Algebraic Approach to Ambiguity-Group Determination in Nonlinear Analog Circuits

Barbara Cannas, Alessandra Fanni, and Augusto Montisci

Abstract—In this paper, a symbolic procedure for ambiguity-group determination, based on the a priori identifiability concept, is proposed. The method starts from the analysis of the occurrence of circuit parameters in the coefficients of the input/output relationship in order to select the potential canonical ambiguity groups. This first step allows one to strongly reduce the problem complexity. In a second step, the obtained nonlinear system that imposes the ambiguity conditions is solved, resorting to Gröbner bases theory. Both of these steps are completely symbolic, thus avoiding round-off errors. Furthermore, the method can be applied to both linear and nonlinear circuits. An alternative approach is also proposed, which extends to nonlinear circuits a method presented in the literature, which can be directly applied only to linear circuits. The methods are illustrated by means of benchmarks regarding well-known linear and nonlinear circuits.

Index Terms—Analog-circuit diagnosis, canonical ambiguity groups (CAGs), nonlinear circuits, testability.

I. INTRODUCTION

In recent years, there has been a growing interest on automatic procedures for analog-circuit fault diagnosis [1]–[8]. To perform this task, it is crucial to have a measure of the circuit solvability, i.e., of the solvability of nonlinear fault diagnosis equations. The circuit solvability is strictly linked to the testability concept. The most popular definition of analog-circuit testability has been introduced in [9] and [10], and it provides a measure of solvability of fault equations. Apart from the method used to solve the fault equations, the testability of a circuit provides the maximum number of simultaneous faults that any diagnostic system could detect.

Let us consider an analog time-invariant linear circuit. For the sake of simplicity, a single-input–single-output circuit is considered. The input–output relationship (IOR) can be expressed in terms of transfer function as

$$H(F, s) = \frac{\sum_{h=0}^m a_h(F) \cdot s^h}{s^n + \sum_{k=0}^{n-1} b_k(F) \cdot s^k} = \frac{N_H}{D_H}$$

where $F = \{F_1, F_2, \ldots, F_L\}$ is the set of circuit parameters.

Testability $T$ can be calculated as the column rank of the Jacobian matrix $\Phi$ whose elements are the partial derivatives of the transfer function with respect to the circuit parameters [9]

$$\Phi(F, s) = \left[ \frac{\partial H}{\partial F_1} \frac{\partial H}{\partial F_2} \cdots \frac{\partial H}{\partial F_L} \right] = \frac{P(F, s)}{D_H(F, s)}$$

where $P$ is the polynomial matrix

$$P(F, s) = \left[ \frac{\partial N_H}{\partial F_1} D_H - \frac{\partial D_H}{\partial F_1} N_H, \ldots, \frac{\partial N_H}{\partial F_L} D_H - \frac{\partial D_H}{\partial F_L} N_H \right].$$

Hence, testability $T$ is the column rank of $P$ [1].

A noteworthy simplification from a computational point of view, has been introduced in [3], where it has been proven that the column rank of $P$ is equal to the rank of a matrix $B_C$, whose elements are the coefficients of the polynomials in $P$.

The methodology for testability evaluation has been further simplified in [4], where it has been shown that the testability matrix $B_C$, whose elements are the derivatives of the coefficients of the network function $H$ with respect to the faulty parameters, has the same rank of the matrix $B$.

In common fault diagnosis applications, the multiple-test-point case is very frequent. In this case, network functions can be formulated with the same denominator [4]; thus, adding a new test point only, the numerator coefficients contribute to the circuit testability.

For low-testability circuits, an important concept is that of an ambiguity group. Roughly speaking, an ambiguity group is a set of components that, if considered as potentially faulty, does not allow diagnosis to yield an unique solution. An ambiguity group that does not contain other ambiguity groups is called canonical ambiguity group (CAG) [5].

Several numerical algorithms and symbolic procedures for evaluating circuit testability and CAGs in the case of linear circuits have been developed [1]–[7]. However, few contributions are present in the literature regarding nonlinear circuit testability.

In [11], a method for optimal analog-test-point selection for fault diagnosis is proposed. The described method uses ambiguity-group concept and evolutionary computations to determine the optimal set of analog test points.

In [12], a symbolic approach for determining the testability value and the ambiguity groups in nonlinear analog circuits is presented. The approach is an extension of the methodologies developed for the linear case to circuits where nonlinear components, such as diodes or transistors, are present. The procedure consists in substituting each nonlinear component with its

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piecewise linear model and in analyzing the resulting linear circuit whose parameters depend on the working point.

In [13], the ambiguity analyses in the dc and ac domains are conducted independently of one another, and the results are combined. In the first case, the nonlinear components are replaced with piecewise linear models, whereas in the ac case, a small signal analysis is performed.

In [14], a fault detection framework for a class of nonlinear networked control systems via a shared communication medium has been proposed. Periodic communication sequences are designed to preserve the plant’s reachability and observability. If the system is reachable and observable, an observer-based fault detection scheme can be obtained to detect the fault.

This paper proposes two symbolic procedures for ambiguity-group determination, which can be applied to both linear and nonlinear circuits.

The first approach is based on the a priori identifiable concept, whereas the second one resorts to the results presented in the literature for linear circuits [1]–[8], extending them to nonlinear cases.

The method starts with the determination of IOR; then, an analysis of the occurrence of parameters in the coefficients of IOR is performed in order to select the potential CAGs. This analysis allows one to strongly reduce the problem complexity. Finally, in the first approach, the nonlinear systems obtained by imposing the ambiguity conditions are solved, resorting to Gröbner bases theory [13], [15], whereas the second approach is based on the calculation of the rank of a reduced testability matrix.

This paper is organized as follows. Section II recalls the definition of CAG and some concepts of differential algebra. In Section III, the procedures for CAG determination are described, whereas in Section IV, the proposed procedures are applied to linear and nonlinear circuits. Section V reports the conclusions.

II. THEORETICAL BASIS

Let us consider an analog time-invariant linear circuit described by

\[
\frac{dx}{dt} = A(F)x(t) + B(F)u(t) \quad \text{y} = C(F)x(t) + Du(t)
\]

where \(x\) is the \(n\)-dimensional state variable vector, \(u\) is the scalar input, \(F = \{F_1, F_2, \ldots, F_L\}\) is the set of \(L\) circuit parameters, \(y\) is the scalar output, and \(A, B, C\) and \(D\) are the matrices depending on \(F\).

The literature reports several definitions of CAGs for linear circuits [2], [6], [16]. Following [6], a first definition of CAG can be given.

**Definition 1:** A set of components, whose parameters are \(\{F_1, F_2, \ldots, F_D\}\), constitutes a CAG of order \(D\) if a variation of \(d < D\) parameters causes a variation in the transfer function, which is indistinguishable from that due to the variation of the remaining \(D-d\) parameters of the set, i.e.,

\[
\Delta H_1(\Delta F_1, \ldots, \Delta F_d) = \Delta H_2(\Delta F_{d+1}, \ldots, \Delta F_D),
\]

where

\[
\Delta H_1(\Delta F_1, \ldots, \Delta F_d) = \frac{\partial H}{\partial F_1} \Delta F_1 + \cdots + \frac{\partial H}{\partial F_d} \Delta F_d,
\]

\[
\Delta H_2(\Delta F_{d+1}, \ldots, \Delta F_D) = \frac{\partial H}{\partial F_{d+1}} \Delta F_{d+1} + \cdots + \frac{\partial H}{\partial F_D} \Delta F_D.
\]

Note that, in most cases, it is not possible to indifferently refer to a set of components or parameters in order to define a CAG, as the characteristic of a component can be described by more than one parameter. For this reason, in the following, we choose to refer to a CAG as a set of parameters rather than a set of components.

In this paper, an equivalent definition of CAG is adopted, which is derived from the following theorem [16].

**Theorem:** Let \(H\) be the transfer function of a linear circuit. A set of \(D\) parameters \(\{F_1, F_2, \ldots, F_D\}\) constitutes a CAG of order \(D\) if a variation \(\Delta F \neq 0\) exists, which does not affect the transfer function value, i.e.,

\[
H|_{F_1, \ldots, F_D} = H|_{F_1 + \Delta F_1, \ldots, F_D + \Delta F_D}.
\]

**Proof:** Let us consider first a group of parameters \(\{F_1, F_2, \ldots, F_D\}\) constituting a CAG, and let \(\Delta H(\Delta F_1, \ldots, \Delta F_D)\) be the transfer function variation due to the variation of the \(d\) parameters of CAG. The modified transfer function is \(H_1 = H + \Delta H\). An opposite variation of \(d\) parameters allows one to obtain, for \(H_1\), the variation \(\Delta H'(\Delta F_1, \ldots, \Delta F_d) = -\Delta H\), obtaining the original transfer function \(H\) (see Fig. 1). \(H\) and \(H_1\) only differ for the values of \(F_1, F_2, \ldots, F_d\). This implies that the CAG \(\{F_1, F_2, \ldots, F_d\}\) is a CAG also for \(H_1\). Thus, a variation of \(D-d\) parameters exists such that the corresponding transfer function variation for \(H_1\) is \(\Delta H_1' = \Delta H'\) (see Fig. 1). Consequently, the global variation of the transfer function is

\[
\Delta H_{\text{global}} = \Delta H(\Delta F_1, \ldots, \Delta F_d)
\]

\[
+ \Delta H'(\Delta F_{d+1}, \ldots, \Delta F_D)
\]

\[
= 0.
\]

Let us now consider a group of \(\{F_1, F_2, \ldots, F_D\}\) parameters. If a variation \(\{\Delta F_1, \ldots, \Delta F_D\}\) exists such that \(\Delta H_{\text{global}} = 0\), then it is possible to obtain the following (see Fig. 1):

1) a variation \(\Delta H\) of \(H\) due to a variation of \(d < D\) parameters:

2) a variation \(\Delta H_1' = -\Delta H\) of \(H_1\) due to a variation of the other \(D-d\) parameters.

Consequently, starting from \(H_1\), a variation of \(d\) parameters provides a variation of the transfer function, which is indistinguishable from the variation produced by modifying the other \(D-d\) parameters of the group.
Thus, \( \{F_1, F_2, \ldots, F_D\} \) is a CAG for \( H_1 \) and, consequently, for \( H \).

Hence, the following definition of CAG can be given.

**Definition 2:** A set of \( D \) parameters \( \{F_1, F_2, \ldots, F_D\} \) constitutes a CAG of order \( D \) if a variation \( \Delta F \not= 0 \) exists, which does not affect the value of the transfer function.

In practice, when the variations in Definitions 1 and 2 are only a finite number, the probability that the corresponding parameters assume exactly the ambiguous values is virtually null. Thus, in real applications, they are not considered as CAGs.

### A. Nonlinear Circuits

Let us consider an analog time-invariant nonlinear circuit described by

\[
\begin{align*}
\frac{dx}{dt} &= f[x(t), u(t), F] \\
y &= h[x(t)]
\end{align*}
\]

(8)

where \( f \) and \( h \) are the algebraic functions in \( x \).

In the case of nonlinear circuit diagnosis, it is necessary to refer to IOR in defining CAG because the transfer function is not defined.

In the case of multiple test points, the coefficients of the corresponding IORs are generally different; thus, adding a new test point, all the new coefficients contribute to the circuit testability.

Definitions 1 and 2 can be generalized as follows.

**Definition 3:** A set of \( D \) parameters \( \{F_1, F_2, \ldots, F_D\} \) constitutes a CAG of order \( D \) if a variation \( \Delta F \not= 0 \) exists such that it does not affect the value of the IOR coefficients.

Generally speaking, if a set of \( D \) parameters constitutes a CAG, system (4) is not identifiable.

In fact, a system in the form of (4) is identifiable through IOR if and only if, for at least a generic set of parameters, the equation

\[
\text{IOR}(F_1, F_2, \ldots, F_L) = \text{IOR}(F_1', F_2', \ldots, F_L')
\]

(9)

has at most a finite number of solutions \( \{(F_1', F_2', \ldots, F_L') \not= (F_1, F_2, \ldots, F_L)\} \) for all input functions \( u \).

For a system in the form of (4), IOR is a nonlinear differential polynomial in \( u, y \), and their derivatives, and it can be expressed as

\[
z(u, y) = z(y, \dot{y}, \ddot{y}, \ldots, u, \dot{u}, \ddot{u}, \ldots, a_1, a_2, \ldots) = 0
\]

(10)

where \( a_t \) denotes the IOR coefficients depending on the set of circuit parameters \( F \).

IOR can be determined by resorting to the “characteristic set” associated to the dynamic state equations. The characteristic set was first introduced by Ritt [17] in 1950, and since 1990, it has been widely used for the study of dynamic systems [18].

The peculiarity of the characteristic set [19] is that it summarizes all the information contained in the differential equations defining a dynamic system. In particular, if a ranking of the variables and their derivatives is chosen, e.g.,

\[
u < \dot{u} < \ddot{u} \cdots < y < x_1 < x_2 < \cdots < y < \dot{x}_1 < \dot{x}_2 \cdots
\]

(11)

for a system in the form of (4), the characteristic set exhibits \( n + 1 \) differential polynomials, that is:

1) IOR \( z(u, y) \);
2) \( n \) differential polynomials, in \( u, x, \) and \( y \), denoted by the \( n \)-dimensional vector \( \mathbf{z}(u, x, y) \).

In some cases, IOR can be easily determined by inspecting and manipulating the dynamic equations. However, in the majority of cases, the procedure to identify IOR may turn out to be rather complex, and resorting to software to calculate the characteristic set is mandatory.

In this paper, the implementation of Ritt’s algorithm for the commercial tool REDUCE described in [20] has been used in order to obtain the coefficients of IOR as symbolic functions of circuit parameters.

The algorithm is directly applicable only to algebraic systems. Nevertheless, some transcendental systems (e.g., systems with trigonometric or exponential nonlinear functions) can be analyzed, resorting to suitable transformations that cancel the transcendental function.

Let us consider a state-space system (8) of order \( \eta \). It is possible to eliminate a transcendental function \( f(x) \) from the system, no matter what the coefficient values are.

1) **Exponential functions:**

\[
f(x) = e^x.
\]

Let us consider an auxiliary variable \( w = e^x \). Any occurrence of the exponential function can be substituted by the auxiliary variable \( w \).

Let us evaluate the derivative of the exponential function \( \dot{w} = \dot{x}e^x = \dot{x}w \). The equation \( \dot{y} = \dot{x}w \), where the exponential function does not appear, is added to the state equations.

This procedure results in a new state-space system that is algebraic and of order \( n + 1 \).

2) **Trigonometric functions:**

\[
f(x) = \sin(x).
\]

Let us consider two auxiliary variables \( w = \sin(x) \) and \( v = \psi \). Any occurrence of the sinusoidal function can be substituted by the auxiliary variable \( w \).

Let us evaluate the first two derivatives of the sinusoidal function

\[
\dot{w} = \cos(x)\dot{x};
\]

\[
\ddot{w} = \dot{\psi} = -\sin(x)\dot{x} + \cos(x)\ddot{x} = -u\dot{x} + \frac{\dot{u}}{x}. \dot{x}^2.
\]

The equations \( \dot{\psi} = u\ddot{x} + (\dot{u}/x)\dot{x} \) and \( \ddot{\psi} = v \), where the trigonometric functions do not appear, are added to the state equations.
This procedure results in a new state-space system that is algebraic and of order \( n + 2 \).

### III. Canonical Ambiguity-Group Determination

The proposed procedure for CAG identification consists of two phases.

In the first phase, a preliminary analysis of the IOR coefficients is performed, which allows one to a priori reduce the number of potential CAGs. For example, two parameters \( F_1 \) and \( F_2 \) cannot constitute a CAG if there is an IOR coefficient such that only one of them (e.g., \( F_1 \)) appears. In fact, in this case, a variation of \( F_2 \) cannot produce the same effect of a variation of \( F_1 \). In particular, the potential CAGs are identified by inspecting an incidence matrix, as described in the following.

Let us consider a circuit described by \( L \) parameters \( F_1, \ldots, F_L \), and let \( a_j \) be the IOR coefficients for \( j = 1, \ldots, K \). An incidence matrix \( \mathbf{W} \) can be written as follows:

\[
\mathbf{W} = 
\begin{bmatrix}
    a_{1,1} & \cdots & a_{1,K} \\
    a_{2,1} & \cdots & a_{2,K} \\
    \vdots & \ddots & \vdots \\
    a_{L,1} & \cdots & a_{L,K}
\end{bmatrix}
\tag{12}
\]

where

\[
w_{i,j} = \begin{cases} 
1, & \text{if the parameter } F_i \text{ appears in coefficient } a_j \\
0, & \text{otherwise}, \\
i = 1, \ldots, L; \ j = 1, \ldots, K.
\end{cases}
\]

The inspection of the incidence matrix allows one to determine if a combination of parameters is a CAG. In particular, a set of parameters \( \{F_1, F_2, \ldots, F_D\} \) is a potential CAG only if at least two of them are present in the same time in all the IOR coefficients where they appear, i.e., if

\[
\sum_i w_{i,j} \neq 1 \ i : F_i \in \text{CAG} \ j = 1, \ldots, K. \tag{13}
\]

It is worth noting that if

\[
\sum_{j=1}^{K} w_{i,j} = 0 \tag{14}
\]

parameter \( F_i \) is not diagnosable. That inspection has to be performed for all the groups of order \( D \leq T \). In fact, according to the definition of testability \( T \), a group of order that is greater than \( T \) is certainly an ambiguity group.

In the second phase of the procedure, for each potential CAG, further analysis is necessary in order to validate it as an actual CAG.

Two alternative approaches are presented in this paper to validate the potential CAGs as the actual CAGs. The first approach makes use of the theorem in Section II, whereas the second one resorts to the results presented in the literature [1]–[8] for linear circuits extending them to nonlinear cases.

#### A. System Identifiability Approach

Let us consider \( D \) parameters that could belong to a CAG of order \( D \). They actually constitute a CAG if a variation \( \Delta F_i \), \( i = 1, \ldots, D \) of these parameters does not modify IOR (see Definition 4).

The application of Definition 4 corresponds to solve a problem of constrained identifiability, where the parameters of the potential CAG are the variables of the problem, whereas the remaining \( L-D \) parameters are considered known.

For each potential CAG, this problem can be expressed by a system of \( J \) equations

\[
\begin{align*}
    a_j(F_1, \ldots, F_L) &= a_j(F'_1, \ldots, F'_L), & j = 1, \ldots, J \\
    \text{subject to } & F'_i = F_i & \forall F_i \notin \text{CAG}
\end{align*}
\tag{15}
\]

where \( a_j \) is the generic coefficient of IOR containing parameters of the potential CAG, \( J \) is the number of such coefficients, and \( F'_i = F_i + \Delta F_i \).

Hence, a set of \( D \) parameters constitutes a CAG if the corresponding algebraic system (15) is not identifiable, i.e., if system (15) has an infinite number of solutions.

Conversely, if system (15) admits a finite number of solutions, the ambiguity holds only for a finite number of sets \( \{F_1, F_2, \ldots, F_D\} \). Such sets of parameters are not considered CAGs because the probability that the parameters assume exactly the ambiguous values is virtually null.

In the case of multiple test points, the procedure is applied to the coefficients of all the IORs. From the complexity point of view, the number of equations introduced by each test point strictly depends on the layout of the circuit.

In summary, in order to determine all the CAGs, system (15) has to be solved for each potential CAG previously determined. If system (15) has an infinite number of solutions, the potential CAG is validated as the actual CAG. In this phase, the lower order potential CAGs are investigated first in order to avoid solving system (15) for the potential CAGs that contain the already-determined CAGs.

Note that when \( D = L \), system (15) allows one to evaluate the circuit testability. In particular, testability \( T \) is equal to the minimum number of equations to which system (15) can be reduced.

In this paper, system (15) has been solved, resorting to Gröbner bases theory [15]. This is the origin of many symbolic algorithms used to manage multiple-variable polynomials. In particular, the commercial tool REDUCE has been used, which implements Buchberger’s algorithm [21]. It makes use of a generalization of the Gaussian elimination method for multivariable linear equations and of Euclid’s algorithm for one-variable polynomial equations.

Note that this approach gives us not only the number of solutions of system (15) but also the relationship among the ambiguous faulty parameters. These expressions, combined with further constraints, e.g., nonnegativity of some circuit parameters, can be exploited in a successive diagnostic phase to solve some ambiguities.

This feature represents an important improvement with respect to the approach described in the following section and to those proposed in the literature for linear circuits [1]–[8], where no information can be deduced on the relationship between indistinguishable faults.
B. Jacobian Rank Approach

The second approach to validate the potential CAGs as the actual CAGs is based on the calculation of the rank of the testability matrix $\mathbf{B}_C$ (see Section I) [4]. An extension to the nonlinear circuits of the CAG evaluation method described in [22] is proposed by introducing a generalized testability matrix, whose elements are the derivatives of the coefficients of IOR with respect to the faulty parameters. In the case of linear circuits, because IOR and the corresponding network function have the same coefficients, the obtained testability matrix coincides with the one in [22].

Let us consider a potential CAG $\{F_1, F_2, \ldots, F_D\}$ of order $D$. A reduced testability matrix $\mathbf{B}_R$ can be considered, whose columns correspond to the $D$ circuit parameters belonging to the potential CAG

$$
\mathbf{B}_R = \begin{bmatrix}
  a_1 & F_1 & \cdots & F_D \\
  \vdots & \vdots & \ddots & \vdots \\
  a_J & b_{1,1} & \cdots & b_{1,D} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_J & b_{J,1} & \cdots & b_{J,D}
\end{bmatrix}
$$

(16)

where $a_j, j = 1, \ldots, J$ is the generic coefficient of IOR containing parameters of the potential CAG and $J$ is the number of such coefficients. If $\mathbf{B}_R$ has no full rank, the potential CAG $\{F_1, F_2, \ldots, F_D\}$ constitutes a CAG.

IV. EXAMPLES

In this section, the proposed procedure is applied to well-known linear and nonlinear circuit benchmarks. In all cases, we assume that the IOR of the circuit is available.

A. Sallen–Key Bandpass Filter

The proposed procedure is applied first to a linear circuit benchmark retrieved from the literature [6], [8]. In particular, the Sallen–Key bandpass filter, shown in Fig. 2, has been considered. The IOR corresponding to the input $u = v_{in}$ and the output $y = v_{out}$ shown in Fig. 2 is

$$
a_1 y + a_2 \frac{dy}{dt} + \frac{a_3}{C_1} \frac{dy}{dt} + a_3 \frac{du}{dt} = 0
$$

(17)

where

$$
a_1 = \frac{G_3}{C_1 C_2} (G_1 + G_2)
$$

Fig. 2. Sallen–Key bandpass filter.

TABLE I

<table>
<thead>
<tr>
<th>CAG</th>
<th>G_1</th>
<th>G_2</th>
<th>G_3</th>
<th>G_4</th>
<th>G_5</th>
<th>C_1</th>
<th>C_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>G_1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>G_2</td>
<td>?</td>
<td>-</td>
<td>?</td>
<td>-</td>
<td>?</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>G_3</td>
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<td>-</td>
</tr>
<tr>
<td>G_4</td>
<td>?</td>
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<td>-</td>
</tr>
<tr>
<td>G_5</td>
<td>-</td>
<td>-</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>C_1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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</tr>
</tbody>
</table>

$$
a_2 = \left[ \frac{G_1 G_5 - G_2 G_4}{G_2 C_2} + \frac{G_3}{C_1 C_2} (C_1 + C_2) \right]
$$

$$
a_3 = -\frac{G_1}{G_5 C_2} (G_4 + G_5).
$$

(18)

In [6], the following CAGs are reported for the same circuit: $\{C_1, G_2, G_3\}, \{G_4, G_5\}$. Incidence matrix $\mathbf{W}$ is

$$
\mathbf{W} = \begin{bmatrix}
  a_1 & a_2 & a_3 \\
  G_1 & 1 & 1 & 1 \\
  G_2 & 1 & 0 & 0 \\
  G_3 & 1 & 1 & 0 \\
  G_4 & 0 & 1 & 1 \\
  C_1 & 1 & 1 & 0 \\
  C_2 & 1 & 1 & 1
\end{bmatrix}.
$$

(19)

We will evaluate first the groups of order two. By inspecting matrix $\mathbf{W}$, it is possible to a priori reduce the potential ambiguity groups of order two to the five couples indicated by a "?" mark in Table I. In fact, for all the other couples, (13) is not verified. For example, $G_1$ and $G_2$ cannot constitute a CAG because in incidence matrix (19), it results to $w_{13} + w_{23} = 1$.

Let us now consider the groups of order three. By inspecting matrix $\mathbf{W}$, it is possible to a priori reduce the potential ambiguity groups of order three to the 18 triples of components indicated by a '?' mark in Table II. In fact, for all the other triples, (13) is not verified. For example, $\{G_1, G_2, G_3\}$ cannot constitute a CAG because [see (7)]

$$
w_{13} + w_{23} + w_{33} = 1.
$$

1) System identifiability approach: Because IOR has three coefficients as a function of circuit parameters, the analysis will be performed only for groups of orders two and three. In fact, all the groups of order that is greater than three contain at least one CAG.

Each potential CAG actually constitutes a CAG iff the corresponding system (15) has an infinite number of solutions.
For example, let us consider the potential CAG \{G_{14}, C_2\}. System (15) consists of \( J = 3 \) equations found in (20), shown at the bottom of the page, where \( G'_{14} \) and \( C'_{2} \) are the fault values, and they represent the system unknowns. By symbolically solving system (20), shown at the bottom of the page, and by using Gröbner analysis, it results that it admits only one solution

\[
\begin{align*}
G'_{14} &= G_1 \\
C'_{2} &= C_2.
\end{align*}
\]

Thus, \( G_1 \) and \( C_2 \) do not constitute a CAG.

Let us now consider the couple \{\( G_{45}, C_{5} \). System (15) consists of \( J = 2 \) equations found in (21), shown at the bottom of the page, where \( G'_{4} \) and \( C'_{5} \) are the fault values, and they represent the system unknowns.

System (21) admits an infinite number of solutions

\[
G'_{4} = \frac{G_4 C_5}{G_5}.
\]

Thus, \( G_4 \) and \( G_5 \) constitute a CAG. This implies that, independently on the diagnostic approach, whatever fault of the component \( G_{14} \) is indistinguishable from a corresponding fault of the component \( G_{5} \).

By performing the same analysis for all the five couples in Table I, it results that the couple \{\( G_{14}, G_{5} \)\} is the only CAG of order two.

Let us consider the potential CAG of order three \{\( G_{11}, G_{22}, G_{41} \). System (15) consists of \( J = 3 \) equations found in (22), shown at the bottom of the page, where \( G'_{11}, G'_{22} \) and \( C'_{1} \) are the fault values, and they represent the system unknowns.

System (22) admits the following two solutions:

\[
\begin{align*}
G'_{11} &= \frac{-G_1 G_4 + G_3 G_5 - G_2 G_4 + G_4 G_5}{G_2} \\
G'_{22} &= \frac{G_3 G_4 + G_4 G_5 - G_3 G_2 + G_2 G_5}{G_5} \\
G'_{41} &= \frac{-G_1 G_2 + G_3 G_4 G_5 + G_2 G_4 - G_2 G_5}{G_5} \\
G'_{12} &= \frac{-G_1 G_2 + G_3 G_4 G_5 + G_2 G_4 - G_2 G_5}{G_5} \\
\end{align*}
\]

where \( \Gamma \) is defined in (23), shown at the bottom of the page.

Thus, group \{\( G_{11}, G_{22}, G_{41} \)\} is not a CAG.

Let us now consider the potential CAG \{\( G_{22}, G_{33}, C_{1} \). System (15) consists of \( J = 2 \) equations found in (24), shown at the bottom of the next page, where \( G'_{22}, G'_{33} \), and \( C'_{1} \) are the fault values.
values, and they represent the system unknowns. System (24) admits an infinite number of solutions

\[
\begin{align*}
G_2 &= \frac{G_2G_4 - G_3G_5 + G_3G_5'}{a_4} \\
C_1' &= \frac{C_1G_4'}{G_3}. 
\end{align*}
\tag{25}
\]

Thus, the group \(\{G_2, G_3, C_1\}\) is a CAG, in the sense that whatever fault occurs in one of the three components, it will be indistinguishable from a fault of both the other two components. This means that, if one assumes the hypothesis of single faults, the ambiguity no longer holds.

By performing this analysis for all the potential CAGs, it results that the group \(\{G_2, G_3, C_1\}\) is the only CAG of order 3. Hence, the Sallen Key filter has the following CAGs: \(\{G_4, G_5\}\), \(\{G_2, G_3, C_1\}\). The same result has been also reported in [6].

2) Jacobian rank approach: Let us consider the testability matrix \(B\) found in (26), shown at the bottom of the page.

Because the rank of the testability matrix is three, groups of components of order that is greater than three certainly constitute a CAG. Singular-value decomposition (SVD) has been performed for submatrices \(B_R\) corresponding to the 23 potential CAGs of order up to three, obtaining not full rank in two cases: \(\{G_4, G_5\}\), \(\{G_2, G_3, C_1\}\).

Because the rank is independent on the component values, the \(B_R\) matrices are built in numerical form by assigning arbitrary values to circuit parameters.

B. Chua’s Circuit

Chua’s circuit, shown in Fig. 3, is a simple circuit that exhibits a chaotic behavior. First introduced in 1983 by L. O. Chua [23], its ease of construction has made it an ubiquitous real-world example of chaotic system, leading some to declare it “a paradigm for chaos.” Chua’s circuit consists of two linear capacitors, one linear resistor, one linear inductor, and one nonlinear resistor. The nonlinear resistor (Chua’s diode) is chosen to have a cubic voltage/current characteristic of the form [24]

\[
i = \gamma_1 \cdot v + \gamma_3 \cdot v^3. \tag{27}\]

Let us consider voltage \(v\) as the output. IOR is

\[
\begin{align*}
d^3v \over dt^3 + a_1 \frac{dv}{dt}^2 + a_2 \frac{dv}{dt} + a_3 \left( \frac{dv}{dt} \right)^2 \\
+ a_4 \frac{d^2v^2}{dt^2} + a_5 \frac{d^2v^3}{dt} + a_7v &= 0 
\end{align*}
\tag{28}
\]

where

\[
\begin{align*}
a_1 &= \frac{C_2(1 + R\gamma_1)}{RC_2} \\
a_2 &= \frac{L\gamma_1}{RC_1C_2} \\
a_3 &= \frac{6\gamma_3}{C_1} \\
a_4 &= \frac{3\gamma_3'}{C_1} \\
a_5 &= \frac{3\gamma_3'}{RC_1C_2} \\
a_6 &= \frac{\gamma_3}{LC_2C_1} \\
a_7 &= \frac{1 + R\gamma_1}{RC_1C_2L}. 
\end{align*}
\tag{29}
\]

Then, incidence matrix \(W\) is

\[
W = \begin{bmatrix}
R & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
C_1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
C_2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
\gamma_1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\gamma_3 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}. \tag{30}
\]

The number of coefficients is greater than the number of parameters. Then, any possible set of components has to be considered as a potential CAG. By following the same procedure used in the linear example and by inspecting the incidence matrix (30), it is possible to a priori exclude the presence of any CAG of order two.

\[
\begin{align*}
\left\{ \frac{G_2}{C_1C_2}(G_1 + G_2) &= \frac{G_2'}{C_1C_2'}(G_1 + G_2') \\
\frac{G_2G_4 - G_3G_5}{G_5C_2} + \frac{G_3G_5'}{C_1C_2}(C_1 + C_2) &= \frac{G_2G_4 - G_3G_5}{G_5C_2} + \frac{G_3G_5'}{C_1C_2}(C_1 + C_2) \right. 
\end{align*}
\tag{24}
\]

\[
B = \begin{bmatrix}
\frac{G_2}{C_1C_2} & \frac{G_2}{C_1C_2} & \frac{G_2G_4}{G_5C_2} & 0 & 0 & \frac{-G_2}{C_1C_2'} & \frac{-G_2}{C_1C_2'} \\
1 & \frac{-G_2}{G_5C_2} & \frac{-G_2}{G_5C_2} & \frac{-G_2}{G_5C_2} & \frac{-G_2}{G_5C_2} & \frac{-G_2}{G_5C_2} & \frac{-G_2}{G_5C_2} \\
\frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} \\
\frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} \\
\frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} \\
\frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} \\
\frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} \\
\frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2} & \frac{G_2G_4}{G_5C_2}
\end{bmatrix}. \tag{26}
\]
Let us now consider the potential CAGs of order three. The analysis of incidence matrix \( W \) yields the results reported in Table III.

Only four potential CAGs of order three have to be examined: 
\[ \{ R, C_1, \gamma_1, \gamma_2 \}, \{ R, C_2, L \}, \{ C_1, C_2, \gamma_3 \}, \text{ and } \{ R, \gamma_1, \gamma_2, \gamma_3 \}. \]

By inspecting incidence matrix \( W \), the following potential ambiguity groups of order that is greater than three are obtained:

1) seven groups of order four: \( \{ R, C_1, C_2, \gamma_3 \}, \{ R, C_1, L, \gamma_3 \}, \{ R, C_2, L, \gamma_2 \}, \{ C_1, C_2, \gamma_1, \gamma_3 \}, \{ C_1, C_2, L, \gamma_1 \}, \{ C_1, L, \gamma_1, \gamma_3 \}, \text{ and } \{ C_1, C_2, L, \gamma_1, \gamma_3 \}; \]

2) four groups of order five: \( \{ R, C_1, C_2, \gamma_3 \}, \{ R, C_1, L, \gamma_1, \gamma_3 \}, \text{ and } \{ C_1, C_2, L, \gamma_1, \gamma_3 \}; \)

3) one group of order six: \( \{ R, C_1, C_2, L, \gamma_1, \gamma_2, \gamma_3 \}. \)

1) System identifiability approach: On the basis of Gröbner analysis, none of the potential CAGs of order three results to be an ambiguity group, as the corresponding system (15) has only one solution.

For each of the 11 potential CAGs of orders four and five, Gröbner analysis has been applied to the corresponding system (15), obtaining only one solution, as well as in the previous cases. Hence, none of these groups constitutes a CAG.

On the contrary, by considering the potential CAG of order six, the corresponding system (15) consists of the following \( J = 7 \) equations:

\[
\begin{align*}
C_0 (1+R\gamma_1 + C_1) + C_2 L &= C_0 L (1+R \gamma_1') + C_2' L
\end{align*}
\]

where \( R', C_1', C_2', L', \gamma_1', \text{ and } \gamma_3' \) are the fault values, and they represent the system unknowns. System (31) admits an infinite number of solutions:

\[
\begin{align*}
C_1' &= -\frac{C_1 R}{R'} \\
C_2' &= -\frac{C_2 R}{R'} \\
\gamma_1' &= -\frac{\gamma_1 R}{R'} \\
\gamma_3' &= -\frac{\gamma_3 R}{R'} \\
L' &= \frac{L R}{R'}
\end{align*}
\]

Thus, \( \{ R, C_1, C_2, L, \gamma_1, \gamma_2, \gamma_3 \} \) is the unique CAG of Chua’s circuit. This means that a fault occurring in a subset of the group is indistinguishable from a corresponding fault in the rest of the group; therefore, a single fault is indistinguishable from a quintuple fault, a double fault is indistinguishable from a quadruple fault, and a triple fault is indistinguishable from another triple fault. Assuming that at the most two components can get faulty, the circuit becomes fully diagnosable.

2) Jacobian rank approach: Let us consider the testability matrix \( B_C \) found in the equation shown at the bottom of the page.

Because the rank of the testability matrix is five, groups of components of order that is greater than five certainly constitute a CAG. Thus, there is one CAG of order six composed by all the circuit components.

SVD has been performed for submatrices \( B_R \) corresponding to the 15 potential CAGs of order up to five, obtaining full rank in all the cases.

C. Common-Emitter Amplifier

In order to show the applicability of the system identifiability approach to circuits with components characterized by transcendental relationships, procedure 1) is applied to a
common-emitter amplifier (Fig. 4). In this case, an exponential nonlinearity appears in the relationship between current $i_B$ and base-emitter voltage $v_{BE}$.

The state equations are

$$
\begin{align*}
\dot{v}_{C1} &= \frac{1}{C_1} \left( -v_{C1} \frac{R_1 + R_2}{R_1 R_2} + i_B + u \frac{R_1 + R_3}{R_1 R_3} - \frac{1}{R_1} U_{CC} \right) \\
\dot{v}_{C2} &= \frac{1}{C_2} \left( -v_{C2} - R_3 i_C + U_{CE} \right) \\
\dot{v}_{C3} &= \frac{1}{C_3} \left( -\frac{v_{CE}}{R_3} + i_B + i_C \right)
\end{align*}
$$

where

$$
\begin{align*}
i_C &= \beta i_B \\
i_B &= I_{BO}(e^{v_{BE}/V_T} - 1)
\end{align*}
$$

(33)

The values of thermal voltage $V_T$, of the resistance of the load $R_L$, and of polarization voltage $U_{CC}$ have been set as follows:

$V_T = 26 \text{ mV}$ $R_L = 10 \text{ kΩ}$ $U_{CE} = 12 \text{ V}$

Let us introduce an auxiliary variable

$$w = e^{v_{BE}/V_T}$$

(34)

and its derivative

$$\dot{w} = \frac{v_{BE}}{V_T} e^{v_{BE}/V_T} = \frac{v_{BE}}{V_T} w.$$  

(35)

The substitution of (34) and the introduction of (35) in system (33) allow one to eliminate the exponential function from the state equations. The fourth-order differential system with algebraic nonlinearities is obtained in (36), shown at the bottom of the page.

\[
\begin{align*}
\dot{v}_{C1} &= \frac{1}{C_1} \left( -v_{C1} \frac{R_1 + R_2}{R_1 R_2} + I_{BO}(w - 1) + u \frac{R_1 + R_3}{R_1 R_3} - \frac{U_{CE}}{R_1} \right) \\
\dot{v}_{C2} &= \frac{1}{C_2} \left( -v_{C2} - R_3 \beta f I_{BO}(w - 1) + U_{CE} \right) \\
\dot{v}_{C3} &= \frac{1}{C_3} \left( -\frac{v_{CE}}{R_3} + I_{BO}(w - 1)(1 + \beta f) \right) \\
\dot{w} &= \frac{w}{V_T} \left( v_{C1} \frac{R_1 + R_2}{R_1 R_2} + v_{C2} \frac{R_2}{R_2} + I_{BO}(w - 1) \left( \frac{1}{C_1} + \frac{1 + \beta f}{C_3} \right) + \dot{u} - u \frac{R_1 + R_3}{R_1 R_3 C_1} + U_{CE} \frac{R_3}{R_3 C_1} \right)
\end{align*}
\]

(36)

Let us consider the input $u = v_{R2}$ and the output $y = v_{C2}$ shown in Fig. 4. The corresponding IOR consists of 62 coefficients. Thus, for the sake of brevity, the incidence matrix and system (15) are not reported.

Incidence matrix $W$ is a matrix of dimension $9 \times 62$. Because the number of coefficients is greater than the number of parameters, any possible set of components has to be considered as a potential CAG, so that 502 groups have to be examined. By following the procedure previously described and by inspecting the incidence matrix, it is possible to a priori exclude the presence of some CAGs, obtaining 456 potential CAGs.

On the basis of Gröbner analysis of the potential CAGs of order two, $[R_1, R_2]$ results to be an ambiguity group, as the corresponding system (15) has an infinite number of solutions. This result allows us to discard all the potential CAGs containing group $[R_1, R_2]$, so that the number of potential CAGs reduces to 329. Finally, by discarding the potential CAGs containing the already-determined CAGs, the number of potential CAGs to be analyzed is 325.

The results of this analysis lead to the following CAGs:

\[
\begin{align*}
&\{R_1, R_2\} \\
&\{R_3, R_4, C_2, C_3, \beta f\} \\
&\{R_2, R_4, C_1, C_3, \beta f, I_{BO}\} \\
&\{R_1, R_4, C_1, C_3, \beta f, I_{BO}\} \\
&\{R_2, R_3, R_4, C_1, C_2, C_3, I_{BO}\} \\
&\{R_1, R_3, C_1, C_2, \beta f, I_{BO}\} \\
&\{R_2, R_3, C_1, C_2, \beta f, I_{BO}\}
\end{align*}
\]

(37)

V. COMMENTS AND CONCLUSION

In this paper, a symbolic procedure for canonical ambiguity-group determination has been proposed, resorting to the theory of a priori identifiability.

The method started with IOR determination; then, an analysis of the occurrence of component parameters in the IOR coefficients was performed in order to select the potential CAGs. In the last step, the corresponding parameter identifiability was investigated.

IOR determination was performed, resorting to an implementation of Ritt’s algorithm. The algorithm was directly applicable only to algebraic systems. Nevertheless, some transcendental systems (e.g., systems with trigonometric or exponential nonlinear functions) can be analyzed, resorting to suitable transformations.
that cancel transcendency. The identifiability problem has been solved, resorting to the Buchberger algorithm for the calculation of Gröbner bases.

An analysis of computational complexity has not been performed to date. It is worth noting that the complexity strongly depends on the chosen ordering of the variables for both Ritt’s and Buchberger’s algorithms. Moreover, a bound for the complexity is not available. The main advantage of the method is that it is independent of the fault value, because it does not introduce any linear approximation, as in the majority of symbolic approaches proposed in the literature. Moreover, the analysis of the IOR coefficients instead of the transfer function coefficients allows one to extend to nonlinear systems the applicability of several methods, presented in the literature only for linear systems.

An extension of the numerical method described in [22] is also presented in order to apply it to nonlinear circuits.

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