The double scroll chaotic attractor in the dynamics of a fixed-price IS-LM model

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Abstract: With the aim of exploring the conditions which determine a chaotic behavior in the long-run properties of an economic model, this paper innovates the existing Keynesian macroeconomic literature by showing that the dynamics of the well-known IS-LM model may generate a double-scroll strange attractor, for a particular set of structural parameters.

Keywords: global bifurcation; Bogdanov-Takens singularity; double scroll attractor; IS-LM model.


Biographical notes: Giovanni Bella’s actual research is inserted in the literature of environmental economics, with particular attention to the different typologies of economic equilibrium that emerge when introducing an environmental variable. His recent interest regards the application of search theory to the problem of pollution control, while particular attention is reserved to the problem of indeterminateness of the equilibrium solutions, and the consequent problems of bifurcation and chaos in the steady state trajectories. This also implies the study of the dynamic problems that characterise the economic systems in the transition towards a sustainable development. A deep analysis of the environmental Kuznets curve hypothesis within OECD countries is also analysed. A further field of analysis regards the application of the predator-prey theory, and its impact on the exploitation of the natural resources and the mass tourism concerns.

Paolo Mattana’s actual research activity mainly involves the analysis of transitional and long-run dynamics arising in variants of the two-sector growth models with particular regard to indeterminacy problems and chaotic solutions. Investigations of this kind are important in economic theory since help mapping the regions of the parameters space in correspondence of which the capacity of the models to produce indications on future economic outcomes starting from given fundamentals is drastically impaired.
Beatrice Venturi’s actual research expertise mainly relates to non-linear analysis, in particular chaos theory and bifurcations, which allowed her to analytically develop various interesting economics and financial models. Her research activity, during these years has been mainly directed to the study of mathematical applications within the endogenous growth theory, the international trade markets, and the dynamics of the standard IS-LM model.

1 Introduction

Recent literature has revived interest in Schinasi’s (1981, 1982) variant of the dynamic IS-LM model to better understand the determinants for the emergence of endogenous fluctuations, and the existence of oscillating solutions in specific regions of the parameters space, to justify the rise of business cycle (e.g., Makovinyiova, 2011; Fanti and Mandredi, 2007; Neri and Venturi, 1999, 2007). Until now, most of the papers mainly focused on a local analysis, which unfortunately does not allow us to get the complete picture of the dynamic evolution of such economic models. To this end, the principles of global bifurcation theory might be useful to gain hints on the properties of the equilibrium, and to investigate the whole set of conditions which lead to global indeterminacy and, eventually, chaotic behavior outside the small neighbourhood of the BGP (see, for example, Mattana et al., 2009; Nishimura and Shigoka, 2006). Of particular interest, seems the case where, within a region of parameters, the equilibrium trajectories move off the steady state stable path, and start to behave chaotically, thus allowing an invariant distribution of oscillating growth rates to emerge (see, for example, Boldrin et al., 2001). This paper aims to give a contribution to this line of research.

Application of chaos theory to Keynesian macroeconomics systems, whose complex behaviour becomes a priori unpredictable, seems an innovative field to work on, even though the emergence of chaos is usually related to very rich non-linear dynamics and highly complicated mathematical analysis. We tackle this problem by investigating the conditions for the existence of two disjoint Rössler attractors that grow in size until eventually they collide and give birth to the so-called double scroll strange attractor, originally observed in Physics, as a simple unfolding of a $\mathbb{R}^3$-vector field, which generates a ‘period-doubling’ bifurcation tree. Simulation results serve us to support our theoretical derivations.

The paper develops as follows. The second section introduces the well-known Schinasi (1981, 1982) IS-LM dynamical system, and studies the long-run properties of the equilibrium. The third section shows the emergence of the double-scroll chaotic attractor, by using some powerful instruments of the global bifurcation analysis. A brief conclusion reassesses the main findings of the paper, and a subsequent the Appendix provides all the necessary proofs.

2 The model

The model proposed in this paper is a simple Schinasi’s variant of the dynamic, fixed price, IS-LM model with pure money financing of the budget deficit, which implies the
following three-dimensional planar system of first order differential equations (see also Sasakura, 1994)\(^1\)

\[
\begin{align*}
\dot{r} &= \delta [L(r, y) - m] \\
\dot{y} &= \alpha \left[ I(y, r) - S(y^D, W) + g - T(y) \right] \\
\dot{m} &= g - T(y)
\end{align*}
\]

(1)

where \( \dot{r} = dr/\,dt, \dot{y} = dy/\,dt, \) and \( \dot{m} = dm/\,dt. \) We also assume that all functions are continuously differentiable at a suitable order. Let \( L(r, y) \) be the liquidity (i.e., money demand) function, depending on \( r, \) the (real) interest rate, and \( y, \) the income level. As commonly found in the related literature, we need that \( L_r > 0 \) and \( L_y < 0. \) Moreover, let \( I(r, y) \) represent the investment function, and let \( S(y^D, W) \) capture savings as function of both disposable income, \( y^D, \) and wealth, \( W, \) which corresponds herein to the total amount of real money balances, \( W = m, \) for we assume no bonds issued, to simplify the analysis. Finally, the tax collection function, \( T(y), \) is assumed to proportionally depend on income \( (T = \tau y). \) As for the set of parameters, let \( g > 0 \) be the (constant) government expenditure, \( \tau \) is the tax share, whereas \( \alpha \) and \( \delta \) are simple scale parameters.

For the sake of a simple representation, and without any loss of generality, we can assume linearity for both the liquidity function \( L = \gamma y - \beta r, \) and the disposable income function \( y^D = y - T(y) = (1 - \tau)y. \)

System (1) thus reduces to

\[
\begin{align*}
\dot{r} &= \delta [\gamma y - \beta r - m] \\
\dot{y} &= \alpha \left[ f(y, r, m) + g - \tau y \right] \\
\dot{m} &= g - \tau y
\end{align*}
\]

(2)

where we find it also convenient to define \( f(y, r, m) = I(y, r) - S(y^D, W) \) as the function of total inventories, which is more suitable to work with in the mathematical proceedings of the paper.

2.1 Steady state

Let \((r^*, y^*, m^*)\) be values of \((r, y, m)\) such that system (2) reaches a long-run steady state, \( \dot{r} = \dot{y} = \dot{m} = 0. \) Simple algebra shows that, in equilibrium, we have

\[
\begin{align*}
\gamma &\left[ f \left( y^*, r^*, m^* \right) \right] - \beta r^* = m^* \\
f \left( y^*, r^*, m^* \right) &= 0 \\
\frac{g}{\tau} &= y^*
\end{align*}
\]

(3)

where \( f(\cdot) \to \mathbb{R} \) is a function conveniently smooth in all its arguments, and fulfils the standard Kaldorian assumptions.\(^2\) In particular, we are interested in the case where \( df/dy = I_y - S_y \) changes sign in its domain, for specific values of \( y. \) Then \( f(\cdot) \) can have multiple intersections with the \( y\)-axis, that is multiple steady states emerge (see Figure 1).

The tools of global analysis become thus necessary to investigate the stability of the equilibrium. More precisely, a global bifurcation analysis will allow us to prove
that the economy described by system (2) can exhibit chaotic fluctuations. We prove so
by showing the set of necessary conditions for the emergence of a double scroll
quasi-attractor, and determine the regions in the parameter space which imply the
existence of chaotic motion. The demonstration is not trivial and requires several steps to
be accomplished.

Figure 1  Function of total inventories (see online version for colours)

2.2 Saddle-foci dynamics

Consider the following linearisation matrix associated with system (2) at the steady state,
that is:

$$
\mathbf{J}^* = \begin{bmatrix}
\delta \beta & \delta \gamma & -\delta \\
\alpha f_y & \alpha \left(f_y - \tau\right) & \alpha f_w \\
0 & -\tau & 0
\end{bmatrix}
$$

(4)

where for the sake of a simple notation, the arguments of the partial derivatives of $f(\cdot)$
have been dropped. Let the characteristic polynomial associated with $\mathbf{J}^*$ be

$$
\det(\lambda I - \mathbf{J}^*) = \lambda^3 - \text{Tr}(\mathbf{J}^*) \lambda^2 + B(\mathbf{J}^*) \lambda - \det(\mathbf{J}^*)
$$

(5)

where $I$ is the identity matrix. $\text{Tr}(\mathbf{J}^*)$, $\det(\mathbf{J}^*)$ and $B(\mathbf{J}^*)$ are the trace, determinant and
sum of principal minors of $\mathbf{J}^*$, respectively. In the Appendix, they are shown to be
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\[ \text{Tr}(J^*) = \alpha \left( f_y' - \tau \right) - \delta \beta \]  
(6.1)

\[ \text{Det}(J^*) = \alpha \delta \tau \left( f_y' - \beta f_m' \right) \]  
(6.2)

\[ B(J^*) = \alpha \tau f_m'' - \alpha \delta \left( \beta \left( f_y' - \tau \right) + \gamma f_m' \right) \]  
(6.3)

In order to check whether system (2) may exhibit a set of periodic trajectories which lead to the formation of a chaotic attractor, we need to apply the well-known Shilnikov criterion:

**Theorem 1:** If the third-order autonomous system (2) has two saddle-foci (of index 2) equilibrium points, \( E_1 \) and \( E_2 \), with eigenvalues associated to (4) given by \( \eta_k \in \mathbb{R} \) and \( \sigma_k + i \omega_k \in \mathbb{C} \), \( k = 1, 2 \), such that \( \sigma_1 \sigma_2 > 0 \) or \( \eta_1 \eta_2 > 0 \), with a further constraint \( |\eta_1| > |\sigma_1| \), and there exists a heteroclinic orbit, connecting \( E_1 \) and \( E_2 \), then the dynamic flow in the neighbourhood of heteroclinic orbit exhibits a Smale horseshoe type of chaos.

**Proof:** See Wang et al. (2009)

In what follows, we show that system (2) fulfils the above theorem. This is obtained by solving (5) with Cardano’s formula, which provides the following three roots:

\[ \lambda_1 = \frac{-\hat{a}}{3} + u + v \]

\[ \lambda_{2,3} = \frac{-\hat{a}}{3} - \frac{u + v}{2} \pm \frac{\sqrt{3}}{2} \left( u - v \right) i \]

where \( i = \sqrt{-1} \) is the imaginary root, \( u = \sqrt{\frac{-q + \sqrt{\Delta}}{2}} \) and \( v = \sqrt{\frac{-q - \sqrt{\Delta}}{2}} \), with \( p = \frac{3\hat{a}^2}{2} \) and \( q = \hat{c} + \frac{3\hat{a}^3}{2} - \frac{27\hat{b}}{4} \), \( \hat{a} = -\text{Tr}(J^*) \), \( \hat{b} = B(J^*) \), and \( \hat{c} = -\text{Det}(J^*) \). Whereas \( \Delta = \left( \frac{q}{2} \right)^2 + \left( \frac{\Delta}{2} \right)^2 \) is the discriminant (see Zhou and Chen, 2008). For the scope of our paper, a saddle-focus (of index 2) emerges when

\[ \Delta > 0 \]

\[ \frac{3}{2} \sqrt{\frac{q}{2} + \frac{1}{2} \sqrt{\Delta}} + \frac{3}{2} \sqrt{\frac{q}{2} - \frac{1}{2} \sqrt{\Delta}} < -\frac{2\hat{a}}{3} \]  
(7)

that is explicitly

\[ \left( \frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} \right)^2 > \left( \frac{\hat{a}^2 - 3\hat{b}}{9} \right)^3 \]  
(8.1)

and

\[ \frac{3}{2} \sqrt{\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} \frac{\hat{b}}{6}} + \sqrt{\Delta} + \frac{3}{2} \sqrt{\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} \frac{\hat{b}}{6} - \sqrt{\Delta}} < -\frac{2\hat{a}}{3} \]  
(8.2)

given
\[ \dot{a} = \delta \beta - a \left( f_y^* - \tau \right) \]
\[ \dot{b} = \alpha \tau f_m^* - a \delta \left[ \beta \left( f_y^* - \tau \right) + \gamma f_y^* \right] \]
\[ \dot{c} = a \delta \tau \left( f_m^* - f_y^* \right) \]

which guarantee the emergence of a scroll (a wing) around the equilibrium points.

2.3 Global analysis and chaos

In general, following Gamero et al. (1999), the dynamics of system (2) can be described through the partial unfolding of a triple-zero eigenvalue bifurcation system in the normal form

\[ \begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \phi \left( w_1, w_2, w_3 \right) \]

where \( \epsilon_1 = -\hat{c}, \ \epsilon_2 = -\hat{b}, \ \text{and} \ \epsilon_3 = -\hat{a} \) are the unfolding parameters, and

\[ \phi() = \sum_{k=2}^{\infty} \left[ \sum_{j=0}^{m} \left( c_j^{(k)} w_1^{k-j} w_2^{j+1} + b_j^{(k)} w_1^{k-2j} w_2^{j+1} \right) \right] \]

is the expansion of second (or higher) order non-linear terms. Therefore, \( \mathcal{P} \) constitutes an organising centre of a codimension-three singularity where the origin exhibits a triple-zero eigenvalue, which in embryo contains several bifurcation singularities. In particular, in what follows, we will concentrate our attention on the possible emergence of chaotic motion. A detailed procedure is described henceforth in the Appendix.

For the sake of a simple mathematical analysis, we expand \( \phi(w_1, w_2, w_3) \) up to the third degree, and annihilate all cross-product coefficients, since they do not affect the scope of our study. This allows us to reduce \( \mathcal{P} \) to the following hypernormal form

\[ \begin{align*}
\dot{w}_1 & = w_2 \\
\dot{w}_2 & = w_3 \\
\dot{w}_3 & = \epsilon_1 w_1 + \epsilon_2 w_2 + \epsilon_3 w_3 + s_1 w_1^2 + s_2 w_2^3
\end{align*} \]

(\( \mathcal{M} \))

which describes a family of three-dimensional autonomous differential equations of the following jerk function

\[ \dot{w}_1 - \epsilon_3 \dot{w}_1 - \epsilon_2 \dot{w}_2 + g(w_1) = 0 \]

(10)

whose global structure is topologically equivalent to the original system (1), where \( g(w_1) = \left( \epsilon_1 w_1 + s_1 w_1^2 + s_2 w_2^3 \right) \) is a third-degree polynomial, and \( s_1 = \alpha \tau f_m^* \). Moreover, without any loss of generality, we can also set \( s_2 = -1 \), such that system \( \mathcal{M} \) resembles the
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A little-known non-linear Arneodo system, a particular convenient normal form with a very rich associated dynamics, crucially depending on parameter $s_1$, including the emergence of a complex chaotic attractor.

Interestingly, both the size and sign of $s_1$ provide very useful economic insights. In detail, $s_1$ represents the second order derivative of $f^r()$ with respect to $r$, which can be interpreted as a measure of the variation (acceleration) of speed adjustment of total inventories with respect to the real interest rate, $f''_r = I^*_r - S^*_r$, which in modulus can be greater (lower) than unity. Basically, this suggests that, given the negative effect of real interest rate on the investment function, the sign on the savings functions remains still uncertain, as suggested by the standard economic theory. For the purpose of our paper, this means that only when $S^*_r < 0$, that is to say standard Kaldorian assumptions are violated, the $s_1$ parameter gets closer (in modulus) to zero, and the double-scroll chaotic scenario is likely to occur. We are able to show these differences numerically in the following examples.

Referring to Makovinyiova (2011), if we consider the set of structural parameters for Slovak economy, namely $(\alpha, \beta, \gamma, \delta, \tau) = (8.32, 0.2, 0.8, 1.23, 0.2)$, conditions (8.1) are verified when $(\epsilon_1, \epsilon_2, \epsilon_3) = (0.45, 0.23, 2.27)$, which entail also $(\epsilon_1, \epsilon_2, \epsilon_3) = (0.45, 0.23, 2.27)$. This allows us to show the equilibrium dynamics when $s_1$ is varied. We finally obtain that

**Example 1:** If $s_1 = -2$, the hyperbolic fixed point of $M$ exhibits a single-scroll scenario (see Figure 2).

![Single-scroll attractor (see online version for colours)](image)

**Example 2:** If $s_1 = -0.2$, the hyperbolic fixed point of $M$ exhibits a double scroll scenario (see Figure 3).
As thoroughly shown by Impram et al. (2003), it appears that a two-dimensional manifold scrolls around a one-dimensional branch, giving rise to the complicated transition from a stable/unstable Hopf bifurcation structure towards the double scroll chaotic motion. More in detail, the irregular oscillatory behaviour in Figure 3 shows how two spiral quasi-attractors unite, including the invariant manifold of a saddle-focus.

Figure 3  The double-scroll attractor (see online version for colours)

3 Conclusions

This paper innovates the literature regarding dynamic IS-LM models of Schinasi’s (1981, 1982) type. First of all, we find that the model admits a dual steady state. Second, and more important, we used the tools of global analysis to prove the emergence, for a class of parameters, of a double scroll strange attractor. In details, the global analysis of the bifurcation set allows us to derive precise details on the behavior of the equilibrium trajectories. To our scopes, we find particularly interesting to show that a heteroclinic orbit, connecting the two steady states, can arise for specific values in the parametric space. The most important implication that can be derived from this phenomenon is that the economy might perpetually oscillate between the two equilibria, one of which is associated to a lower steady state, the other referred to as a higher equilibrium level. Besides its complicate mathematical structure, this technique is able to provide very interesting results, and a promising application both in business cycles or financial and monetary problems. We can thus conclude that local stability analysis is not sufficient, for it might exhibit an attracting area in the vicinity of the steady state (e.g., a fixed point or limit cycle), whereas in the large an unpredictable (chaotic) motion emerges, where the
dynamics of the system randomly fluctuates around the equilibrium, making thus long-term economic forecasting almost impossible.

References


Notes


2 It is assumed that both investment, $I_t$, and savings, $S_t$, are increasing functions with respect to output, $y$, and of an s-shaped form. Therefore, $I_t < S_t$ for lower values of $y$, whereas $I_t < S_t$ for higher levels of $y$. 
Appendix

Normal form reduction

Let the following second-order Taylor expansion of system (7):

\[
\begin{pmatrix}
\dot{\bar{r}} \\
\dot{\bar{y}} \\
\dot{\bar{m}}
\end{pmatrix} = J^* \begin{pmatrix}
\bar{r} \\
\bar{y} \\
\bar{m}
\end{pmatrix} + \frac{1}{2} \alpha \begin{pmatrix}
\beta \bar{y}^2 \
\bar{x}^2 + 2 \bar{f}_{xy} \bar{y} + \bar{f}_{yy} \bar{y}^2
\end{pmatrix}
\]

(A.1)

where $J^*$ is the Jacobian matrix of the linear part, and \( \bar{f}_i(\bar{r}, \bar{y}, \bar{m}) \) are the second order terms:

\[
\begin{pmatrix}
\bar{f}_1(\bar{r}, \bar{y}, \bar{m}) \\
\bar{f}_2(\bar{r}, \bar{y}, \bar{m}) \\
\bar{f}_3(\bar{r}, \bar{y}, \bar{m})
\end{pmatrix} = \left( \frac{1}{2} \alpha \begin{pmatrix}
\beta \bar{y}^2 \
\bar{x}^2 + 2 \bar{f}_{xy} \bar{y} + \bar{f}_{yy} \bar{y}^2
\end{pmatrix} \right)
\]

(A.2)

Assume first system A.1 undergoes a triple-zero eigenvalue structure, which allows us to make the following change of coordinates

\[
\begin{pmatrix}
\bar{r} \\
\bar{y} \\
\bar{m}
\end{pmatrix} = T \begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}
\]

(A.3)

via an appropriate transformation matrix

\[
T = \begin{pmatrix}
1 & 0 & \frac{\bar{r} \tau - \bar{y}}{\tau} \\
0 & \frac{\bar{y}}{\tau} & \frac{\bar{r} \bar{y}}{\tau} \\
-\frac{\beta \bar{y} - \tau}{\tau} & 0 & 0
\end{pmatrix}
\]

(A.4)

whose columns represent the eigenvectors associated to the triple-zero eigenvalues.

We are thus able to put (A.4) in a Jordan normal form

\[
\begin{pmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3
\end{pmatrix} = B \begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix} + F
\]

(A.5)
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1. \( B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \)

2. \( F = \begin{bmatrix} \bar{F}_1 (w_1, w_2, w_3) \\ \bar{F}_2 (w_1, w_2, w_3) \\ \bar{F}_3 (w_1, w_2, w_3) \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta \tau} \left( -\gamma \tau + \beta \gamma^2 \right) \bar{F}_2 (w_1, w_2, w_3) \\ \gamma \bar{F}_2 (w_1, w_2, w_3) \\ \tau \bar{F}_2 (w_1, w_2, w_3) \end{bmatrix} \)

given

\( \bar{F}_2 (w_1, w_2, w_3) \)

\( = \frac{1}{2} \alpha \left[ f_{\gamma \gamma} \left( w_1 + \frac{\gamma \tau - \beta \gamma^2}{\beta} w_3 \right)^2 + 2 f_{\gamma \gamma} \left( w_1 + \frac{\gamma \tau - \beta \gamma^2}{\beta} w_3 \right) \left( \frac{\beta}{\tau} w_2^2 + \frac{\gamma \tau - \beta \gamma^2}{\beta} w_3 \right) + f_{\gamma \gamma} \left( \frac{\beta}{\tau} w_2^2 + \frac{\gamma \tau - \beta \gamma^2}{\beta} w_3 \right)^2 \right] \)

Translation to the origin

Let us define the following parameters translation, \( \delta = \bar{\delta} + v, \, \alpha = \bar{\alpha} + \mu, \, g + \bar{g} + \kappa, \) such that system (7) rewrites

\( \dot{r} = \left( \bar{\delta} + v \right) \left( \gamma r - \beta r - m \right) \)

\( \dot{y} = \left( \bar{\alpha} + \mu \right) \left( f (y, r, m) + (\bar{g} + \kappa) - \tau y \right) \)

\( m = (\bar{g} + \kappa) - \tau y \)

(A.6)

A second-order Taylor expansion of (A.6) leads to

\( \begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{y}} \\ \dot{\hat{m}} \end{pmatrix} = J \begin{pmatrix} \hat{r} \\ \hat{y} \\ \hat{m} \end{pmatrix} + A \begin{pmatrix} \hat{r} \\ \hat{y} \\ \hat{m} \end{pmatrix} + \frac{1}{2} \alpha \left[ f_{\gamma \gamma} \hat{r}^2 + 2 f_{\gamma \gamma} \hat{r} \hat{y} + f_{\gamma \gamma} \hat{y}^2 \right] + \frac{1}{2} \mu \kappa \)  \( (A.7) \)

where

\( A = \begin{bmatrix} -\beta v & \gamma v & -v \\ f_{\gamma \gamma} & f_{\gamma \gamma} & f_{\gamma \gamma} \\ 0 & 0 & 0 \end{bmatrix} \).
Versal deformation matrix

Repeating the same procedure described in Steps 1 and 2, (A.7) can be put in normal form:

\[
\begin{pmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix} + M
\begin{pmatrix}
\dot{w}_1 \\
\dot{w}_2 \\
\dot{w}_3
\end{pmatrix} + \frac{P_1(w_1, w_2, w_3)}{F_1(w_1, w_2, w_3)} + \frac{P_2(w_1, w_2, w_3)}{F_2(w_1, w_2, w_3)} + \frac{P_3(w_1, w_2, w_3)}{F_3(w_1, w_2, w_3)}
\]

(A.8)

given

\[M = \begin{bmatrix}
s_1 & s_2 & s_3 \\
s_4 & s_5 & s_6 \\
s_7 & s_8 & s_9
\end{bmatrix}\]

(A.9)

where

\[
s_1 = \frac{1}{\beta \tau} (-\gamma \tau + \beta \gamma^2) (f_r - \beta f_m) \mu
\]

\[
s_2 = \frac{1}{\tau} \left( (\gamma - \beta \gamma) v + \frac{1}{\beta \tau} (-\gamma \tau + \beta \gamma^2) \left( f_y \frac{\beta}{\tau} + f_m \frac{\beta \gamma - \tau}{\tau} \right) \right) \mu
\]

\[
s_3 = \frac{1}{\beta \tau} \left( -\gamma \tau + \beta \gamma^2 \right) \left( f_r \frac{\gamma}{\beta} + f_y \right) \frac{\tau - \beta \gamma}{\tau^2} \mu
\]

\[
s_4 = \gamma \left( f_r - \beta f_m \right) \mu
\]

\[
s_5 = -\beta \nu + \frac{\gamma}{\beta} \left( f_y \frac{\beta}{\tau} + f_m \frac{\beta \gamma - \tau}{\tau} \right) \mu
\]

\[
s_6 = \gamma \left( f_r \frac{\gamma}{\beta} + f_y \right) \frac{\tau - \beta \gamma}{\tau^2} \mu
\]

\[
s_7 = \gamma \left( f_r - \beta f_m \right) \mu
\]

\[
s_8 = \frac{\beta^2}{\tau - \beta \gamma} v + \left[ f_y \beta + f_m (\beta \gamma - \tau) \right] \mu
\]

\[
s_9 = \left( f_r \frac{\gamma}{\beta} + f_y \right) \frac{\tau - \beta \gamma}{\tau} \mu
\]

We can thus construct a versal deformation of the linear part of (A.8):

\[
V(\tau, \mu, \eta) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} + M = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\epsilon_1 & \epsilon_2 & \epsilon_3
\end{bmatrix}
\]

(A.10)
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given the unfolding parameters:
\[
\begin{align*}
\epsilon_1 &= s_1 \left[ s_2 s_9 - s_8 \left( s_8 + 1 \right) \right] - \left( s_2 + 1 \right) \left[ s_4 s_9 - s_7 \left( s_6 + 1 \right) \right] + s_1 \left[ s_4 s_9 - s_5 s_7 \right] \\
\epsilon_2 &= -s_9 s_5 + s_4 \left( s_2 + 1 \right) - s_9 s_9 + s_8 \left( s_6 + 1 \right) - s_1 s_9 + s_5 s_7 \\
\epsilon_3 &= s_1 + s_5 + s_9
\end{align*}
\]

which reduce in equilibrium to
\[
\begin{align*}
\epsilon_1 &= \text{Det} J^* \\
\epsilon_2 &= -BJ^* \\
\epsilon_3 &= tr J^*
\end{align*}
\]

Moreover, following Gamero et al. (1999), (A.10) can be normalised to
\[
\begin{align*}
\dot{w}_1 &= w_2 \\
\dot{w}_2 &= w_3 \\
\dot{w}_3 &= \epsilon_1 w_1 + \epsilon_2 w_2 + \epsilon_3 w_3 + \phi \left( w_1, w_2, w_3 \right) \tag{A.11}
\end{align*}
\]

where
\[
\phi \left( w_1, w_2, w_3 \right) = \sum_{k=2}^{m} \left\{ \sum_{j=0}^{m} \left( a^{(k)} w_1^{k-j-1} w_3^{j+1} + b^{(k)} w_1^{k-2-j-1} w_2^{j+1} \right) \right\} + \sum_{j=0}^{m} \left( c^{(k)} w_1^{k-2-j} w_2^{j} \right)
\]

which finally describes system \( \mathcal{P} \).