Optimization-based digital redesign of analogue controllers

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Abstract—A novel method is proposed for the digital redesign of analogue controllers with account for the closed-loop system performance in continuous time. The method, which is based on a two-level optimization algorithm, makes it possible to place the closed-loop poles inside a specified region of the complex plane and provides for reduced-order controllers. The effectiveness of the proposed technique is demonstrated by some well-known redesign examples.

Index Terms—Sampled-data systems, Digital redesign, Modal control, Parametric design, Optimization, Numerical methods

I. INTRODUCTION

Digital redesign of analogue controllers is one of the most popular methods used by engineers for designing digital control laws for continuous-time plants. Let there exist a continuous-time system composed of an analogue plant \( P \) and controller \( K_c \) (Fig. 1a).

![Diagram](https://example.com/diagram.png)

Fig. 1. Redesign problem: a) original system; b) redesigned system

It is required to exchange the controller for a digital device, which incorporates an anti-aliasing filter \( F_a \), sampling unit with period \( T \), digital filter \( K \), and a hold circuit \( H \) (Fig. 1b).

In the simplest cases, the controller can be discretized independently of the other elements of the system. Various methods of this group (backward difference, forward difference, step and ramp invariant transformations, pole-zero mapping, triangular hold equivalence, etc.) are described in [1], [2], [3]. The most popular of them is based on applying the Tustin transform, which reduces to the substitution \( s = \frac{2(z - 1)}{T(z + 1)} \) in the transfer function of the analogue controller. A generalization of this approach is presented in [4].

Nevertheless, when the sampling period increases, such an approach often yields unacceptable results and may lead to an unstable system. Better redesign methods take into account the closed-loop system properties. For example, it is necessary to ensure its stability and good performance recovery.

The first redesign methods taking into account closed-loop properties were published in [5], [6] for state-feedback control systems. In [7], Kennedy and Evans proposed a solution to the redesign problem for output feedback systems in form of a two-block digital controller.

Most of modern approaches call for solving certain optimization problems of the form

\[
K_*(\zeta) = \arg \min_{K \in \mathcal{K}} J(K),
\]

where \( K(\zeta) \) denotes the discrete transfer function of the digital controller (written as a function of the backward shift operator \( \zeta = e^{-sT} \)), \( \mathcal{K} \) is a set of admissible controllers, and \( J(K) \) is a cost functional. A method based on fast sampling and \( H_\infty \)-optimization technique was developed by Keller and Anderson [8], [9]. Rattan [10], [11] proposed to design a digital controller such that the frequency responses of the original and redesigned systems are close in a given frequency range. The plant input mapping approach was developed in [12], [13], [14]. Zhang and Chen proved that the \( L_p \) performance of their approach converges to the continuous system performance, when the sampling period tends to zero [15].

Traditionally, the most popular criteria of performance recovery are connected with the step response. The problem of minimizing the integral quadratic error between the responses of the original and redesigned systems has been solved in [16], [17] using lifting technique, and in [18] with the help of the parametric transfer function concept [19]. These methods ensure the stability of the redesigned system and good performance recovery, but the controller order appears to be too high. Moreover, in some cases a stabilizing optimal controller does not exist, because the strictly optimal system is marginally stable [19].

In the present paper a novel digital redesign method is proposed, which is also based on the integral quadratic error criterion. As distinct from [16], [17], [20], [18], the new approach makes it possible to place the closed-loop poles \( \eta_i (i = 1, \ldots, m) \) in a specified region, and to restrict the controller order. Thus, the set \( \mathcal{K} \) is defined as

\[
\mathcal{K} = \{ K : \eta_i \in \Omega (i = 1, \ldots, m); \text{ord } K \leq \ell \} ,
\]
where $\Omega$ is a subregion of the stability region, and \texttt{ord} denotes the order of a rational function (maximum of its numerator or denominator degree).

An analytical solution to the problem (1)–(2) is not known to the authors. The contribution of the paper is twofold. Firstly, we propose a numerical solution based on the two-level optimization procedure of [21] and demonstrate its efficiency on well-known examples. Secondly, to enhance the effectiveness of the search algorithm, for typical modal constraints, we parameterize the set of admissible characteristic polynomials $\Delta$ in terms of a parameter vector, whose components variate independently in $[0,1]$. This parametrization makes it possible to apply the information algorithm of global optimization in a hyperrectangle [22], [23], [24].

The paper is organized as follows. In Sec. 2 the redesign problem is formulated as an optimization problem for a standard sampled-data system. The two-level optimization procedure proposed by the authors in [21] is briefly described in Sec. 3. The new parametrization of the sets of admissible controllers for typical modal constraints are presented in Sec. 4. Finally, the efficiency of the method is demonstrated in Sec. 5 by numerical examples suggested by Katz and Rattan.

II. OPTIMIZATION PROBLEM

Introduce the error signal $\varepsilon(t) = \varphi(t) - \varphi_c(t)$, where $\varphi(t)$ and $\varphi_c(t)$ are, respectively, the responses of the redesigned and original systems to a unit step, having the Laplace transform $R(s) = 1/s$. The block-diagram of the system forming the error $\varepsilon(t)$ is shown in Fig. 2, where $w(t)$ is the impulse input signal (Dirac’s delta-function $\delta(t)$).

The optimal redesign problem is to find a controller $K \in \mathcal{K}$ that minimizes the integral quadratic error (squared $\mathcal{L}_2$-norm of the signal $\varepsilon(t)$):

$$ J = \|\varepsilon\|_2^2 = \int_0^\infty \varepsilon^2(t) \, dt \rightarrow \min . \quad (3) $$

If required, the integral of squared control error $\varepsilon_u(t) = u(t) - u_c(t)$ with some weight can also be incorporated in the cost function (see [17]). But this extension changes nothing in principle, and hereinafter for simplicity we use the functional (3).

The system in Fig. 2 is described by the following operator equations:

\begin{equation}
\begin{align*}
\varepsilon &= -Q(s) R(s) w + P(s) u , \\
y &= F_a(s) R(s) w - F_a(s) P(s) u ,
\end{align*}
\end{equation}

where $y$ denotes the sampled signal, and

$$ Q(s) = \frac{P(s)K_c(s)}{1 + P(s)K_c(s)} $$

is the transfer function of the closed-loop analogue system. These equations define a standard sampled-data system [16], which is shown in Fig. 3. Here $G$ denotes a generalized plant associated with (4).

Thus, the redesign problem reduces to the minimization of the $\mathcal{L}_2$-norm of the output of a standard sampled-data system under the input $w(t) = \delta(t)$.

III. TWO-LEVEL OPTIMIZATION PROCEDURE

The search procedure proposed in [21] is based on a parameterization of the set of $\ell$-th order controllers associated with a fixed characteristic polynomial $\Delta$.

It is known that stability of the closed loop in Fig. 3 is determined by the discretized model of the block $G_{22}(s) = F_a(s) P(s)$ with a hold $H$:

$$ D_{22}(\zeta) = D_{G_{22}H}(T, \zeta, 0) \quad (5) $$

and the transfer function of the controller $K(\zeta)$. In (5), $D_F(T, \zeta, t)$ denotes the discrete Laplace transform for the function $F(s)$ [19], and $H(s)$ stands for the transfer function of the hold.

The functions $D_{22}(\zeta)$ and $K(\zeta)$ can be written as ratios of coprime polynomials

$$ D_{22}(\zeta) = \frac{n(\zeta)}{d(\zeta)} , \quad K(\zeta) = \frac{a(\zeta)}{b(\zeta)} . $$

By $|x|$ we will denote the degree of a polynomial $x$. The number $p = \text{ord} D_{22} = \max\{ |a|, |d| \}$ will be called the order of the plant, and $\text{ord} K = \max\{ |a|, |b| \}$ is the order of the controller.

The characteristic polynomial of the closed loop has the form\footnote{Hereinafter function arguments are often omitted for brevity.}

$$ \Delta(\zeta) = an + bd . \quad (6) $$

For a fixed $\Delta$, equation (6) can be considered as a polynomial equation with respect to the unknown polynomials $a(\zeta)$ and $b(\zeta)$. A corresponding controller $K(\zeta)$ will be called a $\Delta$-controller.

For $|d| \geq |n|$, we define polynomials $a_0(\zeta)$ and $b_0(\zeta)$ as the solution of (6) with $a_0$ of minimal degree. Otherwise, if $|d| < |n|$, these polynomials are defined as the solution of (6) with $b_0$ of minimal degree.
Hereinafter we assume that $\ell \geq p - 1$, therefore $|a_0| \leq \ell$ and $|b_0| \leq \ell$ [25]. As follows from [26], [21], the set of all $\Delta$-controllers with ord $K \leq \ell$ is parametrized as
\[
K(\zeta) = \frac{a_0 + d\xi}{b_0 - n\xi}, \tag{7}
\]
where $\xi(\zeta)$ is zero or a polynomial of degree not higher than $\ell - p$.

Since there is a binomial relation between $\xi$ and $\Delta$ on the one hand, and $K$ on the other hand, the cost function (3) can be presented in the form $J(\xi, \Delta)$. The two-level optimization algorithm proposed in [21] can be symbolically written as
\[
\min_{\eta_i \in \Omega} \min_{|\xi| \leq \ell - p} J(\xi, \Delta).
\]

On the upper level, the polynomial $\Delta$ should be changed so that its roots $\eta_i (i = 1, \ldots, m)$ remain inside the region $\Omega$. On the lower level, the optimal polynomial $\xi$ is determined for a fixed $\Delta$ on the basis of parametrization (7).

For a known $K$, cost function $J$ in (3) can be computed using the lifting technique [16], [17] or parametric transfer functions [19]. The second approach is preferable in many cases, because it allows to take into account pure delay units, including the computational delay of the computer. Moreover, using the frequency domain method of [19], the lower level optimization for quadratic functionals (3) can be performed analytically and reduces to solving a system of linear equations with respect to coefficients of the polynomial $\xi$ [21]. Therefore, the “bottleneck” of this procedure is the upper-level optimization.

Local optimization methods are not effective at the upper level, because the cost functional is not convex. In the following section we present a parametrization of the set of admissible characteristic polynomials, which makes it possible to use the information algorithm of global optimization [22], [23], [24]. For this purpose, the problem should be reformulated into the form
\[
\min_{x \in [0,1]} J(x), \tag{8}
\]
where $x$ is a free vector of independent parameters that belong to $[0,1]$.

IV. PARAMETRIZATION OF THE SET OF ADMISSIBLE CHARACTERISTIC POLYNOMIALS

The stability region in the $\zeta$-plane (the region outside the closed unit disk) is infinite, and this causes serious problems in reformulating the problem in the form (8). Therefore, we change the variable $\zeta$ to $z^{-1}$ in the functions $D_{22}(\zeta)$ and $K(\zeta)$, and construct the characteristic polynomial in the $z$-plane:
\[
\bar{\Delta}(z) = \bar{a} \bar{n} + \bar{b} \alpha,
\]
where the pairs of coprime polynomials $\{\bar{n}(\zeta), \bar{d}(\zeta)\}$ and $\{\bar{a}(\zeta), \bar{b}(\zeta)\}$ are given by the equalities
\[
D_{22}(z^{-1}) = \frac{\bar{n}(z)}{\bar{d}(z)}, \quad K(z^{-1}) = \frac{\bar{a}(z)}{\bar{b}(z)}.
\]

In the $z$-plane, the stability region (the open unit disk) is finite, as well as any subregion.

If the plant and controller are causal systems, then $|\bar{a}| \leq |\bar{d}|$ and $|\bar{b}| \leq |\bar{d}|$ so that $|\bar{\Delta}| = \ell + p$. If the polynomial $\bar{\Delta}(z)$ has roots $\theta_i (i = 1, \ldots, \ell + p)$, the corresponding polynomial $\bar{\Delta}(\zeta)$ can be constructed as
\[
\bar{\Delta}(\zeta) = \prod_{i=1, |\theta_i| \neq 0} (\zeta - \frac{1}{\theta_i}). \tag{9}
\]

Thus, the degree of $\bar{\Delta}(\zeta)$ can be lower than $|\bar{\Delta}|$ if there are zero roots $\theta_i$ (dead-beat modes).

Traditionally, modal constraints are posed in the $s$-plane. We consider two typical regions called “truncated sector”
\[
S_{\alpha,\beta} = \left\{ s : \Re s < -\alpha : \left| \frac{\Im s}{\Re s} \right| \leq \beta \right\},
\]
and “shifted sector”
\[
\bar{S}_{\alpha,\beta} = \left\{ s : \Re s < -\alpha : \left| \frac{\Im s}{\Re s + \alpha} \right| \leq \beta \right\}.
\]

Here $\alpha$ (stability degree) and $\beta$ (oscillation degree) are known nonnegative values. Transforming the constraints to the plane of the variable $z = e^{sT}$, we obtain the complex regions $\bar{D}_{\alpha,\beta}$ and $\bar{D}_{\alpha,\beta}$ shown in Fig. 4.

\[
\bar{\Delta}(z) = \prod_{i=1}^{m_1} (z + q_i) \prod_{i=1}^{m_2} (z^2 + r_{1i}z + r_{2i}), \tag{10}
\]
where $q_i (i = 1, \ldots, m_1)$ and $r_{1i}, r_{2i} (i = 1, \ldots, m_2)$ are real numbers, while $m_1$ and $m_2$ are nonnegative integers.
such that \( m_1 + 2m_2 = \ell + p \). Without loss of generality, we may take \( m_1 = 0 \) for even \( p + \ell \) and \( m_1 = 1 \) otherwise.

Hereinafter for given \( \alpha \) and \( \beta \) we denote
\[
E_\alpha = e^{-\alpha T}, \quad E_\beta = e^{-\pi/\beta}, \quad E_0 = \min \{ E_\alpha, E_\beta \}.
\]
The following propositions define uniquely the set of coefficients \( q_i, r_{1i}, \) and \( r_{2i} \) in terms of a free parameter vector, whose components belong to \([0, 1]\). Proofs are omitted for space limitation.

**Proposition 1:** The sets of all admissible coefficients of a polynomial \( z + q_i \) for the regions \( D_{\alpha \beta} \) and \( \overline{D}_{\alpha \beta} \) can be parametrized as
\[
q_i = q_{ii} + \rho_0(\overline{q}_i - q_{ii}), \quad (11)
\]
where \( \rho_0 \) is a parameter in \([0, 1]\), while \( q_{ii} \) and \( \overline{q}_i \) are given for the respective regions as:
\[
D_{\alpha \beta} : \quad q_{ii} = -E_\alpha, \quad \overline{q}_i = E_0, \quad \overline{D}_{\alpha \beta} : \quad q_{ii} = -E_\alpha, \quad \overline{q}_i = E_\alpha E_\beta.
\]

**Proposition 2:** The set of admissible pairs \( (r_{1i}, r_{2i}) \) of coefficients of a polynomial \( z^2 + r_{1i}z + r_{2i} \) can be parametrized in the form
\[
\begin{align*}
\tau_{2i} &= -E_\alpha E_0, \quad \tau_{2i} = E_\alpha^2, \quad \tau_{1i} = \frac{r_{2i}}{E_0} - E_0, \\
\tau_{1i} &= \begin{cases} 
\frac{r_{2i}}{E_0} + E_\alpha E_\beta, & r_{2i} \leq E_\beta^2 \\
-2\sqrt{r_{2i}} \cos \left[ -\beta \ln \sqrt{r_{2i}} \right], & r_{2i} > E_\beta^2
\end{cases},
\end{align*}
\]
and for the region \( \overline{D}_{\alpha \beta} \)
\[
\begin{align*}
\tau_{2i} &= -E_\alpha E_\beta, \quad \tau_{2i} = E_\alpha^2, \quad \tau_{1i} = \frac{r_{2i}}{E_\alpha} - E_\alpha, \\
\tau_{1i} &= \begin{cases} 
\frac{r_{2i}}{E_\alpha} + E_\alpha E_\beta, & r_{2i} \leq E_\beta^2 \\
-2\sqrt{r_{2i}} \cos \left[ -\beta \ln \sqrt{r_{2i}} \right], & r_{2i} > E_\beta^2
\end{cases}.
\end{align*}
\]
Thus, the set of all admissible characteristic polynomials \( \Delta(z) \) of order \( \ell + p \) can be parametrized in terms of the vector
\[
x = \begin{bmatrix}
\rho_{01} & \ldots & \rho_{0,m_1} & \rho_{11} & \ldots & \rho_{1,m_2} & \rho_{21} & \ldots & \rho_{2,m_2}
\end{bmatrix}^T.
\]
This parametrization opens the possibility to find the optimal vector \( x \) using the information algorithm of global optimization in a hyperrectangle in the space \( \mathbb{R}^{\ell+p} \) without additional nonlinear constraints [22]. For a known \( x \), the associated characteristic polynomial \( \Delta(\zeta) \) can be constructed using the following algorithm.

**Algorithm**

**Input:** Parameter vector \( x \)
**Output:** Characteristic polynomial \( \Delta(\zeta) \)
**Step 1:** Using Propositions 1 and 2, find \( q_i (i = 1, \ldots, m_1), r_{1i} (i = 1, \ldots, m_2) \) and \( r_{2i} (i = 1, \ldots, m_2) \).

**Step 2:** Construct the polynomial \( \tilde{\Delta}(z) \) and find its roots \( \tilde{\psi}_i \).

**Step 3:** Construct polynomial \( \Delta(\zeta) \) (9).

**V. Numerical Examples**

**A. Example 1.**

Consider the problem of redesigning the continuous-time system shown in Fig. 1a for
\[
P(s) = \frac{863.3}{s^3}, \quad K_c(s) = \frac{2940s + 86436}{(s + 294)^2}.
\]
This system was introduced by Katz in [1] and was investigated also in [7], [9], [13]. Here we assume that \( F_n(s) = 1 \), the sampling period is \( T = 0.03 \) sec and the hold is a zero-order hold with \( H(s) = (1 - e^{-sT})/s \).

Applying the Tustin transform to \( K_c(s) \), we obtain
\[
\hat{K}_T(\zeta) = \frac{2.1712(1 - 0.3879\zeta)(1 + \zeta)}{(1 + 0.6303\zeta)^2},
\]
which yields \( J = 0.0012 \).

The optimal fourth order controller, which gives \( J_{\text{min}} = 3.7 \times 10^{-6} \), is
\[
K_{\text{opt}}(\zeta) = \frac{1.7005(1 - 0.3896\zeta)(1 - \zeta)}{(1 + 0.6068\zeta)(1 - 0.7091\zeta)} \\
\times \frac{(1 + 0.1074\zeta + 0.003356\zeta^2)}{(1 + 0.0796\zeta + 0.001936\zeta^2)}.
\]

It is easy to check that in this case the redesigned system is marginally stable due to the pole-zero cancellation at \( \zeta = 1 \) in the product \( D_{22}(\zeta)K_{\text{opt}}(\zeta) \).

Using the methods proposed in the paper and realized in the DIRECT toolbox [27], for \( \alpha = 0.01 \) and \( \beta = \infty \) we obtain the controller
\[
K(\zeta) = \frac{1.7157(1 - 0.971\zeta)(1 - 0.3318\zeta)}{(1 - 0.653\zeta)(1 + 0.6089\zeta)};
\]
for which \( J = 1.52 \times 10^{-5} \).

The step responses of the analogue and digital systems are shown in Fig. 5. Together with the \( K_T(\zeta) \) and \( K(\zeta) \), we consider also the curves for the controllers
\[
K_{KA}(\zeta) = \frac{1.411(1 - 0.7068\zeta)(1 + 0.09586\zeta)}{(1 + 0.1761\zeta)(1 + 0.1079\zeta)};
\]
\[
K_{MH}(\zeta) = \frac{1.3186(1 - \zeta)(1 - 0.4139\zeta)}{(1 - 0.6645\zeta)(1 + 0.3167\zeta)};
\]
obtained in [9] and [13], respectively.

As is evident from Fig. 5, the use of the Tustin transform leads to large oscillations of the output. The controllers \( K_{KA} \) and \( K_{MH} \) yield good performance recovery, but the step response of the system with the controller \( K \) matches that of the original system best of all. In comparison with \( K_{\text{opt}} \), the controller \( K \) has a lower order and ensures asymptotic stability of the closed loop.
B. Example 2.

Now we employ the proposed method to solve the redesign problem for Rattan’s system [10], [9]:

\[ P(s) = \frac{10}{s(s + 1)}, \quad K_c(s) = \frac{0.416s + 1}{0.139s + 1}. \]

It is required to discretize the controller with \( T = 0.157 \) sec, using \( F_a(s) = 1 \) and a zero-order hold.

The Tustin transform yields:

\[ K_T(\zeta) = \frac{2.2736(1 - 0.6825\zeta)}{\zeta - 0.2782}, \]

so that \( J = 0.0538 \). The optimal fourth order controller

\[ K_{opt}(\zeta) = \frac{2.217(1 - 0.8547\zeta)(1 - 0.6855\zeta)}{(1 - 0.7984\zeta)(1 + 0.4523\zeta)} \times \frac{(1 + 0.2375\zeta)(1 + 0.101\zeta)}{(1 - 0.1681\zeta)(1 + 0.01757\zeta)}, \]

obtained by means of the DIRECTSD toolbox, gives \( J = 2.0395 \times 10^{-5} \).

Using the proposed search procedure, we find the digital controller

\[ K(\zeta) = \frac{2.1043(1 - 0.764\zeta)}{\zeta - 0.1664}, \]

which gives \( J = 3.854 \times 10^{-5} \). It is a bit inferior to the optimal controller as regards the cost function, but has order 1 instead of 4.

The step responses of the systems are shown in Fig. 6. In addition to \( K_T \) and \( K \), we consider also step responses for the controllers

\[ K_R(\zeta) = \frac{3.436(1 - 0.6377\zeta)}{1 + 0.239\zeta}, \]

\[ K_{KA}(\zeta) = \frac{2.8926(1 + 0.1681\zeta)(1 - 0.7088\zeta)}{(1 + 0.0173\zeta)(1 + 0.271\zeta)}, \]

proposed in [10] and [9], respectively.

As follows from Fig. 6, the new controller \( K(\zeta) \) yields the best performance recovery.

VI. CONCLUSIONS

The problem of the digital redesign of an analogue controller is considered in the closed-loop framework. A novel optimization-based method is proposed, which makes it possible to place closed-loop poles in a specified region and restrict the controller order.

It is remarkable that the redesign problem is considered as a special case of the general \( L_2 \)-optimization problem for a sampled-data system under a known input signal.

Since the optimization is performed in the frequency domain, the proposed technique can easily be applied to time-delay systems, including those with computational delay (see [18]).

If the controller static gain should be preserved, the method is also applicable, but the minimal controller order equals \( p \) instead of \( p - 1 \), because one degree of freedom must be spent to ensure this condition.

Instead of (3), another cost function can be used, which evaluates the performance recovery, for example, the \( L_2 \)-induced norm (\( H_{\infty} \)-norm) of the system in Fig. 3. Then, as distinct from the considered case, numerical methods should be used for determining the optimal polynomial \( \xi \) at the lower level of the procedure.

Extension of these ideas on the redesign of multivariable controllers is a topic of our current research.

REFERENCES


