \left( cv \right) \left( z_1^{-1}z_2 + z_1z_2^{-1} \right) - \left( ub \right) \left( z_1^{-1} - z_2^{-1} \right).
\]

It can be verified that \( E'(z) \) satisfies Proposition-1, and this gives quadrantal symmetric analysis filters.

With the above predict step, we increased the order of resulting analysis highpass \( H_1(z) \), while maintaining quadrantal symmetry and PR. To increase the order of resulting lowpass filter \( H_0(z) \), we can use the following update stage:

\[
U(z) = s(z_1 + z_2) + t(z_1z_2)
\]

\[
\text{COS}(U(z)) = \begin{bmatrix} 1/2 \\
1/2
\end{bmatrix} = d_0 - d_1.
\]

So this choice of the update step satisfies Lemma-2.
used only a single predict and update step. Further, the symmetry aspect of the filters is not explicitly addressed in [3].

IV. CONCLUSION

In this letter, we presented conditions on the lifting steps to design Quincunx filter-banks with quadrantal or diagonal symmetry. In particular, we showed the following:

a) to design quadrantly symmetric Quincunx filter-banks, the lifting steps should have diagonal symmetry with a constraint on their center of symmetry; and

b) to design diagonally symmetric Quincunx filter-banks, the lifting steps should have quadrantal symmetry with a constraint on their center of symmetry.

We presented examples of quadrantly and diagonally symmetric Quincunx filter-banks, by using lifting steps satisfying the conditions derived in this letter.

REFERENCES