q-Analogues of multiparameter non-central Stirling and generalized harmonic numbers

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Abstract

In this paper we derive q-analogues of the multiparameter non-central Stirling numbers of the first and second kind, introduced by El-Desouky. Moreover, recurrence relations, explicit formulas and a connection between these numbers and generalized q-harmonic numbers are obtained. Furthermore, some important special cases and new combinatorial identities are given. Finally, algorithms of these numbers and matrix representation using Maple are derived.

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1. Introduction

The multiparameter non-central Stirling numbers of first and second kind, respectively were introduced by El-Desouky [12] with

\[
(t)_n = \sum_{k=0}^{n} s(n, k; \underline{x})(t; \underline{x})_k, \tag{1.1}
\]

where \( \underline{x} = (x_0, x_1, \ldots, x_{n-1}) \), \( s(0, 0; \underline{x}) = 1 \), \( s(n, k; \underline{x}) = 0 \) for \( k > n \), and \( (t; \underline{x})_n = \prod_{i=0}^{n-1} (t - x_i) \), and

\[
(t; \underline{x})_n = \sum_{k=0}^{n} S(n, k; \underline{x})(t)_k, \tag{1.2}
\]

where \( S(0, 0; \underline{x}) = 1 \), \( S(n, k; \underline{x}) = 0 \) for \( k > n \). If \( x_i = i \), \( i = 0, 1, \ldots, n - 1 \), then \( (t; \underline{x})_n \) is reduced to \( (t)_n = \prod_{i=0}^{n-1} (t - i) \).

Throughout this article we adopt the following notations from [3,4,14]. With \( 0 < q < 1 \), \( t \) a real number and \( n \) positive integer:

\[
[t]_q = \frac{1 - q^t}{1 - q}, \quad \text{the q-number,}
\]

\[
[t]_q! = [t]_q[t - 1]_q \cdots [1]_q, \quad (t \in \mathbb{N}: = \{1, 2, 3, \ldots\}), \quad \text{the q-factorial of } t, \tag{1.3}
\]

\[
[t]_{n+1} = [t]_q[t - 1]_q \cdots [t - n + 1]_q, \quad \text{the falling factorial of q-number of order } n, \tag{1.4}
\]

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\[ [t]_q = [t]_q [t + 1]_q \cdots [t + n - 1]_q, \quad \text{the rising factorial of } q\text{-number of order } n. \]  

(1.5)

Let the generalized falling and rising factorial of \( q \)-number \([t]_q \) of order \( n \), associated with the sequence \( \mathfrak{r} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \), be defined, respectively, by

\[ [t; \mathfrak{r}]_q = [t - \alpha_0]_q [t - \alpha_1]_q \cdots [t - \alpha_{n-1}]_q \]

and

\[ [t; \mathfrak{r}]_q = [t + \alpha_0]_q [t + \alpha_1]_q \cdots [t + \alpha_{n-1}]_q. \]

The \( q \)-derivative operator \( D_q \) (see [1,14,15]) is defined by

\[ D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad q \neq 1. \]  

(1.6)

The \( q \)-derivative operator \( D_q \) satisfies

\[ D_q (f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x). \]  

(1.7)

The \( q \)-Binomial coefficient is defined as follows:

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}, \quad (k \in \mathbb{N}) \text{ and } \left[ \begin{array}{c} n \\ 0 \end{array} \right]_q = 1. \]  

(1.8)

The \( q \)-Stirling numbers of first and second kind [4] are defined, respectively, by

\[ [t]_q = q \left( \frac{n}{2} \right) \sum_{k=0}^{n} s_q(n, k)[t]_q^k, \]  

where \( s_q(0, 0) = 1, \ s_q(n, k) = 0 \) for \( k > n \), and

\[ [t]_q^n = \sum_{k=0}^{n} q \left( \begin{array}{c} k \\ 2 \end{array} \right) s_q(n, k)[t]_q^k, \]  

(1.10)

where \( S_q(0, 0) = 1 \) and \( S_q(n, k) = 0 \) for \( k > n \).

Let \( s_q(n, k) \) and \( S_q(n, k) \) be the generalized \( q \)-Stirling numbers of first and second kind, associated with sequence \( \mathfrak{r} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \), also called \( q \)-Comtet numbers, are defined, respectively, by

\[ [t; \mathfrak{r}]_q = q \sum_{k=0}^{n} S_q(n, k)[t]_q^k, \]  

(1.11)

where \( s_q(0, 0) = 1, \ s_q(n, k) = 0 \) for \( k > n \), and

\[ [t]_q^n = q \sum_{k=0}^{n} q^k S_q(n, k)[t; \mathfrak{r}]_q^k, \]  

(1.12)

where \( S_q(0, 0) = 1 \) and \( S_q(n, k) = 0 \) for \( k > n \) (see [13]).

Also, \( s_q(n, k; r; \mathfrak{r}) \) and \( S_q(n, k; r; \mathfrak{r}) \), the generalized non-central \( q \)-Stirling numbers of first and second kind, associated with the sequence \( \mathfrak{r} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \), are defined in [13], respectively, by

\[ [t - r; \mathfrak{r}]_q = q \sum_{k=0}^{n} q^k S_q(n, k; r; \mathfrak{r})[t]_q^k, \]  

(1.13)

where \( s_q(0, 0; r; \mathfrak{r}) = 1, \ s_q(n, k; r; \mathfrak{r}) = 0 \) for \( n < k \), and

\[ [t]_q^n = q^k S_q(n, k; r; \mathfrak{r})[t - r; \mathfrak{r}]_q^k, \]  

(1.14)

where \( S_q(0, 0; r; \mathfrak{r}) = 1 \) and \( S_q(n, k; r; \mathfrak{r}) = 0 \) for \( n < k \).

In Sections 2 and 3, we define \( q \)-analogue of multiparameter non-central Stirling numbers of first and second kind. Also, we study a modified approach to these numbers via differential operators. Moreover, we investigate new explicit formulas of some results given by Charalambides [3] and Corcino et al. [8] as special cases of our results. Furthermore, some relations between these numbers and generalized \( q \)-harmonic numbers and many new combinatorial identities are derived. In Section 4, an algorithm and matrix representation of these numbers and some results obtained are given by using Maple.
2. The multiparameter non-central \( q \)-Stirling numbers of first kind

**Definition 2.1.** Let \( t \) be a real number and \( \overline{a} = (a_0, a_1, \ldots, a_{n-1}) \) be a sequence of real numbers. The \( q \)-analogue of multiparameter non-central \( q \)-Stirling numbers of first kind \( s_q(n,k;\overline{a}) \), may be called multiparameter non-central \( q \)-Stirling numbers of first kind, is defined by

\[
[s]_{q,a} = q \left( \frac{n}{2} \right) \sum_{k=0}^{n} q^{\delta_{k,0}} s_q(n,k;\overline{a})[t;\overline{a}]_{q,a},
\]

(2.1)

where \( s_q(0,0;\overline{a}) = 1 \) and \( s_q(n,k;\overline{a}) = 0 \) for \( k > n \).

**Definition 2.2.** Let \( \Theta_q = xD_q \) and \( D_q \) be the \( q \)-differential operator, we define \( (\Theta_q;\overline{a})_{q,a} = \prod_{i=0}^{n-1}(\Theta_q - [a_i]_q) \). If \( a_i = i, i = 0,1,\ldots,n-1 \), then \( (\Theta_q;\overline{a})_{q,a} = \prod_{i=0}^{n-1}(\Theta_q - [i]_q) \).

This leads us to define the \( q \)-analogue of multiparameter non-central Stirling numbers of first kind \( s_q(n,k;\overline{a}) \) via the operator \( \Theta_q \). Therefore, Eq. (2.1) can be represented as

\[
(\Theta_q)_{q,a} = q \left( \frac{n}{2} \right) x^n D_q^n = \sum_{k=0}^{n} s_q(n,k;\overline{a})[\Theta_q;\overline{a}]_{q,a},
\]

hence

\[
x^n D_q^n = q \left( \frac{n}{2} \right) \sum_{k=0}^{n} s_q(n,k;\overline{a})[\Theta_q;\overline{a}]_{q,a}.
\]

(2.2)

**Theorem 2.1.** The numbers \( s_q(n,k;\overline{a}) \) satisfy the recurrence relation

\[
s_q(n+1,k;\overline{a}) = s_q(n,k-1;\overline{a}) + ([a_k]_q - [n]_q) s_q(n,k;\overline{a}),
\]

(2.3)

with initial condition \( s_q(n,0;\overline{a}) = [a_0]_q ([a_1]_q - [1]_q) \cdots ([a_{n-1}]_q - [n-1]_q) \).

**Proof.** Operating on both sides of (2.2) with \( \Theta_q \), we get

\[
(xD_q)(x^n D_q^n) = q \left( \frac{n}{2} \right) \sum_{k=0}^{n} s_q(n,k;\overline{a})[xD_q - [a_k]_q + [a_k]_q][\Theta_q;\overline{a}]_{q,a}
\]

\[
= q \left( \frac{n}{2} \right) \sum_{k=0}^{n} s_q(n,k;\overline{a})[\Theta_q;\overline{a}]_{q,a} + q \left( \frac{n}{2} \right) \sum_{k=0}^{n} s_q(n,k;\overline{a})[a_k]_q[\Theta_q;\overline{a}]_{q,a},
\]

hence

\[
(xD_q)(x^n D_q^n) = q \left( \frac{n}{2} \right) \sum_{k=0}^{n} (s_q(n,k-1;\overline{a}) + [a_k]_q s_q(n,k;\overline{a})[\Theta_q;\overline{a}]_{q,a}.
\]

(2.4)

Furthermore, using (1.8) and (2.2) we obtain

\[
(xD_q)(x^n D_q^n) = q^n x^{n-1} D_q^{n+1} + [n]_q x^n D_q^n = q \left( \frac{n}{2} \right) \sum_{k=0}^{n-1} s_q(n+1,k;\overline{a}) + [n]_q s_q(n,k;\overline{a})[\Theta_q;\overline{a}]_{q,a}.
\]

(2.5)

Equating the coefficients of \( (\Theta_q;\overline{a})_{q,a} \) in (2.4) and (2.5) yields (2.3). \( \square \)

**Special cases:**

(i) When \( a_i = 0, i = 0,1,\ldots,n-1 \), then (3.2) is reduced to

\[
s_q(n+1,k) = s_q(n,k-1) + [n]_q s_q(n,k), \text{ where } s_q(n,k) \text{ denote } q \text{-Stirling numbers of first kind}, \text{ see [3].}
\]

(ii) When \( a_i = -x, i = 0,1,\ldots,n-1 \), then from (2.1) and [4, Eq. (4.1)] we obtain that \( S_q(n,k;\overline{a}) = q^{k-n} s_q(n,k;\overline{a}) \), where \( S_q(n,k;\overline{a}) \) denotes the non-central \( q \)-Stirling numbers of first kind, see [4].

(iii) If \( a_i = 1, i = 0,1,\ldots,n-1 \), then \( s_q(n,k;\overline{a}) = \delta_{n,k}. \)
Theorem 2.2. The numbers \( s_q(x_0, n, k; \mathfrak{u}) \) have the explicit formula

\[
s_q(x_0, n, k; \mathfrak{u}) = \sum_{\sigma_n = k} \binom{x_0}{n} \prod_{i=0}^{n-1} \left[ i_0 + x_i \right]_{q} \left[ i_1 + x_{i_0 + i_1} - 1 \right]_{q} \cdots \left[ i_{n-1} + x_{i_0 + i_1 + \cdots + i_{n-1}} - n + 1 \right]_{q},
\]

where \( \sigma_n = i_0 + i_1 + \cdots + i_{n-1} \) and \( i_0 \in \{0, 1\} \).

Proof. When \( k = 0 \),

\[
s_q(x_0, 0; \mathfrak{u}) = \sum_{\sigma_n = k} \binom{1}{2} \prod_{i=0}^{n-1} \left[ i_0 + x_i \right]_{q} \left[ i_1 + x_{i_0 + i_1} - 1 \right]_{q} \cdots \left[ i_{n-1} + x_{i_0 + i_1 + \cdots + i_{n-1}} - n + 1 \right]_{q},
\]

which is easily verified by using (2.3).

If \( i_{n-1} \in \{0, 1\} \), we have

\[
s_q(x_0, n, k; \mathfrak{u}) = \sum_{\sigma_n = k} \binom{x_0}{n} \prod_{i=0}^{n-2} \left[ i_0 + x_i \right]_{q} \left[ i_1 + x_{i_0 + i_1} - 1 \right]_{q} \cdots \left[ i_{n-2} + x_{i_0 + i_1 + \cdots + i_{n-2}} - n + 2 \right]_{q},
\]

then

\[
s_q(x_0, n, k; \mathfrak{u}) = \sum_{\sigma_n = k} \binom{x_0}{n} \prod_{i=0}^{n-2} \left[ i_0 + x_i \right]_{q} \left[ i_1 + x_{i_0 + i_1} - 1 \right]_{q} \cdots \left[ i_{n-2} + x_{i_0 + i_1 + \cdots + i_{n-2}} - n + 2 \right]_{q},
\]

hence

\[
s_q(x_0, n, k; \mathfrak{u}) = s_q(n - 1, k - 1; \mathfrak{u}) + \left[ x_0 \right]_{q} \left[ n - 1 \right]_{q} s_q(n - 1, k; \mathfrak{u}).
\]

This, by virtue of (2.3), completes the proof of (2.6). \( \square \)

3. The multiparameter non-central \( q \)-Stirling numbers of second kind

**Definition 3.1.** Let \( t \) be a real number and \( \mathfrak{u} = (x_0, x_1, \ldots, x_{n-1}) \) be a sequence of real numbers. The \( q \)-analogue of multiparameter non-central Stirling numbers of second kind \( S_q(n, k; \mathfrak{u}) \) is defined by

\[
[t; \mathfrak{u}]_{q} = \sum_{k=0}^{n} \binom{k}{n} \sum_{\sigma_n = k} S_q(n, k; \mathfrak{u}) [t]_{q},
\]

where \( S_q(0, 0; \mathfrak{u}) = 1 \) and \( S_q(n, k; \mathfrak{u}) = 0 \) for \( k > n \).

**Definition 3.2.** The numbers \( S_q(n, k; \mathfrak{u}) \) are defined via the differential operator \( \Theta_q \) by

\[
(\Theta_q t; \mathfrak{u})_{q} = \sum_{k=0}^{n} \binom{k}{n} \sum_{\sigma_n = k} S_q(n, k; \mathfrak{u}) [t]_{q} x^k D_q^k,
\]

where \( S_q(0, 0; \mathfrak{u}) = 1 \) and \( S_q(n, k; \mathfrak{u}) = 0 \) for \( k > n \).

**Theorem 3.1.** The numbers \( S_q(n, k; \mathfrak{u}) \) satisfy the recurrence relation

\[
S_q(n, k; \mathfrak{u}) = \sum_{i=0}^{n-1} S_q(n-1, k-i; \mathfrak{u}) (1 - q) x_i,
\]

where \( x_i = x_0 + \cdots + x_{i-1} \).
\[ S_q(n + 1, k; \overline{x}) = S_q(n, k - 1; \overline{x}) + ([k]_q - [z_0]_q)S_q(n, k; \overline{x}), \]  
\[ \text{where } S_q(n, 0; \overline{x}) = (-1)^n[z_0]_q[z_1]_q \cdots [z_{n-1}]_q. \] \tag{3.3}

**Proof.** Its proof is similar to that of Theorem 2.1. □

Furthermore, we handle the following special cases:

(i) When \( z_i = 0, \overline{z} = (0, 1, \ldots, n - 1), \) then we have from (3.3)
\[ S_q(n + 1, k) = S_q(n, k - 1) + [k]_qS_q(n, k), \] \tag{3.4}
where \( S_q(n, k) \) denote \( q \)-Stirling numbers of the second kind, see [3].

(ii) When \( z_i = -x, \overline{z} = (0, 1, \ldots, n - 1), \) from (3.1) and [4, Eq.(2.12)] we obtain
\[ \overline{S}_q(n, k; x) = q^{x(n-k)}S_q(n, k; -x), \] \tag{3.5}
where \( \overline{S}_q(n, k; x) \) denote non-central \( q \)-Stirling numbers of the second kind.

(iii) When \( z_i = -i, \overline{z} = (0, 1, \ldots, n - 1) \) and from (3.1) and [3, Eq. (2.7)] we obtain
\[ |L_q(n, k)| = q \left( \frac{n}{2} \right)^k \binom{k}{2} S_q(n, k; -\overline{i}), \] \tag{3.6}
where \( |L_q(n, k)| \) denote signless \( q \)- Lah numbers.

(iv) When \( z_i = \beta i, \overline{z} = (0, 1, \ldots, n - 1), \) from (3.1) and [3, Eq. (2.12)] we obtain
\[ S_q(n, k; \beta i) = R_q(n, k; \beta), \] \tag{3.7}
where \( R_q(n, k; \beta) \) are coefficients of the generalized \( q \)-factorials.

**Theorem 3.2.** The numbers \( S_q(n, k; \overline{x}) \) can be expressed as
\[ S_q(n, k; \overline{x}) = q \sum_{i=0}^{n-1} (-1)^i \binom{k}{2} \frac{1}{[k]_q} \sum_{i=0}^{k} (-1)^{k-i} q \left( \frac{k-r}{r} \right) [r]_q |r; \overline{x}|_q. \] \tag{3.8}

**Proof.** From \( q \)-Newton formula, (see [4]),
\[ f_n(t) = \sum_{k=0}^{n} \frac{1}{[k]_q} [\Delta^k_q f_n(t)]_{t=0} [t]_q \]
and (3.1), we obtain
\[ q \sum_{i=0}^{n-1} (-1)^i \binom{k}{2} S_q(n, k; \overline{x}) = \frac{1}{[k]_q} \Delta^k_q |t; \overline{x}|_q, \]
where \( \Delta^k_q |t; \overline{x}|_q = \sum_{r=0}^{k} (-1)^{k-r} q \left( \frac{k-r}{r} \right) [r]_q |E^r(t; \overline{x})|_q. E \) is the shift operator defined by \( E f(x) = f(x + 1) \) and \( E'f(x) = f(x + r) \). Thus, we obtain (3.8). □

**Theorem 3.3.** The numbers \( S_q(n, k; \overline{x}) \) have the explicit formula
\[ S_q(n, k; \overline{x}) = \sum_{\sigma_n = n-k} q \sum_{j=0}^{n-1} \prod_{j=0}^{k-1} \frac{j - z_j - \sum_{j=1}^{i} i_j}{i_j} \] \tag{3.9}
where \( \sigma_n = i_0 + i_1 + \cdots + i_{n-1}, \overline{i_j} \in \{0, 1\}, j = 0, 1, \ldots, n - 1 \) and \( i_j = 0 \iff z_j = 0. \)

**Proof.** When \( k = 0, \) then
Corollary 3.1. An explicit expression for $R_q(n, k; \mathfrak{x})$, see [3], is

$$R_q(n, k; \mathfrak{x}) = \sum_{\sigma_n=k} q \left( \begin{array}{c} n \\sigma_n-k \end{array} \right) \prod_{j=1}^{\sigma_n-k} \left[ j - j \alpha - \sum_{i=0}^{j-1} i_j \right]$$

(3.11)

where $\sigma_n = i_0 + i_1 + \cdots + i_{n-1}$, $i_j \in \{0, 1\}$ and $j = 0, 1, \ldots, n-1$.

Proof. Putting $\alpha_i = i \alpha$ in (3.9), then we obtain (3.11). $\Box$

From (3.8) and (3.9), we have a new combinatorial identity

$$\sum_{k=0}^{n-1} \binom{n}{2} \frac{1}{[k]_q} \sum_{r=0}^{k} (-1)^{k-r} \binom{k-r}{2} \binom{k}{r} \sum_{\sigma_n=k} q \left( \begin{array}{c} n \\sigma_n-k \end{array} \right) \prod_{j=0}^{\sigma_n-k} \left[ j - j \alpha - \sum_{i=0}^{j-1} i_j \right]$$

(3.10)

Corollary 3.2. For $\alpha_i = i \beta = \left( 0, \frac{\beta}{2}, \ldots, \frac{(n-1)\beta}{2} \right)$ and $\tilde{i} = (0, 1, \ldots, n-1)$, we have

$$S_q\left( n, k, \left( \frac{2}{p} \right) \tilde{i} \right) = \left[ \frac{1}{[\tilde{i}]_q} \right]_{q^{n-k}} \sigma_{q^{n-k}}^{\gamma \beta} (n, k; \alpha; \beta, 0),$$

(3.14)

where $\sigma_{q^{n-k}}^{\gamma \beta} (n, k; \alpha; \beta, 0)$ is the $q$-analogue of generalized Stirling numbers of the first kind when $\gamma = 0$, see [7,8].

Proof. Putting $\alpha_i = i \beta = \left( 0, \frac{\beta}{2}, \ldots, \frac{(n-1)\beta}{2} \right)$ and $\tilde{i} = (0, 1, \ldots, n-1)$ in (3.8), then we have

$$S_q\left( n, k, \left( \frac{2}{p} \right) \tilde{i} \right) = \sum_{\sigma_n=k} q \left( \begin{array}{c} n \\sigma_n-k \end{array} \right) \prod_{j=0}^{\sigma_n-k} \left[ j - j \alpha - \sum_{i=0}^{j-1} i_j \right]$$

(3.13)
\[ S_q \left( n, k; \left( \frac{a}{p} \right) \right) = q^{\frac{a}{p}} \binom{n}{2} \binom{k}{2} \sum_{r=0}^{k} (-1)^{k-r} q^{\binom{k-r}{2}} \frac{1}{r!q^{r-1}} \prod_{j=0}^{r-1} \left( r - \frac{i^{j+1}}{p} \right), \]

since

\[ \prod_{r=0}^{n-1} r - \frac{i^{r+1}}{p} = q^{\frac{a}{p}} \binom{n}{2} \sum_{r=0}^{k} (-1)^{k-r} q^{\binom{k-r}{2}} \frac{1}{r!q^{r-1}} \prod_{j=0}^{r-1} \left( r - \frac{i^{j+1}}{p} \right), \]

and

\[ [k]_q! = \prod_{i=1}^{k} (1 - q^i) = \prod_{i=1}^{k} \frac{1}{\beta_i}, \]

then

\[ S_q \left( n, k; \left( \frac{a}{p} \right) \right) = \left[ 1 \right]_{\beta} \sum_{r=0}^{k} (-1)^{k-r} q^{\binom{k-r}{2}} \frac{1}{r!q^{r-1}} \prod_{j=0}^{r-1} \left( r - \frac{i^{j+1}}{p} \right). \]

from [8, Eq. (3)] when \( \gamma = 0 \) yields (3.14). \( \square \)

Also, we find a relationship between \( q \)-analogue of multiparameter non-central Stirling numbers of the first and second kind and the generalized \( q \)-Stirling numbers of the first kind (see [13]) as follows.

**Corollary 3.3.**

\[ s_q(n, k) = \sum_{k=0}^{n} s_q(n, k; \overline{a}) s_q(\overline{k}, \ell). \]  

(3.15)

**Proof.** Since

\[ [\ell]_q = q^{\frac{a}{p}} \sum_{k=0}^{n} \sum_{i=0}^{k} s_q(n, k; \overline{a}) [\ell]_q^{i}, \]

from (1.9) and (1.11), we have

\[ q^{\frac{a}{p}} \sum_{k=0}^{n} s_q(n, \ell) [\ell]_q^{k} = q^{\frac{a}{p}} \sum_{k=0}^{n} s_q(n, k; \overline{a}) \sum_{\ell=0}^{k} s_q(\overline{k}, \ell) [\ell]_q^{k} = q^{\frac{a}{p}} \sum_{\ell=0}^{n} \sum_{k=0}^{n} s_q(n, k; \overline{a}) s_q(\overline{k}, \ell) [\ell]_q^{k}. \]

Equating coefficients of \([\ell]_q^k\) on both sides yields (3.15). \( \square \)

Similarly, from (3.1), using (1.11) and (1.9), we get

\[ s_q(\overline{n}, \ell) = \sum_{k=0}^{n} s_q(n, k; \overline{a}) s_q(\overline{k}, \ell). \]  

(3.16)

We derive a combinatorial identity which gives a relation between \( s_q(n, k) \) and \( H_{q,n}^{(k)} \), the generalized \( q \)-harmonic number of order \( k \).

**Theorem 3.4.** The numbers \( s_q(n, k) \) satisfy the identity

\[ \sum_{\ell=0}^{m} (-1)^{\ell} q^{-\ell} \binom{\ell}{m} s_q(n, \ell) = (-1)^{n+m} [n]_q \sum_{r=0}^{m} (-1)^{r} \prod_{k=1}^{r} H_{q,n}^{(k)}, \]  

(3.17)

where \( H_{q,n}^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} \) and \( H_{q,0}^{(k)} = 0 \) are the generalized \( q \)-harmonic numbers of order \( k \), see [11].

**Proof.** From (2.1) and (1.11), we have

\[ s_q(n, k; \overline{a}) = \sum_{\ell=0}^{m} (-1)^{\ell} q^{-\ell} \binom{\ell}{m} s_q(n, \ell) \]  

and

\[ H_{q,n}^{(k)} = \sum_{j=1}^{n} \frac{1}{j^k} \]  

for \( k \geq 1 \).
\[ [t - 1]_{\mathbb{L}_{q}} = q^{-\frac{n}{2}} \sum_{k=0}^{n} q^{\frac{n-k}{2}} s_{q}(n, k) [t - 1]_{\mathbb{L}_{q}} \]

and

\[ [t - 1; \mathbb{L}_{q}]_{\mathbb{L}_{q}} = q^{\sum_{k=0}^{n} \sum_{l=0}^{n} (-1)^{l-m} q^{n-l} \binom{\ell}{m} s_{q}(n, k) s_{q}(k, \ell) [t]^m]_{\mathbb{L}_{q}}. \]

hence we get

\[ [t - 1]_{\mathbb{L}_{q}} = q^{-\frac{n+1}{2}} \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} (-1)^{l-m} q^{n-l} \binom{\ell}{m} s_{q}(n, l) \right) [t]^m_{\mathbb{L}_{q}}. \]

Then using (3.15) gives

\[ [t - 1]_{\mathbb{L}_{q}} = q^{-\frac{n+1}{2}} \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} (-1)^{l-m} q^{n-l} \binom{\ell}{m} s_{q}(n, \ell) \right) [t]^m_{\mathbb{L}_{q}}. \]

On the other hand,

\[ [t - 1]_{\mathbb{L}_{q}} = (-1)^{q} q^{-\frac{n+1}{2}} \left[ \sum_{j=1}^{n} \left( 1 - \frac{[t]_{\mathbb{L}_{q}}}{j} \right) \right] = (-1)^{q} q^{-\frac{n+1}{2}} \left[ \sum_{j=1}^{n} \log \left( 1 - \frac{[t]_{\mathbb{L}_{q}}}{j} \right) \right] \]

\[ = (-1)^{q} q^{-\frac{n+1}{2}} \left[ \sum_{j=1}^{n} \left( \sum_{k=0}^{\infty} \frac{[t]_{\mathbb{L}_{q}}^{k}}{k} \right) \right] \]

\[ = (-1)^{q} q^{-\frac{n+1}{2}} \left[ \sum_{k=1}^{\infty} \frac{(-1)^{q} q^{-\frac{n+1}{2}}}{r!} \left( \sum_{k=1}^{\infty} \frac{[t]_{\mathbb{L}_{q}}^{k}}{k^n} \right) \right] \]

Using Cauchy product rule for series, we have

\[ [t - 1]_{\mathbb{L}_{q}} = (-1)^{q} q^{-\frac{n+1}{2}} \left[ \sum_{j=1}^{n} \left( 1 - \frac{[t]_{\mathbb{L}_{q}}}{j} \right) \sum_{k=1}^{\infty} \frac{[t]_{\mathbb{L}_{q}}^{k}}{k^n} \right] \]

Equating the coefficients of \([t]_{\mathbb{L}_{q}}^{m}\), in (3.19) and (3.20), yields (3.17).

For the particular case \(m = 2\) and \(n = 3\), we obtain the identity

\[ \sum_{i=2}^{3} (-1)^{q-1} \binom{\ell}{2} s_{q}(3, \ell) = -[3]_{q} \left( -H_{q,3}^{(2)}/2 + \frac{H_{q,3}^{(1)} H_{q,3}^{(1)}}{2} \right), \]

hence we have

\[ 3 \left( (H_{q,3}^{(1)})^2 - H_{q,3}^{(2)} \right) = 3 + 2q + q^2. \]

Also, for \(m = 3\) and \(n = 4\), we obtain

\[ \sum_{i=3}^{4} (-1)^{q-1} \binom{\ell}{3} s_{q}(4, \ell) = -[4]_{q} \left( -H_{q,4}^{(3)}/3 + \frac{H_{q,4}^{(1)} H_{q,4}^{(2)}}{2} - (H_{q,4}^{(1)})^3/6 \right), \]

hence we have an identity

\[ \frac{1}{3!} \left( -2H_{q,4}^{(3)} - 3H_{q,4}^{(1)} H_{q,4}^{(2)} + (H_{q,4}^{(1)})^3 \right) = \frac{3q + 2q^2 + q^3 + 4}{[4]_{q}^3}. \]

When \(q \rightarrow 1\), (3.21) and (3.23) give

\[ (H_{3}^{(1)})^2 - H_{3}^{(2)} = 2, \]
and (3.17) yields the following interesting identity
\[\sum_{\ell=m}^{n} (-1)^\ell \left(\begin{array}{c} m \\ \ell \end{array}\right) s(n, \ell) = (-1)^{n-m} n! \sum_{\ell=0}^{m} \frac{(-1)^\ell}{\ell !} \prod_{i=1}^{\ell} \frac{H_{i}^{(k_i)}}{i}, \quad (3.24)\]

**Theorem 3.5.** The numbers \(s_{q,n}(n,k)\) can be expressed as
\[s_{q,n}(n,k) = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{j=0}^{\infty} \left(1 - \frac{\epsilon_i}{|x|_q}\right) = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{j=0}^{\infty} \exp \left(\sum_{j=0}^{\infty} \log \left(1 - \frac{\epsilon_i}{|x|_q}\right)\right)\]
where \(H_{q,n}(x) = \sum_{j=0}^{\infty} \frac{1}{j!} \left[H_{q,n}(x)\right]_{j} \frac{H_{q,n}(x)}{j!} - \frac{H_{q,n}(x)}{2} \frac{H_{q,n}(x)}{2} + \ldots\), are called the generalized q-harmonic numbers of order k associated with the sequence \(x = (a_0, a_1, \ldots, a_{n-1})\), see [13].

**Proof.** Since
\[\left[x; x\right]_{q,n} = [x - a_0]_q [x - a_1]_q \cdots [x - a_{n-1}]_q = q^{-a_0}([t]_q - [a_0]_q)q^{-a_1}([t]_q - [a_1]_q) \cdots q^{-a_{n-1}}([t]_q - [a_{n-1}]_q),\]
\[= (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{j=0}^{\infty} \left(1 - \frac{\epsilon_i}{|x|_q}\right) = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{j=0}^{\infty} \exp \left(\sum_{j=0}^{\infty} \log \left(1 - \frac{\epsilon_i}{|x|_q}\right)\right),\]
then using multinomial theorem gives
\[\left[x; x\right]_{q,n} = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{j=0}^{\infty} \left(1 - \frac{\epsilon_i}{|x|_q}\right) = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{j=0}^{\infty} \frac{1}{j!} \left[H_{q,n}(x)\right]_{j} \frac{H_{q,n}(x)}{j!} - \frac{H_{q,n}(x)}{2} \frac{H_{q,n}(x)}{2} + \ldots\]
and from (1.11), this leads to
\[\sum_{k=0}^{n} s_{q,n}(n,k) [t]_q^k = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \frac{H_{q,n}(x)}{m_i} [t]_q^k,\]
Equating the coefficients of \([t]_q^k\) on both sides yields (3.25). □

**Remark 3.1.** It is worth noting that using Cauchy product rule for series in the proof of Theorem 3.5, we get
\[s_{q,n}(n, \ell) = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \frac{H_{q,n}(x)}{m_i}.\]
Setting \(\ell = n\) in (3.26) we have an identity
\[(-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \frac{H_{q,n}(x)}{m_i} = 1.\]
Also, setting \(\ell = n - 1\) in (3.26) we obtain
\[s_{q,n}(n, n-1) = (-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \frac{H_{q,n}(x)}{m_i}.\]
If \(q \to 1\), we get the identities
\[(-1)^n \prod_{i=0}^{n-1} [x]_q \sum_{k=0}^{n} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \frac{H_{n}(x)}{m_i} = 1.\]
where \( H^k_i(x) = \sum_{j=0}^{n-1} \frac{1}{j!} \) and \( H^k_0(x) = 0 \) are called the generalized harmonic numbers of order \( k \) associated with the sequence \( x = (x_0, x_1, \ldots, x_n) \) and \( s_q(n, k) \) are Comtet numbers (see [9,10,13]).

For \( \ell = 1, 2, 3 \) and 4, in (3.26), we respectively get the following identities

\[
s_q(n, 1) = (-1)^{n+1} \prod_{i=0}^{n-1} [x_i]! H^{(1)}_{q, n}(x).
\]

(3.31)

\[
s_q(n, 2) = \frac{(-1)^n}{2!} \prod_{i=0}^{n-1} [x_i]! \left( H^{(1)}_{q, n}(x) \right)^2 - H^{(2)}_{q, n}(x).
\]

(3.32)

\[
s_q(n, 3) = \frac{(-1)^n}{3!} \prod_{i=0}^{n-1} [x_i]! \left( -\left( H^{(1)}_{q, n}(x) \right) + 3 H^{(1)}_{q, n}(x) H^{(2)}_{q, n}(x) - 2 H^{(3)}_{q, n}(x) \right).
\]

(3.33)

\[
s_q(n, 4) = \frac{(-1)^n}{4!} \prod_{i=0}^{n-1} [x_i]! \left( H^{(1)}_{q, n}(x) \right)^4 + 8 H^{(1)}_{q, n}(x) H^{(2)}_{q, n}(x) + 3 \left( H^{(2)}_{q, n}(x) \right)^2 - 6 \left( H^{(1)}_{q, n}(x) \right)^2 H^{(2)}_{q, n}(x) - 6 H^{(4)}_{q, n}(x).
\]

(3.34)

For the special case \( x_i = i + 1, \ i = 0, 1, \ldots, n - 1, \) (3.26) yields

\[
s_q(n + 1, \ell + 1) = (-1)^n [n]! \sum_{k=0}^{\ell} \sum_{m_1 + m_2 + \cdots + m_{\ell} = n} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \left( H^{(m_i)}_{q, n} \right) m_i.
\]

When \( q \to 1, \) we get

\[
s(n + 1, \ell + 1) = (-1)^n n! \sum_{k=0}^{\ell} \sum_{m_1 + m_2 + \cdots + m_{\ell} = n} \frac{(-1)^k}{k!} \prod_{i=1}^{\ell} \left( H^{(m_i)} \right) m_i.
\]

(3.36)

From (3.17) and (3.35), we have an identity

\[
(-1)^m s_q(n + 1, \ell + 1) = \sum_{l=m}^{n} (-1)^l \left( \frac{\ell}{m} \right) s_q(n, \ell).
\]

(3.37)

Furthermore if \( q \to 1, \) gives

\[
(-1)^m s(n + 1, \ell + 1) = \sum_{l=m}^{n} (-1)^l \left( \frac{\ell}{m} \right) s(n, \ell).
\]

(3.38)

Moreover setting \( x_i = i + 1, \ i = 0, 1, \ldots, n - 1, \) then Eqs. (3.31)-(3.34) are, respectively, reduced to

\[
s_q(n + 1, 2) = (-1)^n [n]! H^{(1)}_{q, n}.
\]

(3.39)

\[
s_q(n + 1, 3) = \frac{(-1)^n}{2!} [n]! \left( H^{(1)}_{q, n} \right)^2 - H^{(2)}_{q, n},
\]

(3.40)

\[
s_q(n, 4) = \frac{(-1)^n}{3!} [n]! \left( -\left( H^{(1)}_{q, n} \right) + 3 H^{(1)}_{q, n} H^{(2)}_{q, n} - 2 H^{(3)}_{q, n} \right)
\]

(3.41)

and

\[
s_q(n, 5) = \frac{(-1)^n}{4!} [n]! \left( H^{(1)}_{q, n} \right)^4 + 8 H^{(1)}_{q, n} H^{(2)}_{q, n} + 3 \left( H^{(2)}_{q, n} \right)^2 - 6 \left( H^{(1)}_{q, n} \right)^2 H^{(2)}_{q, n} - 6 H^{(4)}_{q, n}.
\]

(3.42)

In fact, Eqs. (3.39)-(3.42) are \( q \)-analogues of [5, Eq. (2.10)], (see also [6,16]).

4. The matrix representation

Let \( s_q; \ s_q^1; \ s_q^2 \) and \( S_q(x) \) be \( n \times n \) lower triangular matrices whose entries are, respectively, the \( q \)-Stirling numbers of first kind, the \( q \)-Comtet numbers of first kind, the \( q \)-analogue of multiparameter non-central Stirling numbers of the second kind and the non-central \( q \)-Stirling numbers of second kind.
The following algorithm determines the entries of the matrix $S_q(\mathbf{x})$, the $q$-analogue of multiparameter non-central Stirling numbers of the first kind.

**Algorithm.** For a positive integer $n$, the elements of the $n \times n$ lower triangular matrix of $S_q(\mathbf{x})$ can be calculated as follows:

Set $s_q^{-1}(\mathbf{x}) = 1$
For $i = 2$ to $n$ do
Set $s_q^i(\mathbf{x}) = 1$
Calculate $s_q^{i-1}(\mathbf{x}) = \prod_{r=0}^{i-1} (|x_0|_q - r)_q + (|x_1|_q - i)_q$
Next $i$
For $i = 3$ to $n$ do
for $j = 2$ to $i - 1$ do
Calculate $s_q^{i-j-1}(\mathbf{x}) = s_q^{i-1-j}(\mathbf{x}) + (|x_j|_q - i)_q s_q^{i-1}(\mathbf{x})$
Next $j$
Next $i$

An algorithm that generates the entries of the matrix $S_q(\mathbf{x})$ can be written similarly.
A computer program is written using Maple program and executed for calculating $S_q(\mathbf{x})$.

For example, if $n = 3$, we have

$$S_q(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ [x_0] + [x_1] - 1 & 1 & 0 \\ [x_0]^2 - [x_0] + [x_1][x_0] + [x_1]^2 - [x_1] - [2][x_0] - [2][x_1] + [2] & [x_0] + [x_1] - 1 + [x_2] - [2] & 1 \end{pmatrix}$$

Next we find a matrix representation of some results obtained.
It is easily be seen that (3.5) can be represented in matrix form as

$$A(x) = S_q(x)S_q^{-1}(-x),$$

(4.1)

where $A(x)$ is an $n \times n$ lower triangular matrix.

For instance, if $n = 3$, then

$$\begin{pmatrix} 1 & 0 & 0 \\ q^2 + 2[-x] - 1 & 1 & 0 \\ q^{2z} + q^z(1 + [2])(2[-x] - 1) + 9[-x]^2 - 2[-x] - 2[-x][2] + [2] & 3[-x][2] - 1 & 1 \end{pmatrix}$$

For $n = 3$, we have

$$(1) \begin{pmatrix} 1 & 0 & 0 \\ q^2 & 1 & 0 \\ q^{2z} & q^z(1 + [2]) & 1 \end{pmatrix}$$

Eq. (3.15) can also be expressed in matrix form as

$$s_q = s_q(\mathbf{x})S_q^{-1}(\mathbf{x}).$$

This implies

$$s_q(\mathbf{x}) = s_qS_q^{-1}(\mathbf{x}) = S_q(\mathbf{x}).$$

(4.2)

where $S_q(\mathbf{x})$ is an $n \times n$ lower triangular matrix whose entries are the $q$-Comtet numbers of second kind.

For example, if $n = 3$, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ [x_0] + [x_1] - 1 & 1 & 0 \\ [x_0]^2 - [x_0] + [x_1][x_0] + [x_1]^2 - [x_1] - [2][x_0] - [2][x_1] + [2] & [x_0] + [x_1] - 1 + [x_2] - [2] & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ [2] & -1 & 1 \end{pmatrix}$$

Let $L_q = |(L_q)|_y$, be $n \times n$ lower triangular matrix whose entries are the signless $q$-Lah numbers.
Eq. (3.6) can be represented in matrix form as
\[ C = L_q S_q^{-1} (-I), \] 

where \( C \) is an \( n \times n \) lower triangular matrix.

For example, when \( n = 3 \), we have

\[
\begin{pmatrix}
1 & 0 & 0 \\
q^2 + q^4(1 + [2])(-1 + [-1]) + q^6(-1 + [2]) + [-1]^2 - [-1][2] & 1 + [-1] & 0 \\
q^3 + q^6(1 + [2]) & 1 + [-1] & 0 \\
q^4(1 + [2]) & 1 + [-1] & 0 \\
qu^6 & 1 + [-1] & 0 \\
\end{pmatrix}
\]

References