On exact travelling wave solutions for two types of nonlinear \(K(n,n)\) equations and a generalized KP equation

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Abstract

In this paper, we study two types of genuinely nonlinear \(K(n,n)\) equations and a generalized KP equation. By developing a mathematical method based on the reduction of order of nonlinear differential equations, we derive general formulas for the travelling wave solutions of the three equations. The compactons, solitary patterns, solitons and periodic solutions obtained are expressed analytically. It is shown that the \(y\) and \(z\) components of the wave number vectors in the travelling wave solutions of the generalized KP equation remain free and arbitrary constants. The work generalizes the known results of travelling wave solutions for the three equations.

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1. Introduction

Recently, intensive research has been conducted to study the following \(K(n,n)\) evolution equation
\[
 u_t + a(u^n)_x + (u^n)_{xxx} = 0, \quad n \geq 1, \tag{1}
\]
which describes the role of nonlinear dispersion in the formation of patterns in liquid drops (see [10,13]). The studies, as presented in [5–10,12–14], discovered that nonlinear dispersion can compactify solitary waves and generate compactons: solitons with finite wavelength or robust soliton-like solutions characterized by the absence of infinite wings. The discovery of compactons that collide elastically and vanish identically outside a finite core region was made in [10] to specify and establish a scientific explanation that nonlinear dispersion leads to qualitative changes [5] in the nature of some genuinely nonlinear phenomena. The tri-Hamiltonian duality between solitons and compactons was reported in Olver and Rosenau [5]. Rosenau and coworkers [5,6,8,10] found that the collision of two compactons results in the creation of low-amplitude compacton and anticompacton pairs, and they reemerge with same coherent shape.

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Various attempts to study Eq. (1) have been made in recent years. Rosenau and Hyman [10] used the pseudo-spectral method in space and a variable order, variable time-step Adams–Bashford–Moulton method in time to study Eq. (1). Wazwaz [12] made use of the Adomian decomposition method to study Eq. (1), particularly, the cases $K(2, 2)$ and $K(3, 3)$, and derived a general formula of compacton solutions of Eq. (1) for $n > 1$. Ismail and Taha [3] developed a finite difference method and a finite element method to study Eq. (1) for $n = 2$ and 3, and obtained numerical solutions for one compacton which were then compared with the exact solutions to assess the accuracy of the methods. It was shown in [3] that the numerical solutions agreed very well with the exact solutions and that the compactons exhibited true soliton behavior. For more details of the methods to acquire compactons and soliton solutions for nonlinear evolution equations, the reader is referred to [1–3,9,11–15] in which many analytical and numerical methods such as the pseudo-spectral method, the Galerkin method, the finite difference method, the sine–cosine ansatz, and the tanh method are presented.

Rosenau [9] and Wazwaz [13] investigated the following models:

$$u_t + a(u^{n+1})_x + [u(u^n)]_x = 0, \quad a > 0, \quad n \geq 1,$$

(2)

and

$$u_t + a(u^{n+1})_x + [u(u^n)]_x + \nabla^2 u = 0, \quad a > 0, \quad n \geq 1,$$

(3)

where $\nabla^2 = (k(k-1)/k!) \delta_x^2 + (k(k-1)(k-2)/k!) \delta_z^2$, $k = 2$ or 3, in which $k$ represents the dimension of the spatial domain. Rosenau [9] regarded Eq. (2) as another variant of the $K(n, n)$ model which was shown to describe the dispersion of dilute suspensions [9] for $n = 1$. Eq. (3) is a generalized form of the well-known KP equation. In [9], some meaningful results were obtained to explore a number of formal mathematical extensions of solitons supporting equations with the aim of producing compact dispersive structures in higher dimensions, and several non-travelling wave solutions for the generalized KP equation were constructed by a mathematical transformation formula.

Motivated by the form of Eq. (3), we write another generalized type of KP equation expressed in the form

$$u_t + a(u^{n+1})_x + [u(u^n)]_x + b_1 u_{yy} + b_2 u_{zz} = 0, \quad a \neq 0, \quad n \neq 0,$$

(4)

where constants $b_1$ and $b_2$ satisfy $b_1^2 + b_2^2 \neq 0$. Obviously, Eq. (4) reduces to Eq. (3) for $b_1 = k(k-1)/k!$ and $b_2 = k(k-2)$.

In this paper, we further develop the work in [10,9,12,13] for the study of Eqs. (1), (2) and (4). By using a mathematical technique different from those in previous work [5–10,12–15], we obtain general formulas for travelling wave solutions with wave variable $\xi = \mu(x-ct)$ or $\xi = \mu x + \eta y + \zeta z - ct$ for three nonlinear Eqs. (1), (2) and (4). For $K(n, n)$ equations (1) and (2), we find that the exponent $n$ and $a$, positive or negative, determine directly the physical structures of solutions such as compactons, solitons, solitary patterns and periodic solutions.

2. Solving $K(n, n)$ equation with positive and negative $n$

Firstly, we consider the solution of the equation

$$\left( \frac{dW}{dz} \right)^2 = a_0 - b_0 W^2,$$

(5)

where $a_0 \neq 0$ and $b_0 \neq 0$ are constants. When $b_0 > 0$, Eq. (5) admits two solutions

$$W_1 = \pm \sqrt{\frac{a_0}{b_0}} \sin[\sqrt{b_0}(z + A)], \quad W_2 = \pm i \sqrt{\frac{a_0}{b_0}} \cos[\sqrt{b_0}(z + A)],$$

(6)

where $A$ is an arbitrary constant.

When $b_0 < 0$, noticing that $\cosh^2 z - \sinh^2 z = 1$, we know that Eq. (5) has two solutions of the form

$$W_3 = \pm \sqrt{-\frac{a_0}{b_0}} \sinh[\sqrt{-b_0}(z + A)], \quad W_4 = \pm i \sqrt{-\frac{a_0}{b_0}} \cosh[\sqrt{-b_0}(z + A)],$$

(7)

where $i = \sqrt{-1}$.
Remark 1. Here we point out that we solve Eq. (5) in the complex value district and note that the formulas (6) and (7) include all the analytical solutions of Eq. (5). In the following discussions, we let the constant $A = 0$ in formulas (6), (7) and forthcoming analytical expressions of travelling wave solutions.

We list the following formulas to be used later on:

$$\sinh x = \frac{1}{i} \sin ix, \quad \sin x = \frac{1}{i} \sinh ix, \quad \cosh x = \cos ix, \quad \cos x = \cosh ix.$$ 

Assuming the travelling wave solution has the form $u(x, t) = u(\xi)$ with wave variable $\xi = \mu(x - ct)$ ($\mu \neq 0$, $c \neq 0$), we have, from Eq. (1), the following ordinary differential equation:

$$-\mu cu_\xi + au^n + \mu^2 (u^n)_{\xi\xi\xi} = 0. \quad (8)$$

Integrating Eq. (8) once and setting the constants of integration to be zero give rise to the second order ODE

$$-cu + au^n + \mu^2 Z \frac{dZ}{du^n} = 0. \quad \tag{9}$$

Setting $Z = da^n/d\xi$, we have $(u^n)_{\xi\xi} = dZ/d\xi = ZdZ/du^n$, and Eq. (9) becomes the following first order ODE

$$-cu + au^n + \mu^2 \frac{dZ}{du^n} = 0,$$

from which we obtain

$$\mu^2 Z dZ = n(cu^n - au^{2n-1}) du. \quad \tag{10}$$

Integrating Eq. (10) and ignoring the constant of integration, we get

$$\frac{1}{2} \mu^2 Z^2 = n \left( \frac{c}{n + 1} u^{n+1} - \frac{a}{2n} u^{2n} \right). \quad \tag{11}$$

2.1. Compactons, solitary pattern solutions for equation $K(n, n)$ with $n > 1$

It follows from Eq. (11) and the definition of $Z$ that

$$\frac{\mu^2}{2} \left( \frac{2n}{n - 1} \right)^2 \left( \frac{d(u^{n-1/2})}{d\xi} \right)^2 = \frac{nc}{n + 1} - \frac{a}{2} u^{n-1}. \quad \tag{12}$$

Setting $V = u^{(n-1)/2}$, we have

$$\left( \frac{dV}{d\xi} \right)^2 = \frac{2}{\mu^2} \left( \frac{n - 1}{2n} \right)^2 \left( \frac{nc}{n + 1} - \frac{a}{2} V^2 \right). \quad \tag{13}$$

If $a > 0$, it follows from (6), (13) and Remark 1 that

$$u = \left\{ \frac{2nc}{a(n + 1)} \cos^2 \left[ \frac{(n - 1) \sqrt{a}}{2n|\mu|} \xi \right] \right\}^{1/(n-1)} \quad \tag{14}$$

or

$$u = \left\{ \frac{2nc}{a(n + 1)} \sin^2 \left[ \frac{(n - 1) \sqrt{a}}{2n|\mu|} \xi \right] \right\}^{1/(n-1)} \quad \tag{15}$$

We can write $|\mu|$ as $\mu$ since $\cos^2 \phi$ and $\sin^2 \phi$ are even functions. Using $\xi = \mu(x - ct)$, we have the following compactons:

$$\begin{cases} u = \left\{ \frac{2nc}{a(n + 1)} \cos^2 \left[ \frac{(n - 1) \sqrt{a}}{2n} (x - ct) \right] \right\}^{1/(n-1)}, & |x - ct| < \frac{n\pi}{\sqrt{a(n - 1)}}, \\ u = 0 & \text{otherwise}, \end{cases} \quad \tag{16}$$
\[
\begin{align*}
\{ u &= \begin{cases} 
\frac{2nc}{a(n+1)} \sin^2 \left[ \frac{(n-1)\sqrt{a}}{2n} (x-ct) \right] \frac{1}{(n-1)}, & |x-ct| < \frac{2n\pi}{\sqrt{a(n-1)}}, \\
0 & \text{otherwise.}
\end{cases} \\
\end{align*}
\]

(17)

**Remark 2.** When \( a = 1 \), formulas (16) and (17) are in full agreement with the results obtained in [7,12,13].

If constant \( a < 0 \), it follows from (7), (13) and Remark 1 that the solitary pattern solutions exist and have the form

\[
\left\{ u = \begin{cases} 
\frac{2nc}{a(n+1)} \cosh^2 \left[ \frac{(n-1)\sqrt{-a}}{2n} (x-ct) \right] \frac{1}{(n+1)}, & |x-ct| < \frac{\pi}{(n+1)\sqrt{-a}}, \\
0 & \text{otherwise.}
\end{cases} \\
\right.
\]

(18)

and

\[
\left\{ u = \begin{cases} 
\frac{-2nc}{a(n+1)} \sinh^2 \left[ \frac{(n-1)\sqrt{-a}}{2n} (x-ct) \right] \frac{1}{(n+1)}, & 0 < |x-ct| < \frac{2\pi}{(n+1)\sqrt{-a}}, \\
0 & \text{otherwise.}
\end{cases} \\
\right.
\]

(19)

### 2.2. Solitons and periodic solutions for equation \( K(-n,-n) \) with \( n > -1 \)

Consider the following \( K(-n,-n) \) equation:

\[
u_t + a(u^n)_x + (u^m)_{xxx} = 0, \quad n > -1.
\]

(20)

It can be shown that all formulas in Section 2.1 remain valid if we substitute \( n \) by \(-n\).

When \( a > 0 \), we obtain the periodic solutions

\[
\left\{ u = \begin{cases} 
\frac{a(1-n)}{-2nc} \sec^2 \left[ \frac{(n+1)\sqrt{a}}{2n} (x-ct) \right] \frac{1}{(n+1)}, & |x-ct| < \pi/(n+1)\sqrt{a}, \\
0 & \text{otherwise.}
\end{cases} \\
\right.
\]

(21)

for \( 0 < x - ct < 2\pi/(n+1)\sqrt{a} \), and

\[
\left\{ u = \begin{cases} 
\frac{a(1-n)}{-2nc} \csc^2 \left[ \frac{(n+1)\sqrt{a}}{2n} (x-ct) \right] \frac{1}{(n+1)}, & 0 < x - ct < 2\pi/(n+1)\sqrt{a}, \\
0 & \text{otherwise.}
\end{cases} \\
\right.
\]

(22)

When \( a < 0 \), soliton solutions are obtained and have the form

\[
\left\{ u = \begin{cases} 
\frac{a(1-n)}{-2nc} \sech^2 \left[ \frac{(n+1)\sqrt{-a}}{2n} (x-ct) \right] \frac{1}{(n+1)}, & \text{otherwise.}
\end{cases} \\
\right.
\]

(23)

and

\[
\left\{ u = \begin{cases} 
\frac{a(1-n)}{2nc} \csch^2 \left[ \frac{(n+1)\sqrt{-a}}{2n} (x-ct) \right] \frac{1}{(n+1)}, & \text{otherwise.}
\end{cases} \\
\right.
\]

(24)

### 3. Other type of \( K(n,n) \) model

In this section, we investigate the model

\[
u_t + a(u^{n+1})_x + [u^a]_{xxx} = 0, \quad n \neq 0, \quad a \neq 0.
\]

(25)
Rosenau [9] and Wazwaz [13] established some interesting results for Eq. (25). In the case \( a > 0 \) and \( n \geq 1 \), Wazwaz assumed that the general solution of the nonlinear dispersive Eq. (25) takes the form

\[
\begin{align*}
    u(x, t) &= \left( \lambda \sin^2 [\mu(x - ct)] \right)^{1/n} \\
    u &= 0, \quad \text{otherwise}
\end{align*}
\]  

or the form

\[
\begin{align*}
    u(x, t) &= \left( \lambda \cos^2 [\mu(x - ct)] \right)^{1/n}.
\end{align*}
\]  

Wazwaz [13] derived that the wave number \( \mu = \sqrt{a}/2 \) and \( \lambda = 2c/a \) and obtained the following set of compactons solutions:

\[
\begin{align*}
    \begin{cases}
        u(x, t) &= \left( \frac{2c}{a} \sin^2 \left[ \frac{\sqrt{a}}{2} (x - ct) \right] \right)^{1/n}, \quad |x - ct| \leq \frac{2\pi}{\sqrt{a}} \\
        u &= 0, \quad \text{otherwise}
    \end{cases}
\end{align*}
\]  

and

\[
\begin{align*}
    \begin{cases}
        u(x, t) &= \left( \frac{2c}{a} \cos^2 \left[ \frac{\sqrt{a}}{2} (x - ct) \right] \right)^{1/n}, \quad |x - ct| \leq \frac{\pi}{\sqrt{a}} \\
        u &= 0, \quad \text{otherwise}
    \end{cases}
\end{align*}
\]  

We will show, in the following, that the travelling wave solutions of Eq. (25) in the cases where \( a > 0 \) and \( n \geq 1 \) can be expressed by formulas (28) and (29), and that various new solutions can be derived for other values of \( a \) and \( n \).

Letting travelling wave solution \( u(x, t) = u(\xi) \) with wave variable \( \xi = \mu(x - ct) \) \((c \neq 0, \gamma \neq 0)\), we transform Eq. (2) into the following ODE:

\[
-\mu cu_{\xi} + a u^{n+1}_{\xi} + \mu^3 [u(u^n)_{\xi}]_{\xi} = 0.
\]  

Integrating Eq. (30) with respect to \( \xi \) and ignoring integration constant give rise to

\[
-c + au^n + \mu^2 (u^n)_{\xi \xi} = 0.
\]  

Setting \( u^n_\xi = Z_1 \), we get \( u^n_{\xi \xi} = Z_1 dZ_1/du^n \), and (31) becomes

\[
\mu^2 Z_1 dZ_1 = n(cu^{n-1} - au^{2n-1}) du.
\]  

Integrating (32) and setting the integration constant to be zero, we get

\[
\frac{\mu^2 Z_1^2}{2} = n \left( \frac{c}{n} u^n - \frac{a}{2n} u^{2n} \right).
\]  

In terms of \( u \) and its derivative with respect to \( \xi \), Eq. (33) becomes

\[
\left( \frac{du^{n/2}}{d\xi} \right)^2 = \frac{c}{2\mu^2} - \frac{a}{4\mu^2} u^n.
\]  

If we regard \( u^{n/2} \) as \( W \), the above equation has the same form as (5) and thus admits two different types of solutions as given in (6) and (7) depending on the values of \( a \) and \( n \).

**Case 1:** When \( a > 0 \) and \( n > 0 \), it derives from (6) and (34) that

\[
\begin{align*}
    u &= \left( \frac{2c}{a} \sin^2 \left[ \frac{\sqrt{a}}{2} (x - ct) \right] \right)^{1/n}
\end{align*}
\]  

and

\[
\begin{align*}
    u &= \left( \frac{2c}{a} \cos^2 \left[ \frac{\sqrt{a}}{2} (x - ct) \right] \right)^{1/n}.
\end{align*}
\]
Therefore we obtain the compactons solution of Eq. (2) or Eq. (25) as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{2c}{a}\sin^2\left[\frac{\sqrt{a}}{2}(x-ct)\right] \quad , \quad |x-ct| < \frac{2\pi}{\sqrt{a}}, \\
u = 0 \quad \text{otherwise}
\end{array} \right.
\end{align*}
\]

(37)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{2c}{a}\cos^2\left[\frac{\sqrt{a}}{2}(x-ct)\right] \quad , \quad |x-ct| < \frac{\pi}{\sqrt{a}}, \\
u = 0 \quad \text{otherwise}.
\end{array} \right.
\end{align*}
\]

(38)

**Case 2:** If \(a<0\) and \(n>0\), solitary patterns solutions are obtained, namely

\[
\begin{align*}
u &= \left\{ \frac{2c}{a}\sinh^2\left[\frac{\sqrt{-a}}{2}(x-ct)\right] \right\}^{1/n} \\
\text{for} \quad 0 < x - ct < \frac{2}{\sqrt{|a|}}, \quad \text{and} \\

nu &= \left\{ \frac{2c}{a}\cosh^2\left[\frac{\sqrt{-a}}{2}(x-ct)\right] \right\}^{1/n} \\
\text{for} \quad |x - ct| < \frac{\pi}{\sqrt{|a|}}.
\end{align*}
\]

(39) and (40)

**Case 3:** If \(n<0\) and \(a>0\), Eq. (25) has periodic solutions which can be expressed by

\[
\begin{align*}
u &= \left\{ \frac{a}{2c}\csc^2\left[\frac{\sqrt{a}}{2}(x-ct)\right] \right\}^{1/-n} \\
\text{for} \quad 0 < x - ct < \frac{2\pi}{\sqrt{a}}, \quad \text{and} \\

nu &= \left\{ \frac{a}{2c}\sec^2\left[\frac{\sqrt{a}}{2}(x-ct)\right] \right\}^{1/-n} \\
\text{for} \quad |x - ct| < \frac{\pi}{\sqrt{a}}.
\end{align*}
\]

(41) and (42)

**Case 4:** If \(n<0\) and \(a<0\), the solitons for Eq. (25) take the following forms:

\[
\begin{align*}
u &= \left\{ -\frac{a}{2c}\csch^2\left[\frac{\sqrt{-a}}{2}(x-ct)\right] \right\}^{1/-n} \\
\text{and} \\

nu &= \left\{ \frac{a}{2c}\sech^2\left[\frac{\sqrt{-a}}{2}(x-ct)\right] \right\}^{1/-n}.
\end{align*}
\]

(43) and (44)

**Remark 3.** Solutions (37)–(40) agree with those in [13], while solutions (41)–(44) are new solutions which were not derived by the method in [13].

### 4. The generalized KP equation in higher dimensions

Rosenau [9] and Wazwaz [13] studied the non-travelling wave solutions for the following general form of KP equation

\[
\begin{align*}
\{u_t + a(u^{n+1})_x + u(u^n)_x\}_{x} + \nabla^2 u = 0, \quad a > 0, \quad n \geq 1,
\end{align*}
\]

(45)

where \(\nabla^2_\perp = (k(k-1)/k!)\partial^2_x + (k(k-2)/k!)\partial^2_z\), \(k = 2, 3\), where \(k\) denotes the dimension of the spatial domain.

Rosenau [9] derived a non-travelling wave solution of Eq. (45) in terms of the cosine profile only. Wazwaz [13] derived non-travelling wave solutions for Eq. (45) in terms of both the cosine and sine profiles. In this section, we consider a generalized form of the KP equation expressed in the form

\[
\begin{align*}
\{u_t + a(u^{n+1})_x + u(u^n)_x\}_{x} + b_1u_{yy} + b_2u_{zz} = 0, \quad a \neq 0, \quad n \neq 0,
\end{align*}
\]

(46)
where constants $b_1$ and $b_2$ satisfy $b_1^2 + b_2^2 \neq 0$. As mentioned before, Eq. (46) becomes (45) if we set $b_1 = (k - 1)/k!$ and $b_2 = (k - 1)(k - 2)/k!$. Here we state that this work only deals with the travelling wave solutions of Eq. (46).

Let $u = u(\xi)$ with wave variable $\xi = \mu x + \eta y + \zeta z - ct$. Then, Eq. (46) can be transformed into

$$(-c\mu + b_1\eta^2 + b_2\zeta^2)u_{\xi\xi} + a\mu^2(u^{n+1})_{\xi\xi} + \mu^4[u(u^n)_{\xi\xi}]_{\xi\xi} = 0. \tag{47}$$

Integrating Eq. (46) twice with respect to $\xi$ and taking the constants of integration to be zero yield

$$(-c\mu + b_1\eta^2 + b_2\zeta^2)u + a\mu^2(u^{n+1}) + \mu^4[u(u^n)_{\xi\xi}] = 0, \tag{48}$$

which gives rise to

$$(-c\mu + b_1\eta^2 + b_2\zeta^2) + a\mu^2(u^n) + \mu^4(u^n)_{\xi\xi} = 0. \tag{49}$$

Setting $g = c\mu - b_1\eta^2 - b_2\zeta^2$ and $Z = du/d\xi$, we have from the above equation that

$$\mu^4 Z dZ = nu^{n-1}(g - a\mu^2 u^n) du. \tag{50}$$

Integrating Eq. (50) and ignoring the constant of integration result in

$$\left(\frac{du}{d\xi}\right)^2 = \frac{1}{2\mu^4} g - \frac{a}{4\mu^2} u^n. \tag{51}$$

Letting $V = u^{n/2}$, we get

$$\left(\frac{dV}{d\xi}\right)^2 = \frac{1}{2\mu^4} g - \frac{a}{4\mu^2} V^2, \tag{52}$$

which takes the same form as Eq. (5) and thus admits different types of solutions depending on the values of $a$ and $n$.

1. If $a > 0$ and $n > 0$, from (52) and $V = u^{n/2}$, we have

$$u = \left\{\frac{2(c\mu - b_1\eta^2 - b_2\zeta^2)}{a\mu^2} \sin^2 \frac{\sqrt{a}}{2\mu} (\mu x + \eta y + \zeta z - ct)\right\}^{1/n} \tag{53}$$

and

$$u = \left\{\frac{2(c\mu - b_1\eta^2 - b_2\zeta^2)}{a\mu^2} \cos^2 \frac{\sqrt{a}}{2\mu} (\mu x + \eta y + \zeta z - ct)\right\}^{1/n}. \tag{54}$$

It results from (53) and (54) that the compacton solutions of Eq. (46) have the form

$$\left\{\begin{array}{l}
\left(\mu x + \eta y + \zeta z - ct\right) < \frac{2|\mu| \pi}{\sqrt{a}}, \\
\left(\mu x + \eta y + \zeta z - ct\right) \geq \frac{2|\mu| \pi}{\sqrt{a}},
\end{array}\right\} \tag{55}$$

and

$$\left\{\begin{array}{l}
\left|\mu x + \eta y + \zeta z - ct\right| < \frac{|\mu| \pi}{\sqrt{a}}, \\
\left|\mu x + \eta y + \zeta z - ct\right| \geq \frac{|\mu| \pi}{\sqrt{a}},
\end{array}\right\} \tag{56}$$
(2) If \( a < 0 \) and \( n > 0 \), it follows from the solution of (7) and (52) that the solutions of Eq. (46) have the form

\[
\begin{equation}
    u = \left\{ -2(c\mu - b_1\eta^2 - b_2\xi^2) \frac{a\mu^2}{a\mu^2} \sinh^2 \left( \frac{\sqrt{-a}}{2\mu} (\mu x + \eta y + \xi z - ct) \right) \right\}^{1/n} \tag{57}
\end{equation}
\]

and

\[
\begin{equation}
    u = \left\{ \frac{2(c\mu - b_1\eta^2 - b_2\xi^2)}{2(c\mu - b_1\eta^2 - b_2\xi^2)} \cosh^2 \left( \frac{\sqrt{-a}}{2\mu} (\mu x + \eta y + \xi z - ct) \right) \right\}^{1/n} \tag{58}
\end{equation}
\]

(3) If \( a > 0 \) and \( n < 0 \), the periodic solutions of the KP Eq. (46) can be expressed by

\[
\begin{equation}
    u = \left\{ \frac{a\mu^2}{2(c\mu - b_1\eta^2 - b_2\xi^2)} \csc^2 \left( \frac{\sqrt{a}}{2\mu} (\mu x + \eta y + \xi z - ct) \right) \right\}^{1/-n} \tag{59}
\end{equation}
\]

for \( 0 < (\mu x + \eta y + \xi z - ct) < 2|\mu|\pi/\sqrt{a} \), and

\[
\begin{equation}
    u = \left\{ \frac{a\mu^2}{2(c\mu - b_1\eta^2 - b_2\xi^2)} \sec^2 \left( \frac{\sqrt{a}}{2\mu} (\mu x + \eta y + \xi z - ct) \right) \right\}^{1/-n} \tag{60}
\end{equation}
\]

for \(|(\mu x + \eta y + \xi z - ct)| < |\mu|\pi/\sqrt{a} \).

(4) If \( a < 0 \) and \( n < 0 \), the soliton-like solutions of the KP equation (46) are

\[
\begin{equation}
    u = \left\{ \frac{-a\mu^2}{2(c\mu - b_1\eta^2 - b_2\xi^2)} \sinh \left( \frac{\sqrt{-a}}{2\mu} (\mu x + \eta y + \xi z - ct) \right) \right\}^{1/-n} \tag{61}
\end{equation}
\]

and

\[
\begin{equation}
    u = \left\{ \frac{a\mu^2}{2(c\mu - b_1\eta^2 - b_2\xi^2)} \cosh \left( \frac{\sqrt{-a}}{2\mu} (\mu x + \eta y + \xi z - ct) \right) \right\}^{1/-n} \tag{62}
\end{equation}
\]

Remark 4. The \( y \) and \( z \) components of the wave number vectors in the travelling wave solutions (53)–(62) remain free and arbitrary constants.

5. Conclusion

As stated by many authors, nonlinear waves show a rich variety of phenomena and properties. Solving nonlinear differential equations describing nonlinear waves, such as those existing in super deformed nuclei, preformation of cluster in hydrodynamic models and the fission of liquid drops [4], needs sophisticated mathematical tools. A mathematical technique based on reduction of order of differential equations is established to study three types of nonlinear dispersive wave equations in this paper. By studying the cases with different combinations of signs of the constant \( a \) and the exponent \( n \) in these equations, we have shown that the travelling wave solutions have different structures including the sine, cosine, sinh, cosh, csc, sec, csch and sech profiles depending on the values of \( a \) and \( n \). Further work can be done to analyze the \( K(m, n) \) equations \( u_t + a(u^m)_x + b(u^n)_{xxx} = 0 \) where \( a \neq 0, b \neq 0, m \neq 0, n \neq 0, m \neq n \) and other nonlinear evolution equations.

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