The significance of Aristotle’s particularisation in the foundations of mathematics, logic and computability

Cohen and the Axiom of Choice

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Abstract—The logic underlying our current interpretations of all first-order formal languages—which provide the formal foundations for all computing languages—is Aristotle’s logic of predicates. I show, first, that a fundamental tenet of this logic, namely Aristotlean particularisation, is a subjective, and objectively unverifiable, postulation that is ‘stronger’ than the Axiom of Choice; and that, second, any putative model of ZF which appeals to Aristotlean particularisation—under Tarski’s definitions of what constitute’s an interpretation of the language—is not sound.

Keywords: Aristotle, Cohen, \(\varepsilon\)-function, Hilbert, particularisation.

1. Introduction

Classical theory—following the 2000-year old philosophical perspective of Greek philosophers such as Aristotle—asks only that the ‘core’ axioms of a formal language, and its rules of inference, be interpretable—under Tarski’s definitions of the satisfiability, and truth, of the formulas of a formal language under an interpretation—as self-evidently ‘sound’ propositions.

The subjective—and essentially irresolvable—element in the ‘soundness’ of such a viewpoint is self-evident.

In these investigations, I highlight the limitations of such subjectivity, and, in the case of the Peano Arithmetic, PA, show how to avoid them by requiring that the ‘core’ axioms of PA, and its rules of inference, be interpretable as algorithmically (and, ipso facto, objectively) verifiable ‘sound’ propositions, and consider some consequences.

1.1 Preliminary Definitions

**Aristotlean particularisation:** This holds that an assertion such as, ‘There exists an unspecified \(x\) such that \(F(x)\) holds’—usually denoted symbolically by ‘\((\exists x)F(x)\)’—can always be validly inferred in the classical, Aristotlean, logic of predicates ([HA28], pp.58-59) from the assertion, ‘It is not the case that, for any given \(x\), \(F(x)\) does not hold’—usually denoted symbolically by ‘\(\neg(\forall x)\neg F(x)\)’.

**Simple consistency:** A formal system \(S\) is simply consistent if, and only if, there is no \(S\)-formula \([F(x)]\) for which both \([(\forall x)F(x)]\) and \(\neg(\forall x)F(x)\) are \(S\)-provable.

\(\omega\)-**consistency:** A formal system \(S\) is \(\omega\)-consistent if, and only if, there is no \(S\)-formula \([F(x)]\) for which, first, \([\neg(\forall x)F(x)]\) is \(S\)-provable and, second, \([F(a)]\) is \(S\)-provable for any given \(S\)-term \([a]\).

**Soundness:** A formal system \(S\) is sound under an interpretation \(I_S\) if, and only if, every theorem \([T]\) of \(S\) translates as ‘\([T]\)’ is true under \(I_S\)’—under Tarski’s definitions of the satisfaction, and truth, of the formulas of a formal language under an interpretation ([Ta33]).

**Axiom of Choice** (a standard interpretation): Given any set \(S\) of mutually disjoint non-empty sets, there is a set \(C\) containing a single member from each element of \(S\).

2. Hilbert’s formalisation of Aristotle’s particularisation

Now, a fundamental tenet of classical logic—unrestrictedly adopted by formal first-order predicate calculus as axiomatic (see [Hi25], p.382; [HA28], p.48; [SC28], p.515; [Be59], pp.178 & 218; [Co66], p.4)—is Aristotlean particularisation.

In a 1927 address, Hilbert reviewed, as part of his ‘proof theory’, his axiomatisation \(L_\varepsilon\) of classical Aristotlean predicate logic as a formal first-order \(\varepsilon\)-predicate calculus ([Hi27], pp.465-466)—in which he introduced a primitive choice-function symbol, ‘\(\varepsilon\)’, for formalising the existence of the unspecified object in Aristotle’s particularisation ([Hi25], p.382; [Ca62], p.156; see also Appendix A):

‘\(\ldots\varepsilon(A)\) stands for an object of which the proposition \(A(a)\) certainly holds if it holds of any object at all’ . . .

Hilbert showed, moreover, how the quantifiers ‘\(\forall\)’ and ‘\(\exists\)’ are definable using the choice-function ‘\(\varepsilon\)’ (see Appendix A)—and noted that ([Hi27], p.475):

‘. . .The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds.’

More precisely (cf. [Hi25], pp.382-383; [Hi27], p.466(1)): 

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Lemma 1: \( L_\varepsilon \) adequately expresses—and yields, under a suitable interpretation—Aristotle’s logic of predicates if the \( \varepsilon \)-function is interpreted so as to yield the unspecified object in Aristotelian particularisation.

What came to be known later as Hilbert’s Program—which was built upon Hilbert’s ‘proof theory’—can be viewed as, essentially, the subsequent attempt to show that the formalisation was also necessary for communicating Aristotle’s logic of predicates effectively and unambiguously under any interpretation of the formalisation. This goal is implicit in Hilbert’s remarks ([Hi25], p.384; [Hi27], p.475):

“Mathematics in a certain sense develops into a tribunal of arbitration, a supreme court that will decide questions of principle—and on such a concrete basis that universal agreement must be attainable and all assertions can be verified.”

“...a theory by its very nature is such that we do not need to fall back upon intuition or meaning in the midst of some argument.”

2.1 Aristotle’s particularisation is ‘stronger’ than the Axiom of Choice

The difficulty in attaining this goal constructively along the lines desired by Hilbert—in the sense of the above quotes—lies in the fact that, as Rudolf Carnap emphasised in a 1962 paper, “On the use of Hilbert’s \( \varepsilon \)-operator in scientific theories” ([Ca62], pp.157-158; see also Wang’s remarks [Wa63], pp.320-321):

Lemma 2: The Axiom of Choice is derivable as a theorem in the Zermelo-Fraenkel set theory \( \text{ZF}_\varepsilon \), where the \( \varepsilon \)-function serves to formalise the concept of an unspecified object, and the quantifiers are defined (see Appendix A) in terms of the \( \varepsilon \)-function.

2.1.1 Any interpretation of \( \text{ZF} \) which appeals to Aristotle’s particularisation is not sound

The significance of Hilbert’s \( \varepsilon \)-function—interpreted according to Aristotle’s logic of predicates, is a model of \( \text{ZF}_\varepsilon \)—appears to simply denote the concept of an unspecified object, and the quantifiers are defined (see Appendix A) in terms of the \( \varepsilon \)-function.

Corollary 1: Any model of \( \text{ZF}_\varepsilon \), in which the quantifiers are interpreted according to Aristotle’s logic of predicates, is a model of \( \text{ZF}_\varepsilon \) if the expression \( \varepsilon x B(x) \) is interpreted to yield Cohen’s symbol \( \exists \) which denotes an unspecified value of \( x \) for which \( B(\exists) \) is true.

Corollary 2: \( \exists \) has no model that appeals to Aristotle’s particularisation.

We cannot, therefore, conclude that the Axiom of Choice is essentially independent of set theory such as \( \text{ZF} \).

Now, Cohen’s argument—in common with the arguments of many important theorems in standard texts on the foundations of mathematics and logic—appeals to the unspecified object in Aristotle’s particularisation when interpreting the existential axioms of \( \text{ZF} \) (or statements about \( \text{ZF} \) ordinals) in the proof and application of the—seemingly paradoxical (see Skolem’s remarks [Sk22], p.295; also [Co66], p.19)—(downwards) Löwenheim-Skolem Theorem

(see, for instance, Cohen’s proof of this theorem in [Co66], p.19) for legitimising putative models of a language (such as the standard model ‘M’ ([Co66], p.19 & p.82) of \( \text{ZF} \) and its forced derivative ‘N’ ([Co66], p.121), in Cohen’s argument ([Co66], p.83 & p.112-118)).

(Downwards) Löwenheim-Skolem Theorem ([Lo15], p.245, Theorem 6; [Sk22], p.293): If a first-order proposition is satisfied in any domain at all, then it is already satisfied in a denumerably infinite domain.

The significance of Hilbert’s formalisation of Aristotle’s particularisation by means of the \( \varepsilon \)-function is seen in Cohen’s following remarks, where he explicitly appeals to a catalytic—rather than formal—definition of the unspecified object in Aristotle’s particularisation ([Co66], p.112; see also p.4):

“When we try to construct a model for a collection of sentences, each time we encounter a statement of the form \( \exists x B(x) \) we must invent a symbol \( \exists \) and adjoin the statement \( B(\exists) \) . . . when faced with \( \exists x B(x) \), we should choose to have it false, unless we have already invented a symbol \( \exists \) for which we have strong reason to insist that \( B(\exists) \) be true.”

Since Hilbert’s \( \varepsilon \)-function formalises precisely this concept of \( \exists \) as \( \varepsilon \)-function, it follows that:

Theorem 1: Any model of \( \text{ZF} \), in which the quantifiers are interpreted according to Aristotle’s logic of predicates, is a model of \( \text{ZF}_\varepsilon \) if the expression \( \varepsilon x B(x) \) is interpreted to yield Cohen’s symbol \( \exists \) which denotes an unspecified value of \( x \) for which \( B(\exists) \) is true.

Hence Cohen’s argument is also applicable to \( \text{ZF}_\varepsilon \). However, since \( \text{ZF}_\varepsilon \) proves the Axiom of Choice ([Ca62], pp.157-158; see also Wang’s remarks [Wa63], pp.320-321), Cohen’s argument ([Co63] & [Co64] & [Co66])—when applied to \( \text{ZF}_\varepsilon \)—actually shows that:

Corollary 1: \( \text{ZF}_\varepsilon \) has no model that appeals to Aristotle’s particularisation.

Corollary 2: \( \text{ZF} \) has no model that appeals to Aristotle’s particularisation.

2.2 The Continuum Hypothesis

Moreover, the belief in the independence of the Continuum Hypothesis (and hence of the Axiom of Choice) is also questionable on other grounds.

Definitions

The Continuum Hypothesis ([Co66], p.67):

\[ c = \aleph_1 \], where \( c \) is the cardinality of the set
of real numbers, and ℵ₀—the immediately larger cardinal than ℵ₀, the cardinality of the natural numbers—is the cardinality of the constructive ordinals.

The constructive ordinals (cf. [Be59], pp.374-375; [Hi25], p.374): The constructive ordinals are 0, 1, 2, ..., ω, (ω +₁ 1₀), (ω +₁ 2₀), ..., ω₂₀, (ω²₀ +₁ 1₀), ..., ω²₀, ..., ω₃₀, ..., ω₄₀, ..., ω₅₀, ..., <ₐ₀ ϵ₀.

(Here, 0, 1, 2, ..., denote the finite ordinals that correspond 1-1 to the natural numbers 0, 1, 2, ..., and +₁ and <ₐ denote ordinal addition and ordinal inequality respectively.)

For instance, if Ordinal Arithmetic is consistent, then the members of the set of real numbers between 0 and 1—when each real number in the set is expressed in its unique binary form 0.a₁a₂ ... , with aᵢ either 0 or 1—are clearly in some (non-algorithmic) 1-1 correspondence with the members of the proper subset of the constructive ordinals, ((ω¹₀a₁₀ +₁ ω²₀a₂₀ +₁ +₁ ω₅₀)).

(Here, as indicated, each ordinal in the subset is expressed in its unique Cantor normal form, with aᵢ₀ denoting the finite ordinal that corresponds to the natural number denoted by aᵢ.)

Further, the ordinals upto ϵ₀ are constructive, and there is no cardinal number between ℵ₀ (which is defined as the cardinality of the set of finite ordinals) and ℵ₁ (which is defined as the cardinality of the set of constructive ordinals).

Now, ℵ₀—the cardinality of the set of real numbers—is the cardinality of the set of number-theoretic functions of an integral argument whose values are also finite integers (see [Hi25], p.384); and it is also the cardinality of the set of real numbers between 0 and 1. We thus have the curious conclusion that:

**Theorem 2:** In any sound interpretation of Ordinal Arithmetic, ℵ₀ = ℵ₁.

**Proof:** In any sound interpretation of Ordinal Arithmetic, \(\{ (\omega^{₁₀a₁₀ +₁ ω²₀a₂₀ +₁ +₁ ω₅₀) } \) ∈ ω<₁₀ and ω<₂₀ <ₐ₀ ϵ₀.

Now, if Ordinal Arithmetic is consistent then, by Gödel’s completeness theorem, this implies that the Continuum Hypothesis, too, is expressible as a theorem in \(\Pi^1\).

**Gödel’s Completeness Theorem:** In any first-order predicate calculus, the theorems are precisely the logically valid well-formed formulas (i.e., those that are true in every model of the calculus).

This, of course, is at variance with the accepted belief that the Continuum Hypothesis is not provable in Ordinal Arithmetic; a belief that—in no small measure—can be attributed to the point of view Cohen expressed at the conclusion of his lectures on “Set Theory and the Continuum Hypothesis”, delivered at Harvard University in the spring term of 1965.

Thus, whilst, essentially, considering the argument sketched out above, Cohen remarked ([Co66], p.151): “We close with the observation that the problem of CH is not one which can be avoided by not thinking in type to sets of real numbers. A similar undecidable problem can be stated using only the real numbers. Namely, consider the statement that every real number is constructible by a countable ordinal. Instead of speaking of countable ordinals we can speak of suitable subsets of ω. The construction \(α \to F_α\) for \(α \leq α₀\), where \(α₀\) is countable, can be completely described if one merely gives all pairs (α, β) such that \(F_α \in F_β\).

This in turn can be coded as a real number if one enumerates the ordinals. In this way one only speaks about real numbers and yet has an undecidable statement in ZF. One cannot push this farther and express any of the set-theoretic questions that we have treated as statements about integers alone. Indeed one can postulate as a rather vague article of faith that any statement in arithmetic is decidable in “normal” set theory, i.e., by some recognizable axiom of infinity. This is of course the case with the undecidable statements of Gödel’s theorem which are immediately decidable in higher systems.”

Curiously, Cohen appears to assert here that if ZF is consistent, then we can “see” that the Continuum Hypothesis is subjectively true for the integers under some model of ZF, but—and along with the Generalised Continuum Hypothesis—we cannot objectively “assert” it to be true for the integers since it is not provable in ZF, and hence not true in all models of ZF.

However, by this argument, Gödel’s undecidable arithmetical propositions, too, can be ‘seen’ to be subjectively true for the integers in the standard model of PA, but cannot be ‘asserted’ to be true for the integers since the statements are not true in an \(ω\)-consistent PA, and hence they are not true in all models of an \(ω\)-consistent PA!

The latter is, essentially, John Lucas’ well-known Gödelian argument ([Lu61]), forcefully argued by Roger Penrose in his popular expositions, ‘Shadows of the Mind’ ([Pe94]) and ‘The Emperor’s New Mind’ ([Pe90]).

As I have argued in The Reasoner ([An07a]; [An07b]; [An07c]), the argument is plausible, but unsound. It is based on a misinterpretation—of what Gödel actually proved formally in his 1931...
paper—for which, moreover, neither Lucas nor Penrose ought to be taken to account ([An07b]; [An07d]).

The distinction sought to be drawn by Cohen is curious, since we have shown that his argument—which assumes that sound interpretations of ZF can appeal to Aristotle’s particularisation—actually establishes that sound interpretations of ZF cannot appeal to Aristotle’s particularisation: just as I show in a companion paper ([An09a], where I note that, if a Peano Arithmetic has an interpretation that presents and Aristotle’s logic of particularisation, then it cannot appeal to Aristotle’s particularisation.

Loosely speaking, the cause of the undecidability of the Continuum Hypothesis, and of the Axiom of Choice, in ZF as shown by Cohen, and that of Gödel’s undecidable proposition in Peano Arithmetic, is common; it is an interpretation of the existential quantifier under an interpretation as Aristotle’s particularisation.

In Cohen’s case, such interpretation is made explicitly and unrestrictedly in the underlying predicate logic ([Co66], p.4) of ZF, and in its interpretation in Aristotle’s logic of predicates ([Co66] p.112).

In Gödel’s case it is made explicitly—but formally to avoid attracting intuitionistic objections—through his specification of what he believed to be a ‘much weaker assumption’ of ω-consistency for his formal system P of Peano Arithmetic ([Go31], p.9 & pp.23-24).

3. Appendix A: Hilbert’s interpretation of quantification

Hilbert interpreted quantification in terms of his ε-function as follows ([Hi27], p.466):

IV. The logical ε-axiom

13. A(a) → A(ε(A))

Here ε(A) stands for an object of which the proposition A(a) certainly holds if it holds of any object at all; let us call ε the logical ε-function.

... 1. By means of ε, “all” and “there exists” can be defined, namely, as follows:

(i) (∀a)A(a) ↔ A(ε¬A))

(ii) (∃a)A(a) ↔ A(εA)) ...

On the basis of this definition the ε-axiom IV(13) yields the logical relations that hold for the universal and the existential quantifier, such as:

(∀a)A(a) → A(b) ... (Aristotle’s dictum), and:

¬((∀a)A(a)) → (∃a)(¬A(a)) ...(principle of excluded middle).

Thus, Hilbert’s interpretation of universal quantification — defined in (i) — is that the sentence (∀x)F(x) holds (under a consistent interpretation I) if, and only if, F(a) holds whenever ¬F(a) holds for any given a (in I); hence ¬F(a) does not hold for any a (since I is consistent), and so F(a) holds for any given a (in I).

Further, Hilbert’s interpretation of existential quantification — defined in (ii) — is that (∃x)F(x) holds (in I) if, and only if, F(a) holds for some a (in I).

References


References:


