Reliable wideband multichannel spectrum sensing using randomized sampling schemes

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A wideband multichannel spectrum sensing approach that utilizes nonuniform sampling and digital alias-free signal processing (DASP) to reliably sense the spectrum using sampling rates well below the ones used in classical DSP is proposed. The approach deploys a periodogram-type spectral analysis tool to estimate the spectrum of the incoming signal from a finite number of its noisy nonuniformly distributed samples. The statistical characteristics of the adopted estimator are analyzed and its accuracy is assessed. It is demonstrated here that owing to the use of nonuniform sampling, the sensing task can be carried out with the use of arbitrary low sampling rates. Most importantly, general guidelines are provided on the required signal analysis window for a chosen sampling rate to guarantee sensing reliability within a particular scenario. The extra requirement on such recommendations imposed by the presence of noise is given. The analytical results are illustrated by numerical examples. This paper establishes a new framework for multiband spectrum sensing where substantial saving on the used sampling rates can be achieved.

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1. Introduction

Various spectrum sensing applications deal with a number of signals of different sources sent over disjoint spectral bands/channels i.e. multiband environment. The task of the sensing module is to detect meaningful activities within those bands, such as an ongoing transmission or occurrence of some event within a predefined range of frequencies. The application areas include: astronomy [1,2], seismology [3], multiband communication systems [4,5] and many others. Several of these applications are characterised by low spectrum utilization i.e. only a small proportion of the monitored channels are active at any given time instant. Typically, spectrum sensing in such environments departs from the classical methods e.g. power/energy detection within each channel and involves estimating the spectrum of the incoming signal [4,6]. This approach is adopted here where a periodogram-type spectrum analysis tool is deployed.

If no prior knowledge is available on the activity of the examined spectral bands, the uniform sampling rate of the sensing device should exceed twice the total monitored bandwidth regardless of the spectrum occupancy [7]. Failing to do so could result in aliasing and irresolvable detection problems. When dealing with low spectrum utilization scenario, such sampling rates would exceed many times the Landau rate [8] which is the theoretically lowest sampling rate that permits full recovery of the sampled signal and is equal to twice the bandwidth of the concurrently active frequency bands. For many bandpass and multiband signals, sampling at the Landau rate requires deployment of nonuniform sampling as well as a prior knowledge of channels activity [9,10]. Exceeding Landau rate significantly may indicate potential inefficiencies in using the sensing device resources such as power and/or deployment of high-cost, fast
hardware capable of dealing with excessive sampling rates. In this paper, we demonstrate that even if the active channels are unknown we can still detect them by the suitable use of arbitrary low-rate intentional nonuniform sampling (randomized sampling) and appropriate processing of the signal—a methodology referred to as DASP. Few monographs such as [11–14] give an overview on the topic.

1.1. Related work

Lomb–Scargle periodogram [1,2] and its recent more efficient form [15] extended the classical periodogram spectral analysis tool to encompass arbitrary distributed signal samples in order to detect discrete spectral components of the analyzed signal. In [16–18] consistent spectral estimation methods are studied for a number of alias-free randomized sampling schemes where the signal is regarded as deterministic and in [19] these methods are utilized to perform spectrum sensing. In this paper, the proposed approach relies on the user’s ability to prescribe the sampling instants. Deliberately randomized, referred to by randomized for short, rather than arbitrary sampling is used to perform spectral analysis of a random multiband signal disturbed by white noise. The latter assumption, i.e. the processed signal is random, postulates a general stochastic framework that strengthens and generalizes the preliminary results reported in [19].

Although the earliest papers on DASP-type algorithms [20–22] considered the problem of estimating the signal’s power spectral density (PSD) for various sampling schemes, they did not resolve the predicament of the estimator’s consistency for a finite number of samples. This issue was partially addressed by Masry in [23–25] where the expressions for the estimator’s bias and variance are provided under the assumption that the number of processed samples tends to infinity. In this paper, the estimation of the exact signal’s PSD is not the objective and an estimate of a smoothed/windowed PSD that permits detection of the active spectral bands is sufficient. Here we assess the accuracy of the adopted estimator from a finite set of samples in contrary to [23–25] where asymptotic consistency expressions are given. It is noted that sampling schemes other than the ones examined in [20–25] are considered here and guidelines on detection dependability are provided. Sensing methods that entail computationally expensive algorithms e.g. [9,10], communication among possible multiple sensing devices e.g. [4,6] and the use of transforms other than Fourier transform (FT) e.g. [26] are abandoned in this study.

Martin [12, p. 38] and Scargle [1] listed the deficiencies of periodogram spectral analysis tool in nonuniform sampling environment, namely inconsistency, leakage and smeared-aliasing. However, in this study several measures are taken, namely estimate averaging, tapering and alias-free sampling, to combat such drawbacks and appropriate periodogram type of analysis to the handled problem i.e. spectrum sensing and not PSD estimation. Averaging a number of periodograms to improve its accuracy has been utilized long ago e.g. [27–29]. The number of averages needed is commonly overlooked and uniform sampling is used. In this paper the number of averages that provides a required level of detection reliability is sought and nonuniform sampling is utilized.

1.2. Contribution

The contribution of this paper is twofold. First, it studies the statistical analysis of a periodogram-type spectral estimation method that uses total random sampling (TRS) scheme. The bias and the consistency of the adopted estimator for a finite number of samples are evaluated. The DASP nature of the estimator is demonstrated in the context of identifying spectral components of the present signal. Second, the main contribution of this paper is defining the sensing reliability in terms of the used sampling rate, spectrum utilization and signal-to-noise ratio (SNR). A lower limit on the combined values of the average sampling rate and the number of estimate averages is provided in order to guarantee detection reliability. Such recommendation provides a tool to assess the trade-offs between the needed sampling rate and the sensing-time as well as a mean to inspect whether nonuniform sampling is beneficial over the uniform case given a particular scenario. A decision making criterion on the channels activity is also proposed. Generalizing the obtained results to embrace sampling schemes other than TRS is addressed.

The paper is organized as follows. In Section 2, the tackled problem is formulated and the adopted sensing methodology is introduced. The deployed spectral analysis tool, its statistical characteristics and suitability for the sensing task are presented/discussed in Section 3. In Section 4, sensing reliability recommendations are given and the applicability of the results to other sampling schemes is discussed. The advantages of the proposed spectrum sensing method are demonstrated by numerical examples in Section 5 and conclusions are drawn in Section 6.

2. Problem formulation and sensing methodology adopted

The considered system has \( L \) spectral bands, each of them with bandwidth \( B_C \). The total single-sided bandwidth of the monitored frequency range is \( B = L B_C \). The maximum number of simultaneously active channels and their joint bandwidth are given by \( L A \) and \( B_A = L A B_C \), respectively. We assume low channel occupancy i.e. \( L_A \ll L \). Central frequencies of all channels are known and the positions of active channels are unknown beforehand. The main objective is to devise a method that is capable of scanning the monitored bandwidth \( B \) and identifying which channels are active, if any. The algorithm should operate at sampling rates significantly less than \( 2B \) which is theoretically the minimum rate (not always achievable) that could be used when bandpass sampling and classical DSP are deployed. If the average sampling rate of the applied nonuniform sampling scheme is above Landau i.e. \( 2 B A \), it is possible to reconstruct the identified transmitted signals.

We recall that the detection approach adopted here relies on estimating the spectrum of the signal. The sensing
is made upon assessing the magnitude of the estimated spectrum. The processed signal is assumed to be contaminated with zero mean additive white Gaussian noise (AWGN) with a variance of \( \sigma_n^2 \). Hence, \( y(t_n) = x(t_n) + n(t_n) \) is the sum of the signal samples \( x(t_n) \) and the added noise \( n(t_n) \). We perform spectral analysis via endorsing the periodogram-type estimator given by

\[
X_e(f) = \frac{N}{(N-1)\mu} \left| \sum_{n=1}^{N} y(t_n)w(t_n)e^{-j2\pi fn} \right|^2
\]

where \( \mu \) is the energy of the deployed tapering/windowing function \( w(t) \):

\[
\mu = \int_{t_0}^{t_0 + T_0} w^2(t) \, dt
\]

\( N \) is the number of the signal sample points \( t_n \)'s and \( T_0 \) is the length of the signal analysis time window. Windowing is deployed to minimize leakage in the conducted spectral analysis. A comprehensive review on windowing functions is given in each of [30] and [31].

The sampling instants in (2.1) are placed inside the time interval \([t_0, t_0 + T_0] \) where \( t_0 \) is the initial time instant of the window. The tapering function \( w(t) \) is aligned with this interval; for the simplicity of notations \( t_0 \) is omitted from (2.1). In practice, a number of the estimates given by (2.1) need to be averaged using a moving window approach that involves changing \( t_0 \) and the repositioning of \( w(t) \) (see Section 4). We demonstrate here that with the proper use of (2.1) we can identify with the required level of confidence which of the monitored channels are active. This involves selecting the length of the window \( T_0 \), a suitable average sampling frequency and averaging the sufficient number of spectral estimates.

### 3. Properties of the spectrum estimator

The spectrum estimator proposed here exploits an alias-free randomized sampling scheme named TRS introduced in [16]. The sampling instants \( t_n \)'s are independent identically distributed (IID) random variables whose probability distribution functions (PDFs) are given by

\[
p(t) = \begin{cases} \frac{1}{T_0}, & t \in [t_0, t_0 + T_0] \\ 0, & \text{elsewhere} \end{cases}
\]

(3.1)

The processed signal is assumed to be bandlimited zero mean wide sense ergodic (WSE). In the following subsection we show that (2.1) is an unbiased estimator of a frequency representation of the incoming signal that is suitable for the sensing task.

#### 3.1. Targeted frequency representation

The estimator represented by (2.1) has three sources of randomness: signal, sample point process and AWGN. The probabilistic expectation should be taken in terms of those three stochastic processes. Given that the components of the summation (2.1) are independent with respect to the sample points, it is shown in Appendix A that the conditional expectation of the estimator is given by

\[
E[X_e(f)|x(t)] = \frac{N}{(N-1)\mu} \left\{ \int_{t_0}^{t_0 + T_0} [x(t)w(t)]^2 \, dt + \sigma_n^2 \right\} + \frac{1}{\mu} |X_W(f)|^2
\]

(3.2)

where \( \mu = N/T_0 \) is the average sampling rate and

\[
X_W(f) = \int_{t_0}^{t_0 + T_0} x(t)w(t)e^{-j2\pi ft} \, dt
\]

(3.3)

is the FT of the windowed \( x(t) \). In the classical theory of spectral estimation, \( |X_W(f)|^2/\mu \) is a continuous form of periodogram where no sampling is involved. It is noted that

\[
E[|X_W(f)|^2] = \Phi_x(f) * |W(f)|^2,
\]

where \( \Phi_x(f) \) is the PSD of \( x(t) \). \( W(f) \) is the FT of the windowing function and "\(*\)" denotes the convolution operation. PSD is defined as the FT of the autocorrelation function of a wide sense stationary (WSS) signal [32]. As a result,

\[
C(f) = E[X_e(f)] = \frac{N}{(N-1)\mu} \left\{ E[x^2(t)] + \sigma_n^2 \right\} + \frac{1}{\mu} \Phi_x(f) * |W(f)|^2
\]

(3.4)

Away from asymptotic limits and in the context of the problem considered in this paper where the pursued \( \alpha \) is relatively small and \( T_0 \) is finite, the bias of the estimator in terms of the windowed signal PSD i.e. the first term in (3.4) is constant and frequency independent. Assuming that the signal observation window period \( T_0 \) is long enough, the tapered PSD forms an identifiable feature. Consequently, \( C(f) \) comprises of a detectable spectral feature i.e. \((1/\mu)\Phi_x(f) * |W(f)|^2\) plus a constant offset. Hence, \( X_e(f) \) given by (2.1) is an unbiased estimator of \( C(f) \) regardless of the sampling rate. Although the constant offset in (3.4) does not have a visible effect on the detectable feature of \( C(f) \), it deteriorates the spectrum dynamic range which can have a negative effect on the spectrum sensing procedure as discussed in Section 4.

#### 3.2. Length of the signal analysis window

The use of a long signal analysis window \( T_0 \) results in a high resolution spectral analysis. The detection method adopted here relies on sensing the magnitude of the estimated spectrum via assessing a number of spectral points to determine the channels status. Maintaining low spectrum resolution by utilizing short signal time window minimizes the number of needed frequency points or DFT points i.e. save on computations. We aspire to use one frequency point per channel to detect any activity within. Using the theorem of minimum number of zero-crossings of bandlimited signals as discussed in [14, p. 170] and exploiting the frequency-time duality characteristic [32] enable us to deploy the number of zero-crossings per channel as a criterion to describe the resolution of the spectrum. We note that a zero-crossing is perceived as a notable fluctuation in the spectrum magnitude. Having two crossings per active channel is a reasonable assumption and as a result a guideline on the window period can be described by

\[
T_0 \geq \frac{1}{B_C}
\]

(3.5)

Nonetheless, the signal analysis period \( T_0 \) should ensure that \( C(f) \) exhibits distinguishable feature for the
detection task. The chosen $T_0$ should strike a balance between spectrum resolution and usefulness of the $C(f)$. Experimental results showed that exceeding (3.5) a number of times would suffice. In the following sections, $T_0$ is considered as being fixed i.e. predefined by the user.

### 3.3. Accuracy of the estimator

Although $X_e(f)$ is an unbiased estimator of $C(f)$, it will be a suitable tool for assessing the channel's activity only if the difference $\Delta(f) = |C(f) - X_e(f)|$ is small. According to Chebychev's inequality [33], which states that: $Pr(|X-E[X]| \geq \varepsilon \sigma_X) \leq 1/\varepsilon^2$ where $\sigma_X$ is the standard deviation of a random variable $X$ and $\varepsilon > 0$, the difference $\Delta(f)$ can be controlled by the suitable reduction of the variance of $X_e(f)$. In this subsection we derive an expression for the variance of $X_e(f)$ given by

$$\sigma_e^2(f) = \left\{ \frac{N}{(N-1)\mu} \right\}^2 \text{var} \left\{ \frac{T_0}{N} \sum_{n=1}^{N} y(t_n)w(t_n)e^{-2\pi f t_n} \right\}$$ (3.6)

We define

$$|X_{WS}(f)|^2 = \left\{ \frac{T_0}{N} \sum_{n=1}^{N} y(t_n)w(t_n)e^{-2\pi f t_n} \right\}^2 = R_{WS}(f) + I_{WS}(f)$$ (3.7)

where $R_{WS}(f)$ and $I_{WS}(f)$ represent the real and imaginary parts of $X_{WS}(f)$ such that,

$$R_{WS}(f) = \frac{T_0}{N} \sum_{n=1}^{N} y(t_n)w(t_n)cos(2\pi ft_n)$$ (3.8)

$$I_{WS}(f) = \frac{T_0}{N} \sum_{n=1}^{N} y(t_n)w(t_n)sin(2\pi ft_n)$$ (3.9)

It can be noticed that each of $R_{WS}(f)$ and $I_{WS}(f)$ consist of the sum of $N$ statistically independent random variables for every $f$. According to the Central Limit theorem they can be assumed to have a normal distribution for large $N$ [34] ($N \geq 20$ is often perceived as sufficient in practice [4] which is reasonable for the conducted analysis). It can be immediately seen that each of (3.8) and (3.9) are of zero mean i.e. $E[R_{WS}(f)] = E[I_{WS}(f)] = 0$. The sum of the squares of two zero mean normally distributed random processes results in un-normalized chi-squared distribution with two degrees of freedom given that the two variables are independent [34]. Since $R_{WS}(f)$ and $I_{WS}(f)$ are of normal distribution, being uncorrelated at all frequencies is sufficient to fulfill the independency condition. Their correlation is shown in Appendix B to be

$$\rho(f) = \frac{|E[X^2(t)] + \sigma^2_e|E_{WCs}(f)}{\bar{X}} + \frac{N-1}{N}E[R_{WS}(f)I_{WS}(f)]$$ (3.10)

where

$$E_{WCs}(f) = \int_{t_0}^{t_0+T_0} w^2(t)cos(2\pi ft)sin(2\pi ft) dt$$ (3.11)

and

$$E[R_{WS}(f)I_{WS}(f)] = \int_{t_0}^{t_0+T_0} \int_{t_0}^{t_0+T_0} R_X(t_1-t_2)w(t_1)w(t_2) \times cos(2\pi f t_1)sin(2\pi f t_2) dt_1 dt_2$$ (3.12)

whilst $R_W(f) = \int_{t_0}^{t_0+T_0} x(t)w(t)cos(2\pi ft) dt$, $I_W(f) = \int_{t_0}^{t_0+T_0} x(t)w(t)sin(2\pi ft) dt$ and $R_X(t)$ is the signal's autocorrelation function. It can be seen from (3.10) that $R_{WS}(f)$ and $I_{WS}(f)$ are not independent for all $f$. Yet the two latter random variables can be replaced with independent ones without altering $X_e(f)$ since $|X_{WS}(f)\exp(i\theta)|^2 = |X_{WS}(f)|^2$. We can write

$$|X_{WS}(f)|^2 = \tilde{R}_{WS}^2 + \tilde{I}_{WS}^2$$ (3.13)

where

$$\tilde{R}_{WS}(f) = \frac{T_0}{N} \sum_{n=1}^{N} y(t_n)w(t_n)cos(2\pi ft_n - \theta(f))$$ (3.14)

$$\tilde{I}_{WS}(f) = \frac{T_0}{N} \sum_{n=1}^{N} y(t_n)w(t_n)sin(2\pi ft_n - \theta(f))$$ (3.15)

The introduced delay is chosen such that $E[\tilde{R}_{WS}(f)\tilde{I}_{WS}(f)]=0$ and is shown in Appendix B to be

$$\theta(f) = 0.5 \arccot\left( \frac{E[R_{WS}^2(f)]-E[I_{WS}^2(f)]}{2\rho(f)} \right)$$ (3.16)

where

$$E[R_{WS}^2(f)] = \frac{|E[X^2(t)] + \sigma^2_e|E_{WCs}(f)}{\bar{X}} + \frac{N-1}{N}E[R_{WS}^2(f)]$$ (3.17)

$$E[W_{WCs}(f)] = \int_{t_0}^{t_0+T_0} [w(t)cos(2\pi ft)]^2 dt$$ (3.18)

$$E[R_{WS}^2(f)] = \int_{t_0}^{t_0+T_0} \int_{t_0}^{t_0+T_0} R_X(t_1-t_2)w(t_1)w(t_2) \times cos(2\pi f t_1)cos(2\pi f t_2) dt_1 dt_2$$ (3.19)

$$E[R_{WS}^2(f)] = \frac{|E[X^2(t)] + \sigma^2_e|E_{WS}(f)}{\bar{X}} + \frac{N-1}{N}E[R_{WS}^2(f)]$$ (3.20)

$$E[W_{WS}(f)] = \int_{t_0}^{t_0+T_0} [w(t)sin(2\pi ft)]^2 dt$$ (3.21)

$$E[R_{WS}^2(f)] = \int_{t_0}^{t_0+T_0} \int_{t_0}^{t_0+T_0} R_X(t_1-t_2)w(t_1)w(t_2)sin(2\pi f t_1) \times sin(2\pi f t_2) dt_1 dt_2$$ (3.22)

The variance for each of $\tilde{R}_{WS}(f)$ and $\tilde{I}_{WS}(f)$ is needed in order to obtain $\sigma^2_e(f)$. It can be shown that $E[\tilde{R}_{WS}^2(f)] = E[I_{WS}^2(f)] = 0$. Hence, $\sigma^2_{R_{WS}}(f) = E[\tilde{R}_{WS}^2(f)]$ and $\sigma^2_{I_{WS}}(f) = E[I_{WS}^2(f)]$. Similarly to the case of $R_{WS}(f)$ and $I_{WS}(f)$,

$$\sigma^2_{R_{WS}}(f) = \frac{|E[X^2(t)] + \sigma^2_e|E_{WS}(f)}{\bar{X}} + \frac{N-1}{N}E[R_{WS}^2(f)]$$ (3.23)

$$\sigma^2_{I_{WS}}(f) = \frac{|E[X^2(t)] + \sigma^2_e|E_{WS}(f)}{\bar{X}} + \frac{N-1}{N}E[I_{WS}^2(f)]$$ (3.24)

The identities $E_{WCs}(f)$, $E[R_{WS}^2(f)]$, $E_{WS}(f)$ and $E[I_{WS}^2(f)]$ are defined by equations identical to (3.18), (3.19), (3.21) and (3.22) respectively where $t$ is replaced with $t-\theta(f)$. As a result, the variance given by (3.6) for the adopted estimator can be obtained using the characteristics of un-normalized chi-squared random variable. Thus,

$$\sigma^2_e(f) = 2\left\{ \frac{N}{(N-1)\mu} \right\}^2 \left[ \sigma^2_{R_{WS}}(f) + \sigma^2_{I_{WS}}(f) \right]$$ (3.25)

Computing the variance can be seen as a complicated process that demands knowledge of the signal’s PSD.
However, the variance calculations are solely manipulated in ensuring the detection reliability. Spectrum sensing approach proposed here only requires calculating $X_e(f)$.

### 3.4. Numerical example of spectral analysis

In this example we examine a multiband system comprising of 160 channels ($L = 160$). Each of them has 5 MHz bandwidth i.e. $B_e=5$ MHz. The system channels reside in $f \in [0.76, 1.56]$ GHz which is the analyzed range of frequencies. Four adjacent channels are assumed to be active where a wide sense stationary (WSS) signal with a PSD defined by $1$ for $f \in [780, 800]$ MHz and zero elsewhere is generated. A Blackman window of length $T_0 = 0.2\,\mu$s and $a=700$ MHz are used. AWGN is introduced where the SNR is 2 dB. Fig. 1 shows $E[X_e(f)]$ defined by (3.4) and the statistical mean calculated from 10,000 independent experiments. Fig. 2 depicts $\sigma^2_e(f)$ given by (3.23)–(3.25) and the mean squared error (MSE) attained from the aforementioned experiments. As seen in Fig. 1, $C(f)$ poses itself as an adequate spectral representation for the detection pursuit. If uniform sampling was deployed at a similar rate, a replica of the underlying continuous time signal’s tapered PSD would have appeared at 1.49 GHz within the analyzed frequency range indistinguishable from the one present at 790 MHz. Identifying the active bands would have been only possible if the signal spectral support was known. On the other hand, TRS suppressed aliasing in its classical sense and allowed detecting the presence of the signal despite the used sampling rate. This demonstrates the alias-free nature of the adopted estimator. However, further analysis is needed to ensure the consistency of $X_e(f)$ for a single signal realization or experiment. Both Figs. 1 and 2 confirm that there is a good match between theoretical analysis and simulations i.e. assumptions made in theoretical analysis did not affect the accuracy of the results.

### 4. Reliable sensing

In order for the spectrum sensing to be reliable, the peak amplitude(s) of the estimator’s outcome associated with the active channel(s) must be significantly higher than the level of the estimated spectrum in the channels that are not active. According to Chebychev’s inequality, the estimator’s error/inconsistency can be linked to the standard deviation. Based on the desired accuracy, the peak amplitude of $C(f)$ that represents the active channel should be $\eta$ times greater than $\sigma_e(f)$. Inspecting $\sigma_e(f)$ given by (3.23)–(3.25), it can be easily shown that it is nearly constant at frequencies where there is no spectral activity. It exhibits a white-noise-like effect and is denoted by $\sigma_{e,\text{cont}}$ which is inversely proportional to $a$. On the other hand $\sigma_e(f)$ has its highest value where the

![Fig. 1. $E[X_e(f)]$ from equation (solid line) and from simulations (broken line). (a) An overview and (b) zoomed around 790 MHz.](image1)

![Fig. 2. Variance of $X_e(f)$ from equation (solid line) and MSE (broken line).](image2)
signal is present and the substantial part of this inaccuracy is independent of the used sampling rate. A classical method to reduce the latter error is to resort to averaging a number of $X_e(f)$ estimates from $K$ signal windows of length $T_0$. For simplicity non-overlapping signal segments scenario is considered in this study. The variance of the estimator is reduced by a factor of $1/K$ if the data in the non-overlapping signal windows is uncorrelated. Otherwise the reduction is inversely proportional to the level of correlation between these signal segments [27–29]. Therefore, $\alpha$ and $K$ are the utilities available to minimize the present error level and ensure the dependability of detection. The non-overlapping data windows are assumed to be uncorrelated in accordance with the typically adopted approach in literature e.g. [27–29]. Consequently, the sensing procedure proposed here relies on averaging $K$ number of $X_e(f)$ estimates over non-overlapping signal windows of length $T_0$ using

$$
\hat{X}_e(f) = \frac{1}{K} \sum_{i=1}^{K} X_i(f)
$$

(4.1)

4.1. Reliability recommendations

Let $H_U$ be the average peak amplitude of $C(f)$ of the active channel(s) at frequency $f_0$ while $H_L$ is the amplitude of the constant factor in $C(f)$ i.e. $N \left( E[|X^2(t)|] + \sigma^2_i \right) / (N-1) \alpha$. Fig. 3 represents a model of the sought spectrum for a better understanding of the reliability criterion where $C(f)$ and the expected deviations of the estimate for a given signal realization are shown. In general for the active channel(s) to be detectable $H = H_U - H_L$ should be visibly higher than the error level i.e. $H_U - \eta \sigma_e(f_0) \geq H_L + \eta \sigma_{e,\text{const}}$. Hence

$$
H \geq (\sigma_{e,\text{cont}} + \sigma_e(f_0)) / \sqrt{K}
$$

(4.2)

It is noticed that the window averaging effect is included in (4.2). This leads to

$$
H^2 > \eta^2 \left( \sigma_{e,\text{cont}} + \sigma_e(f_0) \right)^2 / K
$$

(4.3)

where $\eta \geq 1$ is a parameter that represents the desired accuracy according to Chebychev’s inequality. Noting that $E[\tilde{R}_{WS}(f)] + E[\tilde{P}_{WS}(f)] = \mu C(f)$ we can write

$$
0.5 \mu^2 C^2(f) \leq \left| E[\tilde{R}_{WS}(f)] \right|^2 + \left| E[\tilde{P}_{WS}(f)] \right|^2 \leq \mu^2 C^2(f)
$$

(4.4)

Since between $E[\tilde{R}_{WS}(f)]$ and $E[\tilde{P}_{WS}(f)]$ the main difference is a $0.5 \pi$ phase shift, the lower bound of (4.4) is considered. It is assumed that $E_W(f_0) = 0.5 T_0 + 0.5 f_0 \cos(4 \pi f_0 t - 2 \theta(f_0)) dt \approx 0.5 T_0$ for a rectangular window i.e. $\mu = T_0$ and similarly $E_{WS}(f_0) \approx 0.5 T_0$. This results in

$$
\sigma^2_{e,\text{cont}} = \frac{(N-1)^2}{N^2} \left( \frac{(N-1)P + P_N}{N} \right) H + \frac{(P_S + P_N)^2}{\alpha^2} (N-1)^2
$$

(4.5)

$$
\sigma^2_{e,\text{cont}} = \left( \frac{N}{N-1} \right)^2 \left( \frac{N-1}{N} \right) \left( \frac{(N-1)P + P_N}{N} \right) H + \frac{(P_S + P_N)^2}{\alpha^2} (N-1)^2
$$

(4.6)

$$
\sigma_e(f_0) \sigma_{e,\text{cont}} = \left( \frac{N}{N-1} \right)^2 \left( \frac{N-1}{N} \right) \left( \frac{(N-1)P + P_N}{N} \right) H + \frac{(P_S + P_N)^2}{\alpha^2} (N-1)^2
$$

(4.7)

where $P_S = E[|X|^2(t)]$ and $P_N = \sigma^2_i$ denote the signal and the noise powers respectively. Hence

$$
H^2 > \frac{2N^2}{K} \left( \frac{N}{N-1} \right)^2 \left( \frac{2(P_S + P_N)}{\alpha^2} + \frac{(N-1)(P_S + P_N)}{N} \right) H + \frac{(N-1)^2}{N^2}
$$

(4.8)

Provided that $T_0$ is long enough such that $(1/\mu) \Phi_{X}(f) + |W(f)|^2$ resembles to the signal’s PSD, the processed signal power can be described by $P_S \leq 2B_H$. Due to the conservative nature of the recommendations, (4.8) leads to

$$
K > 2 \eta^2 \left( \frac{N}{N-1} \right)^2 \left( \frac{2B_H (1 + 2SNR^{-1} + SNR^{-2})}{2^2} + \frac{4B_H (N-1)(1 + SNR^{-1})}{N^2} \right) + \eta^2
$$

(4.9)

where $SNR = P_S / P_N$ is the signal to noise ratio. It is noted that $K$ is an integer.

Formula (4.9) gives a conservative lower bound on the number of needed window averages which is a function of the channel occupancy, average sampling rate and signal to noise ratio. This recommendation can also be used to decide on the needed average sampling rate given a decided number of estimate averages possibly imposed by practical constraints (e.g. latency) in a continuous processing environment. It is a clear indication of the trade-off between the sampling rate and the number of averages needed in relation to achieving dependable sensing. According to (4.9), we can use arbitrary low sampling rates for the sensing operation at the expense of using considerably long signal analysis window i.e. $K$ can be arbitrarily large. This observation confirms/demonstrates results presented in the first paper on DASP i.e. [20,21] but for a finite $N$ and signal observation window $KT_0$. It is noted that the deployment of tapering functions other than rectangular has no effect on (4.9) and correlated as well as overlapping signal windows can be easily incorporated into the analysis conducted above by using existing results in literature on the variance reductions e.g. [28].

4.2. Numerical examples of the derived limit

Assessing whether identity (4.3) is satisfied for a given $\eta$ is an effective way to value the legitimacy or the level of conservativeness of the lower bound given by (4.9). The multiband system described in the numerical example in
Section 3 is reexamined in this subsection. Again a WSS signal with a PSD defined by $1$ for $f \in [780, 800]$ MHz and zero elsewhere is generated i.e. four adjacent active channels where $f_1=20$ MHz and $f_2=790$ MHz. An average sampling rate of $\nu=800$ MHz, $T_S=0.2$ us and $\eta=4$ ($\approx 94\%$ success rate) are used. According to (4.9), the minimum number of averaged estimates that leads to reliable detection is $K_{\text{min}} \geq 20$ for the noise free case and $K_{\text{min}} \geq 25$ for SNR $=-1$ dB. Fig. 4 compares the square root of both sides of (4.3) using simulations where in each of the two shown plots 10,000 independent experiments are conducted to approximate the probabilistic expectations.

It is noticed from Fig. 4 that for $K \geq K_{\text{min}}$, condition (4.3) is satisfied. As $K$ increases the region of additional reliability widens. However, at $K=K_{\text{min}}$ the error level in some cases e.g. noisy case in Fig. 4 can be marginally higher than the signal spectral peak. This is due to the fact that the data within the average estimates is not totally uncorrelated as assumed. To avoid such situation, the user is advised to use values that slightly exceed $K_{\text{min}}$. This example confirms the credibility of the derived limit given by (4.9) and its reasonable level of conservativeness. In the following subsection we propose an effective decision making criterion for the activity of the systems spectral bands.

### 4.3. Detection threshold

In order to be able to detect the active channels via assessing their spectral magnitude, a threshold level should be set and any channel that have a spectral point(s) higher than that threshold is regarded as being active. The detection procedure consists of three steps: (1) compute $M$ number of frequency points across the monitored channels, (2) find spectral peaks and (3) compare those peaks with a threshold $\gamma$. This postulates a binary hypothesis testing problem given by

$$
H_0 : \hat{X}(f_k) < \gamma, \quad k = 1, 2, \ldots, M \\
H_1 : \hat{X}(f_k) \geq \gamma
$$

where $H_0$ represents the absence of an activity in the channel i.e. the spectrum is made of the present noise and the estimators’ inaccuracies while $H_1$ depicts the presence of the active channel(s). The frequencies $f_k$’s are the assessed spectral points where using the minimum number of those frequency points is preferred-one per channel i.e. $M = L$.

Setting the threshold $\gamma$ correctly is essential to the reliability of the detection procedure. A practical approach to determining $\gamma$ is to anticipate/decide the level of the present error in the spectral analysis of the processed signal realization based on the accuracy of the adopted estimator and the sought success rate i.e. $\eta$. We recall that the estimator’s standard deviation is given by $\sigma_{\text{cont}}$ at frequencies where there is no channel activity which is identical to $C(f)$ amplitude at such frequencies. Hence, the detection threshold can be expressed by

$$
\gamma = \frac{N(P_S + P_N)}{(N-1)\nu} \left( 1 + \frac{\eta}{\sqrt{K}} \right)
$$  \hspace{1cm} (4.11)

In order to utilize (4.11), the signal and noise powers i.e. $P_{SN} = P_S + P_N$ need to be calculated from the set of captured noisy signal samples. The combined signal and noise powers within each of the analyzed signal windows can be calculated via

$$
\hat{P}(k) = \frac{1}{T_0} \sum_{n=1}^{N} y^2(\hat{t}_n)D_n, \quad k = 1, 2, \ldots, K
$$  \hspace{1cm} (4.12)

where $\hat{t}_n$ is the set of $N$ collected samples per window arranged in an ascending order whilst $D_n$ is the distant between two successive $\hat{t}_n$ samples i.e. $D_n = \hat{t}_n - \hat{t}_{n-1}$. Approximation (4.12) belongs to the subject of integration approximation over a finite interval. Now $P_{SN}$ can be calculated from the number of analysis signal time segments via: $P(k) = (1/K) \sum_{k=1}^{K} \hat{P}(k)$. The averaging effect is expected to eliminate/suppress the additional component in (4.12) i.e. $\sum_{n=1}^{N} x(\hat{t}_n)D_n D_n$ as $E[x(t)]E[y(t)] = 0$. The effectiveness of this thresholding approach is demonstrated in the next section. We note that the detection threshold given by (4.11) is chosen in such a way that the chances/probabilities of a false detection of a non-active channel and that of a missed detection of an active band are bounded by the same limit i.e. $1/\eta^2$. However, in some applications the users may want to reduce one of these probabilities at the expense of increasing the other one. Such requirements can be accommodated in the proposed

**Fig. 4.** Reliability condition, (a) noise-free case and $K_{\text{min}} = 20$ and (b) SNR $= -1$ dB and $K_{\text{min}} = 25$. 
approach by using Chebychev’s inequality. For example if the probability of false detection must not exceed \( P_f \), then \( \eta \) in (4.11) should be replaced with \( P_f^{-\frac{1}{2}} \).

As commonly known, high spectrum dynamic range eases the requirement on the spectral analysis device and forms a safety margin for any inaccuracies that might be incurred. The dynamic range of the pursued spectrum i.e. \( C(f) \) can be set by choosing appropriate \( \alpha \). Due to practical sensing issues discussed in Section 3, \( T_o \) is assumed to be fixed and hence the spectrum dynamic range depends on the number of samples per window i.e. \( N \). Referring to Fig. 3 and for a desired dynamic range, we can write: \( H_L \geq mH_k \) where \( m > 1 \). Adopting a conservative approach the number of samples needed to safeguard \( H \geq (m-1)H_k \) should comply with

\[
N \geq 2B_M(k-1)T_0(1+\text{SNR}^{-1}) + 1 \tag{4.13}
\]

The guideline given by (4.13) although not necessary for reliable detection gives the user an indication of the needed average sampling rate if the spectrum dynamic range is an issue.

4.4. Applicability of the adopted approach to other sampling schemes

In this subsection we discuss the applicability of the proposed sensing approach to randomized sampling schemes other than TRS. The detailed analysis of such generalizations along with considering cyclostationary signals lieu of WSS/WSE will be reported in a separate paper.

In principle, any sampling scheme that produces \( \text{var}(X_e(f))\text{xt}(t) \) lower than or equal to that of TRS and its \( E[X_e(f)] \) is a form of \( C(f) \) given by (3.4) is deemed to be adequate for the proposed approach. This criterion fits a certain set up of stratified and stratified antithetical sampling schemes reported in [17,18] where the sampling point process is defined by a uniformly distributed PDF within an equally partitioned signal window i.e. a form of jittered random sampling (JRS). However, applying jittered-stratified sampling (JSS) and jittered-antithetical sampling (JAS) to \( X_e(f) \) in its current form should be approached with caution as the spectrum dynamic range of the targeted spectrum can affect the sensing reliability for such schemes. This is referred to the high smeared-aliasing suppression ability of JRS, with jitter uniformly distributed within \( 1/\alpha \) period, within the active spectrum band(s) [12,13]. The spectrum dynamic range is locally improved within the frequency band of the active channel(s) but it deteriorates across the overall monitored bandwidth. Hence, the expected value of the estimator no longer guarantees detectable feature for any \( \alpha \), provided that (4.9) is satisfied, as the spectrum dynamic range deteriorates as \( \alpha \) or/and SNR drops. In this case (4.13) can be deployed to avoid such adversity of using JSS and JAS.

5. Numerical examples

Consider a multiband system comprising 50 channels \( (L=50) \) that are 2 MHz each \( (B_c=2 \text{ MHz}) \). The system channels are located in \( f \in [800,900] \text{ MHz} \). A rectangular window of width \( T_o=2 \mu s \) is used. A zero mean WSS signal is generated for each of the active bands with a PSD of amplitude 1 in the passband and zero elsewhere. Channel occupancy of 10% i.e. \( B_N=10 \text{ MHz} \) is assumed and all active channels are of similar power levels. The sensing procedure is based on taking a single frequency point at the middle of each of the systems channels to assess its activity. Fig. 5 shows the average sampling rate \( \alpha \) versus the number of averaged estimates \( K \) given by (4.9) for a rate of success of 96% i.e. \( \eta = 5 \). The asterisks in this figure mark the tested \( \alpha \) values in Figs. 6 and 7. The latter two show the results of the sensing operation along with the chosen threshold level for each of TRS and JSS in noise-free environment and with \( \text{SNR} = 3 \text{ dB} \).

It is evident from Figs. 6 and 7 that the five active channels are detectable using the adopted estimator and the recommendation given by (4.9) with an \( \alpha \) as low as 30 MHz for TRS and JSS in the presence as well as absence of noise. In both figures, spectral sample points belonging to non-active channels are below the set threshold level which illustrates the effectiveness of the thresholding level defined by (4.11). For the JSS it is noticed that the set threshold is more

![Fig. 5. K versus \( \alpha \). Noise-free case (solid-line) and SNR of 3 dB (dashed-line).](image-url)
conservative in comparison to the TRS counterpart due to the smeared-aliasing suppression capability of JSS within and near the active band(s). It is clearly seen in Figs. 6 and 7 that as $\alpha$ or/and SNR drops the spectrum dynamic range deteriorates despite imposing extra requirement on $K$. In terms of spectrum sensing discrepancies between TRS and JSS, those numerical
experiments demonstrate that such differences are marginal for noise-free and noisy signals provided a reasonable spectrum dynamic range. Further experimental results (not shown here) showed that the distribution of simultaneously active channels across the scanned bandwidth does not hinder the performance of the detection procedure.

An interesting case is to assess the detection precision in case none of the channels are active. Fig. 8 shows the detection results for the two test cases shown in Fig. 5 where SNR = 3 dB in case all system channels are ideal/silent whilst the system designer assumed 10% spectrum utilization. It is clear in Fig. 8 that the adopted spectrum sensing approach along with the thresholding procedure is immune against such situations.

If uniform sampling is deployed the minimum bandpass sampling rate that would avoid aliasing within the monitored bandwidth is 225 MHz (bandpass sampling). Hence, more than 85% saving on the used sampling rate could be achieved with the use of the proposed approach in this paper. However, comparing sampling rates of the uniform and nonuniform cases can be misleading as the latter case would demand more estimate averages i.e. larger $K$. In the uniform sampling case, the number of needed averages is $K_{\text{US}} \geq \eta^2$ for noise-free environments (following dependability analogy described in Section 4).

As a result, around 25% saving on the needed number of signal samples to perform sensing is obtained by deploying the proposed detection method in those numerical experiments. Therefore, the proposed approach offers substantial savings over uniform sampling in terms of the sampling rate and number of processed samples for low spectrum occupancy. Each scenario should be evaluated individually to assess benefits of nonuniform over uniform sampling in terms of the number of needed signal samples. In the latter case the suitable rates are constraint by $B$ and its central frequency; the minimum possible rate is $2B$ regardless of the spectrum occupancy. With the low spectrum utilization assumption, it is unambiguously clear that the use of the proposed method would bring substantial savings in terms of the sampling rate and number of processed samples compared to the classical DSP. In fact for the proposed method extending the monitored bandwidth, assuming constant SNR e.g. the sampling is preceded by a filter to limit the noise bandwidth, would not impose any additional requirements on the needed sampling rates or estimate averages provided that the bandwidth of the concurrently active channels does not change.

Maintaining the average sampling rates above Landau prompts researching into effective reconstruction algorithms of noisy nonuniformly sampled data. This can be utilized to recover the conveyed signals/messages or to model the detected channel(s) prior to extracting them sequentially to reveal weaker present component(s) in case the power levels of the concurrently active bands vary significantly. This can be a more computationally efficient extraction method in comparison to existing ones such as SECOEX [35]. Besides, various applications of spectrum sensing in multiband communication systems e.g. cognitive radio stimulate investigating the adequacy of the proposed approach to communication signals given their cyclostationary/cycloergodic nature.

### Appendix A. Proof of (3.2)

We start with

$$E[X(f)] = \frac{T_0^2}{N(N-1)\mu} E\left[ \sum_{n=1}^{N} y(t_n) w(t_n) e^{-j2\pi f t_n} \right]^2$$

The estimator can be expressed by

$$X(f) = \frac{T_0^2}{N(N-1)\mu} \left\{ \sum_{n=1}^{N} y^2(t_n) w^2(t_n) \right\}
+ \sum_{m=1}^{N} \sum_{n=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} y(t_n) w(t_n) y(t_m) w(t_m) e^{-j2\pi f(t_n - t_m)}$$

Given that the components in the double summation are IID random variables

$$E[X(f) | X(t)] = \frac{T_0^2}{N(N-1)\mu} \left[ \sum_{n=1}^{N} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} y(t_n) w(t_n) e^{-j2\pi f(t_n - t_m)} \right) \right]$$

$$+ \sum_{m=1}^{N} \sum_{n=1}^{N} \sum_{i=1}^{N} \sum_{j=1}^{N} y(t_n) w(t_n) y(t_m) w(t_m) e^{-j2\pi f(t_n - t_m)}$$

### 6. Conclusion

A method that utilizes DASP methodology to detect the active channel(s) in a multiband environment is proposed. The adopted approach can use arbitrary low sampling rates which are independent of the width of the monitored bandwidth or its central frequency. However in order to preserve the reconstructability of the detected signals, it is recommended that the sampling rates exceed twice the total bandwidth of the concurrently active channels $B_A$. This feature compares favorably with uniform sampling based spectrum sensing methods where the required sampling rates grow proportionally to the number of the monitored channels. In the latter case the suitable rates are constraint by $B$ and its central frequency; the minimum possible rate is $2B$ regardless of the spectrum occupancy. With the low spectrum utilization assumption, it is unambiguously clear that the use of the proposed method would bring substantial savings in terms of the sampling rate and number of processed samples compared to the classical DSP. In fact for the proposed method extending the monitored bandwidth, assuming constant SNR e.g. the sampling is preceded by a filter to limit the noise bandwidth, would not impose any additional requirements on the needed sampling rates or estimate averages provided that the bandwidth of the concurrently active channels does not change.
Since the signal and the zero mean AWGN are independent
\[
E[X(t)|x(t)] = \frac{T_0}{N(N-1)\mu} \left\{ N\varepsilon t_n^2 w_n^2(t_n) + \sigma_0^2 \mu / \Delta x + \frac{N(N-1)}{T_0^2} |X_w(f)|^2 \right\}
\]
(A.1)
By calculating the expected values in (A.1) in terms of \( p \) given by (3.1), the expression given by (3.2) is obtained.

Appendix B. Proof of (3.16)

Since \( R_{W_1}(f) \) and \( I_{W_1}(f) \) are zero mean, their correlation is given by \( p(f) = E[R_{W_1}(f)I_{W_1}(f)] \). First we split the double summation into two
\[
R_{W_1}(f)/I_{W_1}(f) = \frac{T_0}{N^2} \sum_{n=1}^{N} y_n^2(t_n) w_n^2(t_n) \cos(2\pi f t_n) \sin(2\pi f t_n)
\]
Similar to the \( E[X(t)|x(t)] \) shown in Appendix A, we obtain
\[
E[R_{W_1}(f)/I_{W_1}(f)|x(t)] = \frac{R_0 + \frac{T_0}{N} \varepsilon t_n^2 w_n^2(t_n) \cos(2\pi f t_n) \sin(2\pi f t_n)}{2}
\]
and subsequently (3.10). Following the delay introduction we can write
\[
R_{W_1}(f) = \cos(\theta(f)) R_{W}(f) + \sin(\theta(f)) I_{W}(f)
\]
\[
I_{W}(f) = -\sin(\theta(f)) R_{W}(f) + \cos(\theta(f)) I_{W}(f)
\]
The value of \( 0(f) \) would be chosen such that
\[
E[R_{W_1}(f)/I_{W_1}(f)] = 0,
\]
consequently
\[
0.5 \sin(2\theta(f)) [E[R_{W}^2(f)] - E[R_{W}^2(f)]] + \cos(2\theta(f)) E[R_{W}^2(f)I_{W}(f)] = 0
\]
(B.2)
By manipulating (B.2), we attain (3.16).

References