On the Boolean-like Law $I(x, I(y, x)) = 1$

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The well-known Boolean-law “$\alpha \to (\beta \to \alpha) = 1$” can be generalized to fuzzy context as $I(x, I(y, x)) = 1$, where $I$ is a fuzzy implication. In this paper we show the necessary and sufficient conditions under which this generalization holds in fuzzy logics. We focus the investigation on the following classes of fuzzy implication: $(S, N)$-, $R$-, $QL$-, $D$- and $(N, T)$-implications. In addition, we demonstrate that a fuzzy implication $I$ satisfies such Boolean-like law if, and only if, its $\Phi$-conjugate also satisfies it.

Keywords: Fuzzy implication; Boolean-like laws; automorphisms.

1. Introduction

Boolean laws have been generalized and studied as functional equations or inequations in fuzzy logics. These are called Boolean-like laws and they are not usually satisfied in any standard structure $([0, 1], T, S, N, I)$, where $T$ is a t-norm, $S$ is a t-conorm, $N$ is a fuzzy negation and $I$ is a fuzzy implication. In this scenario, researchers have proposed investigations about the conditions under which a Boolean-like law holds (e.g. Refs. 1–5). This paper reports a further investigation exposed in.\textsuperscript{6} Here, we will produce necessary and sufficient conditions under which the Boolean-like law (1) holds for the following classes of fuzzy implications: $(S, N)$-, $R$-, $QL$-, $D$- and $(N, T)$-implications.

$$I(x, I(y, x)) = 1, \text{ for all } x, y \in [0, 1].$$

(1)
From the formal logic point of view, this law is the fuzzy rewriting of the classical, intuitionistic, Łukasiewicz axiom \( \mathit{x \Rightarrow (y \Rightarrow x)} \) (\(^a\)). In Boolean logics, (\(^a\)) is called “Weakening”, since it is said to be the weakening of \( \mathit{x \Rightarrow x} \). In a similar sense, assuming \( \Rightarrow \) as the material implication, we observe that (\(^a\)) is also a weakening formalization of Aristotle’s Law of Excluded Middle \( \mathit{\neg x \lor x} \) — since, \( \mathit{x \Rightarrow (y \Rightarrow x)} \) would be equivalent to \( \mathit{\neg x \lor (\neg y \lor x)} \).

Another theoretical justification to investigate (\(^1\)) lies in the fact that definitions of fuzzy implication try to comprehend the correct notion (common sense) of the actual essence of a logical implication. Therefore, the structure of fuzzy implication classes will provide some information about what about the meaning of an implication in fuzzy context. Besides, the definition of such classes is generally motivated by the implication notion in distinct logics, for example: \((S, N)\)-, \(R\)- and \(QL\)-implication generalize the implication of classical, intuitionistic and quantum logics, respectively. Hence, the investigation of conditions under which logical laws hold in each of those classes also contributes to characterize the approximate reasoning in accordance with each implication notion.

Moreover, since (\(^a\)) is valid in fundamental logical systems, it is likely that a Fuzzy Rule-Based System (FRBS) must consider such property true in every case. Consequently, this system must use fuzzy operators that guarantee the validity of (\(^a\)). Therefore, an investigation about the validity of such statement in fuzzy logics, regarding distinct classes of fuzzy implications, also contributes the selection of a correct and more appropriate semantics \((T, S, N, I)\) for a FRBS.

The rest of this paper is organized as follows. Section 2 recalls basic definitions and some properties of t-norms, t-conorms and fuzzy negations. Section 3 does the same with respect to (w.r.t., for short) fuzzy implications, their classes — namely: \((S, N)\)-, \(R\)-, \(QL\)-, \(D\)- and \((N, T)\)-implications — and automorphisms on fuzzy implications. Section 4 works out on the sufficient and necessary conditions under which those classes and \(\Phi\)-conjugate implications satisfy (\(^1\)). Finally, we present a discussion about the results of this paper in Sec. 5.

2. Preliminaries

In order to make this paper self-contained, this section summarizes some basic definitions.

**Definition 1.** A function \( T : [0, 1]^2 \rightarrow [0, 1] \) is a t-norm if it satisfies Commutativity (T1), Associativity (T2), Monotonicity (T3) and 1-identity (T4) — \( T(1, y) = y \), where \( y \in [0, 1] \).

**Remark 1.** The minimum t-norm \( T_M \) is the only idempotent t-norm.\(^7\) I.e. \( T_M(x, x) = x \), for all \( x \in [0, 1] \).

\(^a\)“\( x \Rightarrow (y \Rightarrow x) \)” is also a theorem in basic and product logics.

\(^b\)Also note that “\( x \Rightarrow x \)” would be equivalent to \( \neg x \lor x \).
Definition 2. A function \( S : [0, 1]^2 \to [0, 1] \) is a t-conorm if it satisfies Commutativity (S1), Associativity (S2), Monoticity (S3) and 0-identity (S4) — \( S(0, y) = y \), where \( y \in [0, 1] \).

Remark 2. Considering the partial order on the family of all t-conorms induced from the order on \([0,1]\) on the family of all t-conorms, \( S_M(x, y) = \max(x, y) \) is the least t-conorm. So for any t-conorm \( S \), \( S(x, y) \geq S_M(x, y) \geq y \). Moreover, \( S_M(x, 1) = S_M(1, x) = 1 \), so \( S(x, 1) = S(1, x) = 1 \) for any t-conorm \( S \).

Definition 3. A function \( N : [0, 1]^2 \to [0, 1] \) is a fuzzy negation if \( N(0) = 1 \) and \( N(1) = 0 \) (N1), and \( N \) is non-increasing (N2). Besides, a fuzzy negation is called strong, if it is involutive, i.e., \( N(N(x)) = x \), such that \( x \in [0, 1] \).

As examples of fuzzy negations, we cite the least fuzzy negation \( N_\bot \) and the greatest fuzzy negation \( N_\top \):

\[
N_\bot(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0, & \text{if } x \in [0, 1]
\end{cases}
\]

\( N_\bot(x) \) (2)

\[
N_\top(x) = \begin{cases} 
0, & \text{if } x = 1 \\
1, & \text{if } x \in [0, 1]
\end{cases}
\]

\( N_\top(x) \) (3)

A t-conorm \( S \) is distributive over a t-norm \( T \) if

\[
S(x, T(y, z)) = T(S(x, y), S(x, z)), \text{ for all } x, y, z \in [0, 1].
\]

(4)

An important result about distributivity in Boolean-like laws is given next.

Proposition 1.\(^7\) Let \( T \) be a t-norm and \( S \) a t-conorm, \( S \) is distributive over \( T \) iff \( T = T_M \).

Let \( S \) be a t-conorm and \( N \) a fuzzy negation, the pair \((S, N)\) satisfies the Law of Excluded Middle (LEM, for short) if

\[
S(N(x), x) = 1, \text{ for all } x \in [0, 1].
\]

(LEM)

Note that any t-conorm \( S \) with \( N_\top \) satisfies (LEM). Also, no t-conorm with the \( N_\bot \) satisfies (LEM).\(^8\)

3. Fuzzy Implications

Currently, there are different acceptable definitions of fuzzy implications (e.g. Refs. 4, 9 and 10). In this paper, we opt for a well-accepted definition:

Definition 4. A function \( I : [0, 1]^2 \to [0, 1] \) is called a fuzzy implication if it satisfies:

\(^{\text{Some demonstrations will refer this remark.}}\)
11. Boundary conditions: \( I(0,0) = I(0,1) = I(1,1) = 1 \) and \( I(1,0) = 0 \).

Other potential properties are acceptable for some fuzzy implications:

I2. Left antitonicity: if \( x_1 \leq x_2 \) then \( I(x_1, y) \geq I(x_2, y) \), for all \( x_1, x_2, y \in [0,1] \);
I3. Right isotonicity: if \( y_1 \leq y_2 \) then \( I(x,y_1) \leq I(x,y_2) \), for all \( x, y_1, y_2 \in [0,1] \);
I4. Left boundary condition: \( I(0,y) = 1 \), for all \( y \in [0,1] \);
I5. Right boundary condition: \( I(x,1) = 1 \), for all \( x \in [0,1] \);
I6. Identity property: \( I(x,x) = 1 \), for all \( x \in [0,1] \).

There is also a property that relates fuzzy implications and negations:

I7. Contrapositive: \( I(x,y) = I(N(y), N(x)) \), for all \( x, y, \in [0,1] \) and \( N \) being a fuzzy negation.

There are three main classes of fuzzy implications, namely: \((S,N)\)-, \(R\)- and \(QL\)-implications. Other classes can be generated from those ones, namely: \(D\)- and \((N,T)\)-implications are generated from \(QL\)- and \((S,N)\)-implications, respectively.

In the following definition we are going to recall them.

**Definition 5.** Let \( T \) be a t-norm, \( S \) a t-conorm and \( N \) a fuzzy negation, then:

- A function \( I : [0,1]^2 \to [0,1] \) is called an \((S,N)\)-implication (denoted by \( I_{S,N} \)) if
  \[
  I(x,y) = S(N(x), y) .
  \tag{5}
  \]
- A function \( I : [0,1]^2 \to [0,1] \) is called an \( R \)-implication (denoted by \( I_T \)) if
  \[
  I(x,y) = \text{sup}\{ t \in [0,1] | T(x,t) \leq y \} .
  \tag{6}
  \]
- A function \( I : [0,1]^2 \to [0,1] \) is called a \( QL\)-implication (denoted by \( I_{S,N,T} \)) if
  \[
  I(x,y) = S(N(x), T(x,y)) .
  \tag{7}
  \]
- A function \( I : [0,1]^2 \to [0,1] \) is called a \( D\)-implication (denoted by \( I_{S,T,N} \)) if
  \[
  I(x,y) = S(T(N(x), N(y)), y) .
  \tag{8}
  \]
- A function \( I : [0,1]^2 \to [0,1] \) is called a \((N,T)\)-implication (denoted by \( I_{N,T} \)) if
  \[
  I(x,y) = N(T(x,N(y))) .
  \tag{9}
  \]

**Remark 3.**

(i) Every \( I_T \) satisfies I1-I6.\(^{8,11}\)
(ii) The concept of \( QL\)-implications and \( QL\)-operators can be found in a review of the literature. However, in this paper we do not have this distinction since our implication definition (Def. 4) considers only the boundary conditions (I1).
(iii) Although \( D\)-implications are generally defined from strong negations,\(^{13-16}\) according to our definition (Def. 4) the \( D\)-operator defined from a non-strong negation is also a fuzzy implication, thus our \( D\)-implication definition does not require the strong negations.
Another way to obtain a D-implication from a QL-implication — or a QL- from a D-implication — is by (LEM) and (4) (see the following demonstrations).

**Lemma 1.** Given a D-implication $I_{S,T,N}$ and a QL-implication $I_{S,N,T}$. If $(S, N)$ satisfies (LEM) and $T = T_M$, then $I_{S,T,N} = I_{S,N,T}$.

**Proof.** Since $S$ is distributive over $T$ (Eq. (8)) if $T = T_M$ (Proposition 1), so:

$$I_{S,T,N}(x, y) = S(T(N(x), N(y)), y) \quad \text{by (8)}$$

$$= T(S(y, N(x)), S(y, N(y))) \quad \text{by (S1) and (4)}$$

$$= T(S(N(x), y), S(N(x), y)) \quad \text{by (S1) and (LEM)}$$

$$= S(N(x), T(y, x)) \quad \text{by (4)}$$

$$= I_{S,N,T}(x, y) \quad \text{by (T1) and (7).}$$

**Proposition 2.** Given a D-implication $I_{S,T,N}$ and a QL-implication $I_{S,N,T}$. If $I_{S,T,N} = I_{S,N,T}$ then $(S, N)$ satisfies (LEM).

**Proof.** If $I_{S,T,N} = I_{S,N,T}$, then $S(N(0), T(0, y)) = S(T(N(0), N(y)), y)$. On the left side, $S(N(0), T(0, y)) = 1$ and on the other side $S(T(N(0), N(y)), y) = S(N(y), y)$. Hence $S(N(y), y) = 1$, i.e., $(S, N)$ satisfies (LEM).

It is possible to generate fuzzy implications using automorphisms: Let $\Phi$ be the set of automorphisms, fuzzy implications $I$ and $J$ are $\Phi$-conjugate, if there exists a $\varphi \in \Phi$ such that $J = I_\varphi$, where

$$I_\varphi(x, y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \text{ for all } x, y \in [0, 1].$$

4. Solutions of (1) in $(S, N)$-, $R$-, $QL$-, $D$- and $(N, T)$-Implications

4.1. $(S, N)$-implications

**Theorem 1.** An $(S, N)$-implication $I_{S,N}$ satisfies (1) iff $(S, N)$ satisfies (LEM).

**Proof.** $\Leftarrow$: 

$$I_{S,N}(x, I_{S,N}(y, x)) = S(N(x), S(N(y), x)) \quad \text{by Eq. (5)}$$

$$= S(N(x), S(x, N(y))) \quad \text{by S1}$$

$$= S(S(N(x), x), N(y)) \quad \text{by S2}$$

$$= S(1, N(y)) \quad \text{by (LEM)}$$

$$= 1 \quad \text{by Remark 2.}$$

$\Rightarrow$: For all $x \in [0, 1]$, $I(x, I(1, x)) = 1$ and $I(1, x) = S(N(1), x) = S(0, x) = x$, so $S(N(x), x) = I(x, I(1, x)) = 1$. Hence $(S, N)$ satisfies (LEM).
4.2. \( R \)-implications

Recognizably, the theory of residuated lattices demonstrates that when \( I \) is an \( R \)-implication defined from a left-continuous t-norm \( T \), in which, case (\( T, I \)) is an adjoint pair,\(^8\) then \( I \) satisfies (1). In the following theorem, we prove that the left-continuity of underlying t-norm is not required to an \( R \)-implication satisfy (1).

**Theorem 2.** Every \( R \)-implication satisfies (1).

**Proof.** Let \( T \) be any t-norm and \( I_T \) an \( R \)-implication generated from it. Fixing arbitrarily \( x, y \in [0, 1] \), it is well known that \( T(y, x) \leq \min(y, x) \leq x \), so \( I_T(y, x) = \sup \{ t \in [0, 1] | T(y, t) \leq x \} \geq x \). We have already seen (Remark 3(i)) that \( R \)-implications satisfy I3 and I6.\(^8\) Therefore

\[
I_T(x, I_T(y, x)) \geq I_T(x, x) = 1.
\]

Hence \( I_T(x, I_T(y, x)) = 1 \). \( \square \)

4.3. \( QL \)-implications

**Lemma 2.** A \( QL \)-implication \( I_{S,N,T} \) satisfies (1), whenever \( (S, N) \) satisfies (LEM) and \( T = T_M \).

**Proof.** Since \( T = T_M \) iff \( S \) is distributive over \( T \) (Proposition 1), then

\[
\begin{align*}
I(x, I(y, x)) &= S(N(x), T(x, S(N(y), T(y, x)))) & \text{by Eq. (7)} \\
&= S(N(x), T(x, T(S(N(y), y), S(N(y), x)))) & \text{by (4)} \\
&= S(N(x), T(x, T(1, S(N(y), x)))) & \text{by (LEM)} \\
&= S(N(x), T(x, S(N(y), x))) & \text{by } T4 \\
&= T(S(N(x), x), S(N(x), S(N(y), x))) & \text{by (4)} \\
&= T(1, S(N(x), S(x, N(y)))) & \text{by (LEM) and } S1 \\
&= S(S(N(x), x), N(y)) & \text{by } T4 \text{ and } S2 \\
&= S(1, N(y)) & \text{by (LEM)} \\
&= 1 & \text{by Remark 2.} \square
\end{align*}
\]

**Lemma 3.** If a \( QL \)-implication \( I_{S,N,T} \) generated by a strictly increasing t-conorm \( S \) in \([0, 1]\), a t-norm \( T \) and a fuzzy negation \( N \) satisfies (1), then \((S, N)\) satisfies (LEM) and \( T = T_M \).

**Proof.** Assume that \( I_{S,N,T} \) is a \( QL \)-implication which satisfies (1), so \( I_{S,N,T}(1, I_{S,N,T}(y, 1)) = 1 \). Therefore, for any \( y \in [0, 1] \), \( S(N(1), T(1, S(N(y), T(y, 1)))) = 1 \) and:

\[
\begin{align*}
S(N(1), T(1, S(N(y), T(y, 1)))) &= S(0, T(1, S(N(y), T(1, y)))) & \text{by } N1 \text{ and } T1 \\
&= S(N(y), y) & \text{by } S4 \text{ and } T4.
\end{align*}
\]
Thus $S(N(y), y) = 1$. Hence $(S, N)$ satisfies (LEM). Moreover since $I_{S,N,T}$ is a QL-implication which satisfies (1) then $I_{S,N,T}(x, I_{S,N,T}(1, x)) = 1$. Therefore, for any $x \in [0, 1]$, $S(N(x), T(x, S(N(1), T(1, x)))) = 1$ and:

\[
S(N(x), T(x, S(N(1), T(1, x)))) = S(N(x), T(x, S(0, T(1, x))))
\]

by N1

\[
= S(N(x), T(x, T(1, x)))
\]

by S4

\[
= S(N(x), T(x, x))
\]

by T4.

Thus $S(N(x), T(x, x)) = 1$. Since $(S, N)$ satisfies (LEM), for any $x \in [0, 1]$, $S(N(x), T(x, x)) = 1 = S(N(x), x)$. Case $x = 1$, so, trivially $T(x, x) = x$. Case $x < 1$, since $S$ is strictly increasing in $[0, 1]$, $S(N(x), T(x, x)) = S(N(x), x)$ implies $T(x, x) = x$. Therefore $T(x, x) = x$ for any $x \in [0, 1]$, i.e., $T$ is an idempotent t-norm. By Remark 1, $T_M$ is the only idempotent t-norm. Hence, if $I$ is a QL-implication — generated by a strictly increasing t-conorm $S$ in $[0, 1]$, a t-norm $T$ and a fuzzy negation $N$ — that satisfies (1) then $(S, N)$ satisfies (LEM) and so $T = T_M$.

**Theorem 3.** Let $I_{S,N,T}$ be a QL-implication generated by a strictly increasing t-conorm $S$ in $[0, 1]$, a fuzzy negation $N$ and a t-norm $T$. $I$ satisfies (1) iff $(S, N)$ satisfies (LEM) and $T = T_M$.

**Proof.** Straightforward from Lemmas 2 and 3.

### 4.4. D-implications

**Corollary 1.** Let $I_{S,T,N}$ be a D-implication generated by a strictly increasing t-conorm $S$ in $[0, 1]$, a t-norm $T$ and a fuzzy negation $N$ such that $(I, N)$ satisfies the contrapositive (I7). Then $I$ satisfies (1) iff $(S, N)$ satisfies (LEM) and $T = T_M$.

**Proof.** Since D-implication is the contrapositive of QL-implication and by Theorem 3.

The result of the previous theorem is fairly trivial. But are there other conditions to guarantee that (1) will be satisfied for D-implications? The following lemmas and theorem answer this question positively.

**Lemma 4.** Given a D-implication $I_{S,T,N}$ generated by a strictly increasing t-conorm $S$ in $[0, 1]$, a t-norm $T$ and a fuzzy negation $N$. $I$ satisfies (1), if $(S, N)$ satisfies (LEM) and $T = T_M$.

**Proof.** Straightforward from Lemma 1 and Theorem 3.

**Lemma 5.** Given a D-implication $I_{S,T,N}$ generated by a strictly increasing t-conorm $S$ in $[0, 1]$, a t-norm $T$ and a continuous fuzzy negation $N$. If $I_{S,T,N}$ satisfies (1), then $(S, N)$ satisfies (LEM) and $T = T_M$. 
Proof. Let $I_{S,T,N}$ be a D-implication which satisfies (1), so $I_{S,T,N}(0, I_{S,T,N}(y, 0)) = 1$. Therefore, $S(T(N(0), N(S(T(N(y), N(0)), 0))), S(T(N(y), N(0)), 0)) = 1$ for any $y \in [0, 1]$. By N1, T4 and S4, $S(T(N(y), N(0)), 0) = N(y)$ (1), so:

\[
\begin{align*}
S(T(N(0), N(S(T(N(y), N(0)), 0))), S(T(N(y), N(0)), 0)) &= S(T(N(0), N(N(y))), N(y)) \quad \text{by (1)} \\
&= S(N(N(y)), N(y)) \quad \text{by N1 and T4.}
\end{align*}
\]

Since $N$ is continuous, then for all $y' \in [0, 1]$ there exists $y \in [0, 1]$ such that $N(y) = y'$. Therefore, $S(N(y'), y') = S(N(N(y)), N(y)) = 1$. Hence $(S, N)$ satisfies (LEM).

Again by (1), $I_{S,T,N}(x, I_{S,T,N}(1, x)) = 1$, and $I_{S,T,N}(1, x) = S(T(0, N(x)), x) = x$. Then $I_{S,T,N}(x, x) = S(T(N(x), N(x)), x)$. It is known that $T(N(x), N(x)) \leq N(x)$. If $T(N(x), N(x)) < N(x)$ and since $(S, N)$ satisfies (LEM) and $S$ is strictly increasing in $[0, 1]$, then $S(T(N(x), N(x)), x) < S(N(x), x) = 1$. However, $S(T(N(x), N(x)), x) = 1$. So $T(N(x), N(x))$ must not be less than $N(x)$. Thus $T(N(x), N(x)) = N(x)$. Since $N$ is continuous, for all $x \in [0, 1]$ there exists $x \in [0, 1]$ such that $N(x) = x'$. Therefore, $T(x', x') = x'$ for all $x' \in [0, 1]$, i.e. $T$ is an idempotent t-norm. Hence $T = T_M$ — by Remark 1.

\[\square\]

Theorem 4. Let $I_{S,T,N}$ be a D-implication generated by a strictly increasing t-conorm $S$ in $[0, 1]$, a t-norm $T$ and a continuous fuzzy negation $N$. $I_{S,T,N}$ satisfies (1) iff $(S, N)$ satisfies (LEM) and $T = T_M$.

Proof. Straightforward from Lemmas 4 and 5.

\[\square\]

4.5. $(N, T)$-implications

Theorem 5. Let $N$ be a strong negation and $T$ a t-norm. An $(N, T)$-implication $I_{N,T}$ satisfies (1) iff for the $N$-dual t-conorm $S$ of $T$, $(S, N)$ satisfies (LEM).

Proof. Straightforward from Remark 3(iv) and Theorem 1.

\[\square\]

4.6. Automorphisms

Theorem 6. Let $\varphi \in \Phi$. A fuzzy implication $I$ satisfies (1) iff $I_{\varphi}$ satisfies (1).

Proof. $\Rightarrow$: Assume that $\varphi \in \Phi$ and a fuzzy implication $I$ satisfies (1). Now by (10) we have that $I_{\varphi}(x, I_{\varphi}(y, x)) = \varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1}(I(\varphi(y), \varphi(x)))))$ and trivially $\varphi^{-1}(I(\varphi(x), \varphi \circ \varphi^{-1}(I(\varphi(y), \varphi(x))))) = \varphi^{-1}(I(\varphi(x), I(\varphi(y), \varphi(x))))$. By (1), $I(\varphi(x), I(\varphi(y), \varphi(x))) = 1$, so $\varphi^{-1}(I(\varphi(x), I(\varphi(y), \varphi(x)))) = \varphi^{-1}(1) = 1$. Therefore, $I_{\varphi}(x, I_{\varphi}(y, x)) = 1$. Hence, $I_{\varphi}$ satisfies (1). The converse follows straightforward from $\varphi^{-1}((I_{\varphi})) = I$.

\[\square\]
5. Final Remarks

This paper provides necessary and sufficient conditions under which the Boolean-like law $I(x, I(y, x)) = 1$, referred by (1), holds for $(S, N)$-, $R$-, $QL$-, $D$- and $(N, T)$-implications.

The main results of this paper are demonstrated in Theorems 1, 2, 3, 4, 5 and 6, and Corollary 1. These results show that a generalization of a classical implication to a fuzzy context — an $(S, N)$-implication — satisfies (1) iff $(S, N)$ satisfies (LEM).

The dual implication of this generalization must obey similar necessary and sufficient conditions: A $(N, T)$-implication satisfies (1) iff $N$ is strong and for a $N$-dual of $T$ t-conorm $S$, $(S, N)$ satisfies (LEM). Every intuitionistic implication generalized to a fuzzy context — an $R$-implication — satisfies (1) iff (I6). However, a quantum implication in fuzzy context — $QL$-implication — satisfies (1) iff $(S, N)$ satisfies (LEM) and $T = T_{M}$, regarding that its underlying t-conorm must be strictly increasing in $[0,1]$. The contrapositive of the $QL$-implication — $D$-implication — has an additional condition: its underlying negation must be continuous. Table 1 summarizes these results.

Those observations lead us to conclude that (1) cannot be indiscriminately adopted in any computational system based on fuzzy concepts. In addition to the property (1) being adopted in Fuzzy-Based Systems, other properties must be considered. On the one hand, a relevant contribution of this paper is to show which properties and fuzzy operators, software engineers have to choose in order to consider (1) as true in their computational system. On the other hand, in a more general scenario, this paper contributes to understand what an implication actually means in the truth domain $[0,1]$ and it also contributes to determine the meaning behind each fuzzy implication class, i.e., which properties they accept.

In the introduction section, we describe the Boolean relation among LEM, “$x \Rightarrow (y \Rightarrow x)$” and “$x \Rightarrow x$”. In a fuzzy context, this is a relation among (1), (LEM) and (I6) which can be analyzed in each fuzzy implication class:

- $I_{S,N}$ satisfies (I6) iff it satisfies (LEM) iff it satisfies (1).
- Every $R$-implication satisfies (I6) and (1).
- $I_{S,N,T}$ satisfies (I6), if $N = N_{T}$ or $T = T_{M}$ or $S = S_{D}$.\(^{8}\) For any $S$, $(S, N_{T})$ satisfies (LEM); and $(S_{D}, \mathbb{J})$ satisfies (LEM), such that $\mathbb{J}(x) = 0$ iff $x = 1$.\(^{8}\) Moreover, $I_{S,N,T}$ also satisfies (1).\(^{4}\)
- $I_{T,S,N}$ satisfies (I6), if $T = T_{M}$ and $(S, N)$ satisfies (LEM). Regarding this same sufficient condition, $I_{T,S,N}$ also satisfies (1).
- If $N$ is strong and there is an $S$ such that $S$ is $N$-dual of $T$, then: $(S, N)$ satisfies (LEM) iff $I_{N,T}$ satisfies (I6); and $(S, N)$ satisfies (LEM) iff $I_{N,T}$ satisfies (I6).

\(^{4}\) $I_{S_{D},N,T}$ does not satisfy (1) for any $T.$
Finally, we have also demonstrated that if a new implication is defined through an automorphism, this new implication will satisfy (1) iff the original fuzzy implication also satisfies it — Theorem 6.

<table>
<thead>
<tr>
<th>Implication Class</th>
<th>Sufficient and Necessary Conditions to Satisfy (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S, N)$-implication</td>
<td>$(1)$ iff $(S, N)$ satisfies (LEM).</td>
</tr>
<tr>
<td>$R$-implication</td>
<td>Every $R$-implication satisfies $(1)$.</td>
</tr>
<tr>
<td>$QL$-implication</td>
<td>$S$ is strictly increasing in $[0,1]$;</td>
</tr>
<tr>
<td></td>
<td>$(1)$ iff $(S, N)$ satisfies (LEM) and $T = T_M$.</td>
</tr>
<tr>
<td>$D$-implication</td>
<td>$S$ is strictly increasing in $[0,1]$ and $(I, N)$ satisfies (I7): $(1)$ iff $(S, N)$ satisfies (LEM) and $T = T_M$;</td>
</tr>
<tr>
<td></td>
<td>or</td>
</tr>
<tr>
<td></td>
<td>$S$ is strictly increasing in $[0,1]$ and $N$ is continuous: $(1)$ iff $(S, N)$ satisfies (LEM) and $T = T_M$.</td>
</tr>
<tr>
<td>$(N, T)$-implication</td>
<td>$N$ is strong: $(1)$ iff $(S, N)$ satisfies LEM, for a $N$-dual of $T$ t-conorm $S$.</td>
</tr>
<tr>
<td>$\Phi$-conjugate implications</td>
<td>$I$ satisfies $(1)$ iff $I_\Phi$ satisfies $(1)$.</td>
</tr>
</tbody>
</table>

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References


