DMT-optimal, Low ML-Complexity STBC-Schemes for Asymmetric MIMO Systems

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Abstract—For an \( n_t \) transmit, \( n_r \) receive antenna \((nt \times n_r)\) MIMO system with quasi-static Rayleigh fading, it was shown by Elia et al. that space-time block code-schemes (STBC-schemes) which have the non-vanishing determinant (NVD) property and are based on minimal-delay STBCs (STBC block length equals \( n_t \)) with a symbol rate of \( n_t \) complex symbols per channel use (rate-\( n_t \), STBC) are diversity-multiplexing gain tradeoff (DMT)-optimal for arbitrary values of \( n_t \). Further, explicit linear STBC-schemes (LSTBC-schemes) with the NVD property were also constructed. However, for asymmetric MIMO systems (where \( n_r < n_t \)), with the exception of the Alamouti code-scheme for the \( 2 \times 1 \) system and rate-1, diagonal STBC-schemes with NVD for an \( n_t \times 1 \) system, no known minimal-delay, rate-\( n_r \) LSTBC-scheme has been shown to be DMT-optimal. In this paper, we first obtain an enhanced sufficient criterion for an STBC-scheme to be DMT optimal and using this result, we show that for certain asymmetric MIMO systems, many well-known LSTBC-schemes which have low ML-decoding complexity are DMT-optimal, a fact that was unknown hitherto.

Index Terms—Asymmetric MIMO systems, diversity-multiplexing tradeoff, linear space-time block codes, low ML-decoding complexity, non-vanishing determinant, outage-probability, STBC-schemes.

I. INTRODUCTION AND BACKGROUND

Space-time coding (STC) \([1]\) for multiple-input, multiple-output (MIMO) antenna systems has extensively been studied as a tool to exploit the diversity provided by the MIMO fading channel. MIMO systems have the capability of permitting reliable data transmission at higher rates compared to that provided by the single-input, single-output (SISO) antenna system. In particular, when the delay requirement of the system is less than the coherence time (the time frame during which the channel gains are constant and independent of the channel gains of other time frames) of the channel, Zheng and Tse showed in their seminal paper \([2]\) that for the Rayleigh fading channel with STC, there exists a fundamental tradeoff between the diversity gain and the multiplexing gain (see Definition \([3]\) and Definition \([4]\) Section \([II]\), referred to as the diversity-multiplexing gain tradeoff (DMT). The optimal DMT was also characterized with the assumption that the block length of the space-time block codes (STBC) of the scheme (see Definition \([2]\) Section \([II]\) for a definition of “STBC-scheme”) is at least \( n_t+n_r-1 \), where \( n_t \) and \( n_r \) are the number of transmit and receive antennas, respectively. The first explicit DMT-optimal STBC-scheme was presented in \([3]\) for 2 transmit antennas and subsequently, in another landmark paper \([4]\), explicit DMT-optimal STBC-schemes consisting of both square (minimal-delay) and rectangular STBCs from cyclic division algebras were presented for arbitrary values of \( n_t \) and \( n_r \). In the same paper, a sufficient criterion for achieving DMT-optimality was proposed for general STBC-schemes. For a class of STBC-schemes based on linear STBCs \([5]\) (LSTBCs) which employ QAM constellations and transmit \( n_t \) complex information symbols per channel use, this criterion translates to the non-vanishing determinant property (see Definition \([8]\) Section \([IV]\), a term first coined in \([6]\), being sufficient for DMT-optimality. It was later shown in \([7]\) that the DMT-optimal LSTBC-schemes constructed in \([4]\) are also approximately universal for arbitrary number of receive antennas. In literature, there exist several other LSTBC-schemes with NVD - for example, see \([8]\), \([9]\), \([10]\) and references therein.

A. Motivation for our results

The Alamouti code-scheme \([11]\) has the NVD property and is known to be DMT-optimal for the \( 2 \times 1 \) MIMO system. For any \( n_t \times 1 \) system, diagonal STBC-schemes with NVD that consist of STBCs transmitting 1 complex symbol per channel use are known to be DMT-optimal \([7]\). STBC-schemes based on fast-decodable LSTBCs \([12]\) from division algebra that transmit \( n_r \) complex symbols per channel use for asymmetric MIMO systems (for which \( n_r < n_t \)) have been shown to have the NVD property, but have not been reported to be DMT-optimal. There exist several other LSTBCs which transmit less than \( n_t \) independent complex symbols per channel use, and STBC-schemes consisting of these LSTBCs have neither been reported to have the NVD property, nor have been claimed to be DMT-optimal in literature. Examples of such LSTBCs are the full-diversity quasi-orthogonal STBC (QOSTBC) \([13]\), STBCs from co-ordinate interleaved orthogonal designs \([14]\) and four-group decodable STBCs \([15]\)-\([18]\), which all transmit one independent complex symbol per channel use and are characterized by low ML-decoding complexity. For these LSTBC-schemes, the sufficient criterion provided in \([4]\) for DMT-optimality, which requires that LSTBCs transmit \( n_t \) independent complex symbols per channel use irrespective of the number of receive antennas, is not applicable. Hence, there is a clear need for obtaining a new DMT-criterion that can take into account LSTBC-schemes with NVD that are

\[1\]In literature, linear STBCs are popularly called linear dispersion codes \([5]\).
based on LSTBCs having a symbol rate of less than $n_t$ independent complex symbols per channel use.

Further, for asymmetric MIMO systems, the standard sphere decoder [19] or its variations (see, for example, [20], [21] and references therein) cannot be used in entirety to decode rate-$n_r$ LSTBCs (henceforth in this paper, a rate-$p$ LSTBC means an LSTBC that transmits $p$ independent complex symbols per channel use. This LSTBC is said to have a symbol rate of $p$ complex symbols per channel use. See Definition 3, Section IV). For an $n_t \times n_r$ MIMO system, the standard sphere decoder can be used to decode LSTBCs that transmit at most $n_{min} = \min(n_t, n_r)$ complex symbols per channel use. Recent results on fixed complexity sphere decoders [23], [24] are extremely promising from the point of view of low complexity decoding. In particular, it has been shown analytically in [24] that the fixed complexity sphere decoder, although provides quasi-Maximum Likelihood (ML)-performance, helps achieve the same diversity order of ML-decoding with a worst-case complexity of the order of $M^{\sqrt{K}}$, where $M$ is the number of possibilities for each symbol (or the size of the signal constellation employed when each symbol is encoded independently) and $K$ is the dimension of the search, while an exhaustive ML-search would incur a complexity of the order of $M^K$. In the same paper, it has also been shown that the gap between quasi-ML performance and the actual ML-performance approaches zero at high signal-to-noise ratio, independent of the constellation employed. This motivates one to seek DMT-optimal LSTBC-schemes whose LSTBCs are entirely sphere decodable, i.e., transmit at most $n_{min}$ independent complex symbols per channel use.

In literature, only certain rate-$n_r$ LSTBC-schemes (STBC-schemes that are based on rate-$n_t$ LSTBCs) have been known to be DMT-optimal for asymmetric MIMO systems - the Alamouti code-scheme [11] for the $2 \times 1$ system [2], rate-1, diagonal STBC-schemes with NVD for any $n_t \times 1$ system [2], and rate-$n_r$, rectangular LSTBC-schemes for $n_r = 2$ and $n_t = n_r - 1$ [25]. In this paper, we prove the DMT-optimality of many LSTBC-schemes that are based on well-known rate-$n_r$ LSTBCs [13], [13], [12] for asymmetric MIMO systems.

B. Contributions and paper organization

The contributions of this paper are the following.

1. We present an enhanced criterion for DMT-optimality of general STBC-schemes. This criterion enables us to encompass all rate-$n_{min}$ LSTBC-schemes with NVD, which was not possible using the DMT-criterion in [4].

2. In the context of LSTBCs, we show that transmission of $n_{min}$ complex symbols per channel use is necessary for LSTBC-schemes to be DMT-optimal, and for asymmetric MIMO systems, we show that STBC-schemes that are based on rate-$n_r$ LSTBCs whose real symbols take values from PAM constellations are DMT-optimal if they have the NVD property.

3. Finally, we show that some well known low ML-decoding complexity LSTBC-schemes (STBC-schemes based on LSTBCs with low ML-decoding complexity) are DMT-optimal for certain asymmetric MIMO systems (see Table II).

The rest of the paper is organized as follows. Section II deals with the system model and relevant definitions. Section III presents the main result of the paper, which is an enhanced sufficient criterion for DMT-optimality, while Section IV gives a brief introduction to linear STBCs along with a few relevant definitions, and provides a new criterion for DMT-optimality of LSTBCs for asymmetric MIMO systems. A discussion on the DMT-optimality of several low ML-decoding complexity LSTBC-schemes is presented in Section V and concluding remarks constitute Section VI.

Notations: Throughout the paper, bold, lowercase letters are used to denote vectors and bold, uppercase letters are used to denote matrices. For a complex matrix $X$, the Hermitian, the transpose, the trace, the determinant and the Frobenius norm of $X$ are denoted by $X^H$, $X^T$, $tr(X)$, $det(X)$ and $||X||$, respectively. The set of all real numbers, complex numbers and integers are denoted by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$, respectively. The real and imaginary parts of a complex-valued vector $x$ are denoted by $x_1$ and $x_Q$, respectively. $|S|$ denotes the cardinality of the set $S$, $S \times T$ denotes the Cartesian product of sets $S$ and $T$, meaning which $S \times T = \{(s,t)|s \in S, t \in T\}$, and $S \subset T$ denotes that $S$ is a proper subset of $T$. The $T \times T$ identity matrix is denoted by $I_T$ and $O$ denotes the null matrix of appropriate dimension. For a complex number $x$, $x^*$ denotes its complex conjugate and the $(\cdot)$ operator acting on $x$ is defined as

$$\tilde{x} \triangleq \begin{bmatrix} x_1 & -x_Q \\ x_Q & x_1 \end{bmatrix}.$$
<table>
<thead>
<tr>
<th>LSTBC</th>
<th>No. of Tx</th>
<th>Block length $T$</th>
<th>Symbol rate (in complex symbols per channel use)</th>
<th>No. of Rx antennas $n_r$ for which STBC-scheme is DMT-optimal</th>
<th>Constellation used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alamouti Code [11]</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>QAM</td>
</tr>
<tr>
<td>Yao-Wornell Code [13]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dayal-Varanasi Code [26]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Golden code [6]</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>any $n_r$</td>
<td>QAM</td>
</tr>
<tr>
<td>Silver code [27, 28]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Serrad-Sari code [29]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Srinath-Rajan code [30]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>perfect codes [19]</td>
<td>2, 3, 4, 6</td>
<td>$n_t$</td>
<td>$n_t$</td>
<td>any $n_r$</td>
<td>QAM/HEX</td>
</tr>
<tr>
<td>Kiran-Rajan codes [3]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Codes from CDA [4]</td>
<td>any $n_t$</td>
<td></td>
<td>$n_t$</td>
<td>any $n_r$</td>
<td>QAM</td>
</tr>
<tr>
<td>Codes from CDA [4]</td>
<td></td>
<td></td>
<td>$n_t$</td>
<td>any $T &gt; n_t$</td>
<td>QAM/HEX</td>
</tr>
<tr>
<td>perfect STBCs [10]</td>
<td>any $n_t$</td>
<td></td>
<td>$n_t$</td>
<td>any $n_r$</td>
<td>QAM</td>
</tr>
<tr>
<td>Diagonal STBCs with NVD [7]</td>
<td>any $n_t$</td>
<td>$n_t$</td>
<td>1</td>
<td>1</td>
<td>QAM</td>
</tr>
<tr>
<td>Lu-Hollanti [25]</td>
<td>any $n_t &gt; 2$</td>
<td>$T &gt; n_t$</td>
<td>2</td>
<td>2</td>
<td>QAM</td>
</tr>
<tr>
<td>Lu-Hollanti [25]</td>
<td>any $n_t &gt; 2$</td>
<td>$T &gt; n_t$</td>
<td>$n_r = 1$</td>
<td>$n_r = 1$</td>
<td>QAM</td>
</tr>
<tr>
<td>STBCs from CIOD (Subsection 5-B)</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>Rotated QAM</td>
</tr>
<tr>
<td>QOSTBC (Subsection 5-A)</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>Rotated QAM</td>
</tr>
<tr>
<td>4-group decodable STBCs [15, 18]</td>
<td>$n_t = 2^n$, $n \in \mathbb{Z}^+$</td>
<td>$n_t$</td>
<td>1</td>
<td>1</td>
<td>QAM</td>
</tr>
<tr>
<td>Fast-decodable STBCs [30, 12]</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>$n_r \leq 2$</td>
<td>QAM</td>
</tr>
<tr>
<td>Fast-decodable asymmetric STBCs [12]</td>
<td>any $n_t$</td>
<td>$n_t$</td>
<td>$n_r &lt; n_t$</td>
<td>$n_r &lt; n_t$</td>
<td>QAM</td>
</tr>
<tr>
<td>Punctured perfect STBCs $^*$ for asymmetric MIMO</td>
<td>any $n_t$</td>
<td></td>
<td>$n_r &lt; n_t$</td>
<td>$n_r &lt; n_t$</td>
<td>QAM</td>
</tr>
</tbody>
</table>

$^*$ refers to rate-$n_r$ STBCs obtained from rate-$n_t$ perfect STBCs [10] (which transmit $n_t^2$ complex symbols in $n_t$ channel uses) by restricting the number of complex symbols transmitted to be only $n_t n_r$.

**TABLE I**

A TABLE OF DMT-OPTIMAL LINEAR STBC-SCHEMES

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II. SYSTEM MODEL

We consider an $n_t$ transmit antenna, $n_r$ receive antenna MIMO system ($n_t \times n_r$ system) with perfect channel-state information available at the receiver (CSIR) alone. The channel is assumed to be quasi-static with Rayleigh fading. The system model is

$$\mathbf{Y} = \mathbf{H} \mathbf{X} + \mathbf{N},$$

where $\mathbf{Y} \in \mathbb{C}^{n_r \times T}$ is the received signal matrix, $\mathbf{X} \in \mathbb{C}^{n_t \times T}$ is the codeword matrix that is transmitted over a block of $T$ channel uses, $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix with its entries independently and identically distributed (i.i.d.) circularly symmetric complex Gaussian random variables with zero mean and unit variance, and $\mathbf{N} \in \mathbb{C}^{n_r \times T}$ is the noise matrix with its entries being i.i.d. Gaussian random variables with zero mean and unit variance. The average signal-to-noise ratio at each receive antenna is denoted by $SNR$.

**Definition 1:** (Space-time block code) A space-time block code (STBC) of block-length $T$ for an $n_t$ transmit antenna MIMO system is a finite set of complex matrices of size $n_t \times T$.

An example is the Alamouti code [11] with its complex symbols taking values from 4-QAM, given by

$$\mathcal{X}_{Alamouti} = \left\{ \left[ \begin{array}{c} x_1 \\ -x_2^* \\ x_2 \\ x_1^* \end{array} \right] \mid x_1, x_2 \in 4\text{-QAM} \right\}.$$

**Definition 2:** (STBC-scheme) An STBC-scheme $\mathcal{X}$ is defined as a family of STBCs indexed by $SNR$, each STBC of block length $T$ so that $\mathcal{X} = \{ \mathcal{X}(SNR) \}$, where the STBC $\mathcal{X}(SNR)$ corresponds to a signal-to-noise ratio of $SNR$ at each receive antenna.

So, at a signal-to-noise ratio of $SNR$, the codeword matrices of $\mathcal{X}(SNR)$ are transmitted over the channel. Assuming that all the codeword matrices of $\mathcal{X}(SNR) \triangleq \{ \mathbf{X}_i(SNR), i = 1, \ldots, |\mathcal{X}(SNR)| \}$ are equally likely to be transmitted, we have

$$\frac{1}{|\mathcal{X}(SNR)|} \sum_{i=1}^{|\mathcal{X}(SNR)|} ||\mathbf{X}_i(SNR)||^2 = T \cdot SNR. \quad (2)$$
It follows that for the STBC-scheme $X$,
$$\|X_i(SNR)\|^2 \leq SNR, \quad \forall i = 1, 2, \cdots, |X(SNR)|. \quad (3)$$
The bit rate of transmission is $\frac{\log |X(SNR)|}{SNR}$ bits per channel use. Henceforth in this paper, a codeword $X_i(SNR) \in X(SNR)$ is simply referred to as $X_i \in X(SNR)$.

Definition 3: (Multiplexing gain) Let the bit rate of transmission of the STBC $X(SNR)$ in bits per channel use be denoted by $R(SNR)$ (so that $R(SNR) = (1/T) \log_2 |X(SNR)|$). Then, the multiplexing gain $r$ of the STBC-scheme is defined \cite{2} as
$$r = \lim_{SNR \to \infty} \frac{R(SNR)}{\log_2 SNR}. \quad \text{(4)}$$
Equivalently, $R(SNR) = r \log_2 SNR + o(\log_2 SNR)$, where, for reliable communication, $r \in (0, n_{\min})$ \cite{2}.

Definition 4: (Diversity gain) Let the probability of codeword error of the STBC $X(SNR)$ be denoted by $P_e(SNR)$. Then, the diversity gain $d(r)$ of the STBC-scheme corresponding to a multiplexing gain of $r$ is given by
$$d(r) = \lim_{SNR \to \infty} \frac{\log_2 P_e(SNR)}{\log_2 SNR}. \quad \text{(5)}$$
For an $n_t \times n_r$ MIMO system, the maximum achievable diversity gain is $n_t n_r$.

Definition 5: (Optimal DMT curve) \cite{2} The optimal diversity-multiplexing gain curve $d^*(r)$ that is achievable with STBC-schemes for an $n_t \times n_r$ MIMO system is a piecewise-linear function connecting the points $(k, d(k))$, $k = 0, 1, \cdots, n_{\min}$, where
$$d(k) = (n_t - k)(n_r - k). \quad \text{(6)}$$

Theorem 1: \cite{2} For a quasi-static $n_t \times n_r$ MIMO channel with Rayleigh fading and perfect CSIR, an STBC-scheme $X$ that satisfies (3) is DMT-optimal for any value of $n_r$ if for all possible pairs of distinct codewords $(X_1, X_2)$ of $X(SNR)$, the difference matrix $X_1 - X_2 = \Delta X \neq 0$ satisfies
$$\det(\Delta X^H) \geq SNR^{n_t(1- \frac{1}{n_r})}. \quad \text{(7)}$$

Proof: To prove the theorem, we first show that the STBC-scheme $X$ is DMT-optimal when each codeword difference matrix $\Delta X \neq 0$ satisfies
$$\det(\Delta X^H) \geq SNR^{n_t(1- \frac{1}{n_r})}, \quad \text{(7)}$$
and then conclude the proof taking aid of Theorem 1. Towards this end, we assume without loss of generality that the codeword $X_1$ of $X(SNR)$ is transmitted. It is also assumed that $T > n_t$, which is a prerequisite for achieving a diversity gain of $n_t n_r$ when the bit rate of the STBC-scheme is constant with $SNR$ (a special case of the $r = 0$ condition). Let $\Delta X_1 = X_1 - X_2$, where $X_2$, $l = 2, \cdots, |X(SNR)|$, are the remaining codewords of $X(SNR)$. It is to be noted that the bit rate of transmission is $r \log_2 SNR + o(\log_2 SNR)$ bits per channel use and so, $|X(SNR)| = SNR^{2T}$, with $r \in (0, n_{\min})$. Considering the channel model given by (1), with ML-decoding, the probability that $X_1$ is wrongly decoded to $X_2$ for a particular channel matrix $H$ is given by
$$P_e(X_1 \rightarrow X_2|H) = Q\left(\frac{\|H\Delta X_1\|}{\sqrt{2}}\right). \quad \text{(8)}$$
So, the probability that $X_1$ is wrongly decoded conditioned on $H$ is given by
$$P_e(X_1|H) = \sum_{l=2}^{|X(SNR)|} Q\left(\frac{\|H\Delta X_l\|}{\sqrt{2}}\right). \quad \text{(9)}$$

The probability of codeword error averaged over all channel realizations is given by
$$P_e = E_H \{P_e(X_1|H)\} = \int p(H) P_e(X_1|H)dH,$$
where $p(.)$ denotes “probability density function (pdf) of”. Let $\mathcal{E} := \text{event that there is a codeword error}$ and consider the set of channel realizations $\mathcal{O}$ defined in (5) at the top of the next page. Now,
$$P_e = \int_{\mathcal{O}} p(H) P_e(X_1|H)dH + \int_{\mathcal{O}^c} p(H) P_e(X_1|H)dH = \mathcal{P}(\mathcal{O}, \mathcal{E}) + \mathcal{P}(\mathcal{O}^c, \mathcal{E}), \quad \text{(9)}$$
where $\mathcal{P}(.)$ denotes “probability of” and $\mathcal{O}^c = \{H|H \notin \mathcal{O}\}$. $\mathcal{P}(\mathcal{O})$ is the well-known probability of outage $\cite{2}$ and $\mathcal{P}(\mathcal{E}|\mathcal{O})$ is the probability of codeword error given that the channel is in outage. Both $\mathcal{P}(\mathcal{O})$ and $\mathcal{P}(\mathcal{E}|\mathcal{O})$ have been derived in $\cite{2}$.

Theorem 2: For a quasi-static $n_t \times n_r$ MIMO channel with Rayleigh fading and perfect CSIR, an STBC-scheme $X$ that satisfies (3) is DMT-optimal for any value of $n_r$ if for all possible pairs of distinct codewords $(X_1, X_2)$ of $X(SNR)$, the difference matrix $X_1 - X_2 = \Delta X \neq 0$ is such that,
$$\det(\Delta X^H) \geq SNR^{n_t(1- \frac{1}{n_r})}. \quad \text{(7)}$$
\[
\mathcal{O} \triangleq \left\{ \mathbf{H} \mid \log_2 \det \left( \mathbf{I}_{n_t} + \frac{SNR}{n_t} \mathbf{H} \mathbf{H}^H \right) \leq r \log_2 SNR + o(\log_2 SNR) \right\}
\]

(5)

\[
\tilde{\mathcal{O}} \triangleq \left\{ \mathbf{H} \mid \sum_{i=1}^{n_r} \log_2 \left( 1 + \frac{SNR}{n_t} \|\mathbf{h}_i\|^2 \right) > r \log_2 SNR + o(\log_2 SNR) \right\}
\]

(6)

to be

\[
P(\mathcal{O}) = SNR^{-d^*(r)},
\]

(10)

\[
P(\mathcal{E}|\mathcal{O}) = SNR^0,
\]

(11)

where \(d^*(r)\) is given in Definition 5. So, the DMT curve of an STBC is determined completely by \(P(\mathcal{O}', \mathcal{E})\), which is the probability that there is a codeword error and the channel is not in outage. To obtain an upper bound on \(P(\mathcal{O}', \mathcal{E})\), we proceed as follows. Note that \(\mathbf{I}_{n_t} + (SNR/n_t) \mathbf{H} \mathbf{H}^H\) is a positive definite matrix. Denoting the rows of \(\mathbf{H}\) by \(\mathbf{h}_i\), \(i = 1, \ldots, n_r\), we have,

\[
\log_2 \det \left( \mathbf{I}_{n_r} + \frac{SNR}{n_t} \mathbf{H} \mathbf{H}^H \right) \leq \sum_{i=1}^{n_r} \log_2 \left( 1 + \frac{SNR}{n_t} \|\mathbf{h}_i\|^2 \right),
\]

which is due to the Hadamard's inequality which states that the determinant of a positive definite matrix is less than or equal to the product of its diagonal entries. Define the set of channel realizations \(\tilde{\mathcal{O}}\) as shown in (6) at the top of the page. Now, clearly, \(\mathcal{O}' \subseteq \tilde{\mathcal{O}}\) and hence

\[
P(\mathcal{O}', \mathcal{E}) \leq P(\tilde{\mathcal{O}}, \mathcal{E}).
\]

(12)

Hence, using (12) in (9), we have

\[
P_e \leq P(\mathcal{O})P(\mathcal{E}|\mathcal{O}) + P(\tilde{\mathcal{O}}, \mathcal{E}).
\]

(13)

We now need to evaluate \(P(\tilde{\mathcal{O}}, \mathcal{E})\). Denoting the entries of \(\mathbf{H}\) by \(h_{ij}\), \(i = 1, \ldots, n_r\), \(j = 1, \ldots, n_t\), we observe that

\[
\sum_{i=1}^{n_r} \log_2 \left( 1 + \frac{SNR}{n_t} |h_{ij}|^2 \right) = \sum_{i=1}^{n_r} \log_2 \left( \frac{1}{n_t} \sum_{j=1}^{n_t} \left( 1 + SNR |h_{ij}|^2 \right) \right)
\]

\[
\geq \frac{1}{n_t} \sum_{i=1}^{n_r} \sum_{j=1}^{n_t} \log_2 \left( 1 + SNR |h_{ij}|^2 \right),
\]

(16)

with (16) following from the concavity of \(\log(.)\) and Jensen’s inequality.

We now define two disjoint sets of channel realizations \(\tilde{\mathcal{O}}\) and \(\tilde{\mathcal{O}}\) as shown in (14) and (15) at the top of the next page. Clearly, \(\tilde{\mathcal{O}}\) is the disjoint union of \(\mathcal{O}\) and \(\tilde{\mathcal{O}}\). Therefore,

\[
P(\tilde{\mathcal{O}}, \mathcal{E}) = P(\tilde{\mathcal{O}}, \mathcal{E}) + P(\tilde{\mathcal{O}}, \mathcal{E})
\]

\[
= P(\mathcal{O})P(\mathcal{E}|\mathcal{O}) + P(\tilde{\mathcal{O}}, \mathcal{E})
\]

\[
\leq P(\mathcal{O}) + P(\tilde{\mathcal{O}}, \mathcal{E}).
\]

(17)

In Appendix A, it is shown that

\[
P(\tilde{\mathcal{O}}) = SNR^{-n_t(n_r - r)}.
\]

(18)

So, we are now left with the evaluation of \(P(\tilde{\mathcal{O}}, \mathcal{E})\), which is done as follows.

\[
P(\tilde{\mathcal{O}}, \mathcal{E}) = \int_{\tilde{\mathcal{O}}} p(\mathbf{H}) P_e(\mathbf{X}_1|\mathbf{H}) d\mathbf{H}
\]

\[
= \sum_{l=2}^{\infty} \int_{\tilde{\mathcal{O}}} p(\mathbf{H}) Q \left( \frac{\|\mathbf{H} \Delta \mathbf{X}_l\|}{\sqrt{2}} \right) d\mathbf{H}
\]

\[
= \sum_{l=2}^{\infty} \int_{\tilde{\mathcal{O}}} p(\mathbf{H}) Q \left( \frac{\|\mathbf{H} \mathbf{U}_l \mathbf{D}_l \mathbf{V}_l^H\|}{\sqrt{2}} \right) d\mathbf{H}
\]

\[
= \sum_{l=2}^{\infty} \int_{\tilde{\mathcal{O}}} p(\mathbf{H}) Q \left( \frac{\|\mathbf{H} \mathbf{D}_l\|}{\sqrt{2}} \right) d\mathbf{H},
\]

(19)

where (19) is obtained using (8) and \(\Delta \mathbf{X}_l = \mathbf{U}_l \mathbf{D}_l \mathbf{V}_l^H\), obtained upon SVD, with \(\mathbf{U}_l \in \mathbb{C}^{n_t \times n_t}, \mathbf{D}_l \in \mathbb{R}^{n_t \times T}, \mathbf{V}_l \in \mathbb{C}^{T \times T}\). In (20), \(\mathbf{H}_l = \mathbf{H} \mathbf{U}_l\) and

\[
\mathcal{O}_l \triangleq \left\{ \mathbf{H}_l \mid \sum_{i,j} \log_2 \left( 1 + SNR |h_{ij}|^2 \right) > n_r \log_2 SNR + o(\log_2 SNR) \right\}.
\]

Denoting the entries of \(\mathbf{H}_l = \mathbf{H} \mathbf{U}_l\) by \(h_{ij}(l)\), we define the set \(\mathcal{O}_l\) as

\[
\mathcal{O}_l \triangleq \left\{ \mathbf{H}_l \mid \sum_{i,j} \log_2 \left( 1 + SNR |h_{ij}(l)|^2 \right) > n_r \log_2 SNR + o(\log_2 SNR) \right\}.
\]

In Appendix B, it is shown that \(\mathcal{O}_l = \mathcal{O}_l'\) almost surely. As a result, (20) becomes

\[
P(\tilde{\mathcal{O}}, \mathcal{E}) = \sum_{l=2}^{\infty} \int_{\mathcal{O}_l'} p(\mathbf{H}_l) Q \left( \frac{\|\mathbf{H}_l \mathbf{D}_l\|}{\sqrt{2}} \right) d\mathbf{H}_l.
\]

(21)

Now, we evaluate each of the summands of (21). We have

\[
\int_{\mathcal{O}_l'} p(\mathbf{H}_l) Q \left( \frac{\|\mathbf{H}_l \mathbf{D}_l\|}{\sqrt{2}} \right) d\mathbf{H}_l \leq \int_{\mathcal{O}_l'} p(\mathbf{H}_l) Q \left( \frac{\|\mathbf{H}_l \mathbf{D}_l\|_{\text{min}}}{\sqrt{2}} \right) d\mathbf{H}_l
\]

\[
\leq Q \left( \frac{\|\mathbf{H}_l \mathbf{D}_l\|_{\text{min}}}{\sqrt{2}} \right),
\]

(22)

where \(\|\mathbf{H}_l \mathbf{D}_l\|_{\text{min}} \triangleq \inf_{\mathcal{O}_l'} \{\|\mathbf{H}_l \mathbf{D}_l\|\}\) (inf stands for ‘infimum of’). Denoting the non-zero entries of \(\mathbf{D}_l\) by \(d_j(l)\), \(j = 1, 2, \ldots, n_t\) (it is to be noted that these are the singular values of \(\Delta \mathbf{X}_l\) and we assume that \(\Delta \mathbf{X}_l\) is full-ranked, i.e. of rank...
where $\lambda > SNR^0$. Therefore, from (22) and the Chernoff bound $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$, $x \geq 0$, we obtain as $SNR \to \infty$,
\[
\int_{c_{\widehat{D}}} p(H) Q \left( \frac{||H_{\widehat{D}}||}{\sqrt{2}} \right) dH \leq \frac{1}{2} e^{-SNR^d}, \quad \delta \to 0^+ . \tag{29}
\]

Using (29) in (21), we have for any $\delta > 0$, as $SNR \to \infty$,
\[
P(\widehat{\Theta}, \mathcal{E}) \leq \frac{|X(SNR)|}{2} e^{-SNR^d}
= (cSNR^{Te} + o(\log_2 SNR)) e^{-SNR^d} \tag{30}
= SNR^{-\infty} , \tag{31}
\]
where (30) is due to the fact that $|X(SNR)| = cSNR^{Te} + o(\log_2 SNR)$ with $c$ a positive constant and (31) follows from the definition of exponential equality $\overset{\widehat{\sim}}{\sim}$. So, using (13) and (31) in (17), we obtain
\[
P(\widehat{\Theta}, \mathcal{E}) \lesssim SNR^{\max(-n_t(n_r-r),-\infty)}
= SNR^{-n_t(n_r-r)} . \tag{32}
\]

Using (10), (11) and (32) in (15), we arrive at
\[
P_c \overset{\overset{\circ}{\circ}}{\approx} \begin{cases} SNR^{\max(-d^*(r),-n_t(n_r-r))} \\ SNR^{-d^*(r)} \end{cases}
\]

throughout the paper, $\delta \to 0^+$ implies that $\delta$ is an infinitesimal positive real number, i.e., immeasurably close to zero on the positive real line but not zero.
where $d^*(r)$ is as defined in Definition 5 and this proves the DMT-optimality of the STBC-scheme when (7) is satisfied.

Now, combining the above result and that of Theorem 1, we see that an STBC-scheme is DMT-optimal if for each codeword difference matrix $\Delta X \neq O$,

$$det(\Delta X \Delta X^H) \geq SNR^{\min\{n_s(1-\frac{r}{n_t}), n_t(1-\frac{r}{n_t})\}}$$

$$= SNR^{n_s(1-\frac{r}{n_t})}.$$

This completes the proof of the theorem. \[\blacksquare\]

Note: Theorem 1 can also be proved using the steps of the proof of Theorem 2. To do so, we need to redefine $O$ given by (5) as $O \triangleq \{H \log_2 det(n_{ni} + SNR H^H H) \leq r \log_2 SNR + o(\log_2 SNR)\}$. Redefining $O$ this way is justified because $det(I+AB) = det(I+BA)$. With $O$ thus redefined, proceeding as in the proof of Theorem 2 from (5) onwards helps us arrive at the proof of Theorem 1.

The implication of Theorem 2 is that for asymmetric MIMO systems, the requirement demanded by Theorem 1 on the minimum of the determinants of the codeword difference matrices of STBCs that the STBC-scheme consists of is relaxed. In the following section, we show the usefulness of Theorem 2 in the context of LSTBCs for asymmetric MIMO systems.

IV. DMT-optimality Criterion for LSTBC-Schemes

In its most general form, an LSTBC $\mathcal{X}_L$ is given by

$$\mathcal{X}_L = \left\{ \sum_{i=1}^{k} (s_{1i}A_{1i} + s_{1Q}A_{1Q}) \mid \begin{array}{c} s_{1i}, s_{1Q}, \cdots, s_{kI}, s_{kQ} \in A \subset \mathbb{R}^{2k \times 1}, \\ A_{1i}, A_{1Q} \in \mathbb{C}^{n_i \times T} \end{array} \right\},$$

(33)

where $A_{1i}$ and $A_{1Q}$ are complex matrices, called weight matrices [14], associated with the real information symbols $s_{1i}$ and $s_{1Q}$, respectively. It is to be noted that $\{A_{1i}, A_{1Q}, i = 1, \cdots, k\}$ forms a linearly independent set over $\mathbb{R}$. In the case of most known LSTBCs, either all the real symbols $s_{1i}, s_{1Q}$, respectively take values independently from the same signal set $A'$, in which case

$$A = A' \times A' \times \cdots \times A',$$

2k times

or, each symbol pair $(s_{1i}, s_{1Q})$ jointly takes values from a real constellation $\mathcal{A}'' \subset \mathbb{R}^{2 \times 1}$ (the same can be viewed as each complex symbol $s_i = s_{1i} + js_{1Q}$ taking values from a complex constellation that is subset of $\mathbb{C}$), independent of other symbol pairs, in which case

$$A = A'' \times A'' \times \cdots \times A''.$$ $k$ times

Definition 6: (Symbol rate) The symbol rate [5] of the LSTBC given by (33) is $k/T$ complex symbols per channel use. Such an STBC is called a rate-$k/T$ STBC.

For the LSTBC given by (33), the system model given by (1) can be rewritten as

$$\vec{\text{vec}}(Y) = \begin{pmatrix} I_T \otimes H \end{pmatrix} G + \vec{\text{vec}}(N),$$

(34)

where $G \in \mathbb{R}^{2Tn_i \times 2k}$ is called the Generator matrix of the STBC and $s \in \mathbb{R}^{2k \times 1}$, both defined as

$$G \triangleq \begin{bmatrix} \vec{\text{vec}}(A_{11}) \vec{\text{vec}}(A_{1Q}), \cdots, \vec{\text{vec}}(A_{kI}) \vec{\text{vec}}(A_{kQ}) \end{bmatrix},$$

(35)

$$s \triangleq \begin{bmatrix} s_{11}, s_{1Q}, \cdots, s_{kI}, s_{kQ} \end{bmatrix}^T,$$

(36)

with $\mathbb{E}_s \left( \text{tr} \left( G s s^T G^T \right) \right) \leq T \text{SNR}$. A necessary condition for an LSTBC given by (33) to be sphere-decodable [19] is that the constellation $\mathcal{A}$ should be a finite subset of a $2k$-dimensional real lattice with each of the real symbols taking $|\mathcal{A}| \geq \frac{k}{2k} = \frac{k}{2}$ possible values. Further, if $k/T \leq 2n_{min}$, all the symbols of the STBC can be entirely decoded using the standard sphere-decoder [19] or its variations [20], [21]. However, when $k/T > min(n_i, n_q)$, for each of the $|\mathcal{A}|^{\frac{1}{2-\frac{k}{2}}} \geq |\mathcal{A}|^{\frac{1}{2}}$ possibilities for any $2(2k-n_{min}T)$ real symbols, the remaining $2n_{min}T$ real symbols can be evaluated using the sphere decoder. Hence, the ML-complexity of the rate-$\frac{k}{T}$ STBC in such a scenario is approximately $|\mathcal{A}|^{\frac{1}{2-\frac{k}{2}}} \approx |\mathcal{A}|^{\frac{1}{2}}$ times the sphere-decoding complexity of a rate-$n_{min}$ STBC.

Definition 7: (LSTBC-scheme) A rate-$k/T$ LSTBC-scheme $\mathcal{X}$ is defined as a family of rate-$k/T$ LSTBCs (indexed by $SNR$) of block length $T$ so that $\mathcal{X} \triangleq \{X_L(SNR)\}$, where the STBC $X_L(SNR)$ corresponds to a signal-to-noise ratio of $SNR$ at each receive antenna.

For an LSTBC $X_L(SNR)$ of the form given by (33) with the $2k$-dimensional real constellation denoted by $A(SNR)$, from (3), we have that for each codeword matrix $X_i \in X_L(SNR)$, $i = 1, 2, \cdots, |X_L(SNR)|$,

$$|X_i|^2 = |G s|^2 \leq SNR,$$

where $G$ and $s$ are as defined in (35) and (36), respectively. For convenience, we assume that

$$\max_{s \in A(SNR)} \{||G s||^2\} = SNR$$

and hence,

$$\max_{s_{1i}}|s_{1i}|^2 \leq SNR, \quad \max_{s_{1Q}}|s_{1Q}|^2 \leq SNR \quad \forall \ i = 1, \cdots, k.$$ (37)

When the bit rate of $X_L(SNR)$ is $r \log_2 SNR + o(\log_2 SNR)$ bits per channel use, we have $|A(SNR)| \leq SNR^{\frac{k}{T}}$. Further, when each of the $2k$ real symbols takes values from the same real constellation $\mathcal{A}'(SNR)$, it follows that

$$|\mathcal{A}'(SNR)| \leq SNR^{\frac{k}{T}}.$$ (38)

For the special case of $\mathcal{A}'(SNR) = \mu \mathcal{A}_{PAM}$, where $\mu$ is a scalar normalizing constant designed to satisfy the constraints in (37), $\mathcal{A}_{M-PAM}$ is the regular $M$-PAM constellation given by

3In literature, “symbol rate” is referred to simply as ‘rate’. In this paper, to avoid confusion with the bit rate, which is $\frac{\log_2 |A|}{rT}$ bits per channel use, we have opted to use the term “symbol rate”.
\[ A_{\text{M-PAM}} = \left\{ 2l - 1, l = \left\lfloor \frac{M}{2} \right\rfloor + 1, \left\lfloor \frac{M}{2} \right\rfloor + 2, \ldots, \left\lceil \frac{M}{2} \right\rceil \right\}, \] (39)

and \( \mu_{A_{\text{M-PAM}}} = \{ \mu_a | a \in A_{\text{M-PAM}} \} \), we have from (38) and (37),
\[
M \triangleq \text{SNR}^{\frac{2}{2}}, \mu M \triangleq \text{SNR}^{3}
\]
and hence \( \mu^2 \triangleq \text{SNR}^{(1 - \frac{2}{2})} \).

For an LSTBC scheme \( \mathcal{X} \) that satisfies (3) and has a bit rate of \( r \log_2 \text{SNR} + o(\log_2 \text{SNR}) \) bits per channel use with the real symbols of its LSTBCs taking values from a scaled \( M \)-PAM, the LSTBCs \( \mathcal{X}_{L}(\text{SNR}) \) can be expressed as \( \mathcal{X}_{L}(\text{SNR}) = \{ \mu \mathbf{x} | \mathbf{x} \in \mathcal{X}_{L}(\text{SNR}) \} \), where \( \mu^2 \triangleq \text{SNR}^{(1 - \frac{2}{2})} \) and \( \mathcal{X}_{L}(\text{SNR}) \) is the unnormalized (so that it does not satisfy the energy constraint given in (3)) LSTBC given by
\[
\mathcal{X}_{L}(\text{SNR}) = \left\{ \sum_{i=1}^{k} (s_{i1} A_{i1} + s_{iQ} A_{iQ}) \mid s_{i1}, s_{iQ} \in A_{M-PAM}, M = \text{SNR}^{\frac{2}{2}}, \right\}.
\] (40)

With \( \mathcal{X}_{L}(\text{SNR}) \) and \( \mathcal{X}_{L}(\text{SNR}) \) thus defined, we define the non-vanishing determinant property of an LSTBC scheme as follows.

**Definition 8:** (Non-vanishing determinant) An LSTBC scheme \( \mathcal{X} \) is said to have the non-vanishing determinant property if the codeword difference matrices \( \Delta \mathbf{X} \) of \( \mathcal{X}_{L}(\text{SNR}) \) are such that
\[
\min_{\Delta \mathbf{X} \neq \mathbf{0}} \det (\Delta \mathbf{X} \Delta \mathbf{X}^H) \triangleq \text{SNR}^0.
\]

A necessary and sufficient condition for an LSTBC scheme \( \mathcal{X} = \{ \mathcal{X}_{L}(\text{SNR}) \} \), where \( \mathcal{X}_{L}(\text{SNR}) \) has weight matrices \( \mathbf{A}_{i1}, \mathbf{A}_{iQ} \), \( i = 1, \ldots, k \) and encodes its real symbols using PAM, to have the non-vanishing determinant property is that the design \( \mathcal{X}_z \), defined as
\[
\mathcal{X}_z = \left\{ \sum_{i=1}^{k} (s_{i1} A_{i1} + s_{iQ} A_{iQ}) \mid s_{i1}, s_{iQ} \in \mathbb{Z}, i = 1, 2, \ldots, k, \right\},
\] (41)
is such that for any non-zero matrix \( \mathbf{X} \) of \( \mathcal{X}_z \),
\[
\det (XX^H) \geq C,
\]
where \( C \) is some strictly positive constant bounded away from 0.

**Remark:** Any LSTBC is completely specified by a set of weight matrices (equivalently, its generator matrix, defined in (35)) and a 2\( k \)-dimensional real constellation \( \mathcal{A} \) that its real symbol vector takes values from, as evident from (33). However, for an LSTBC, the set of weight matrices (equivalently, its generator matrix) and the 2\( k \)-dimensional constellation need not be unique. As an example, consider the perfect code for 3 transmit antennas, which encodes 9 independent complex symbols, and can be expressed as
\[
\mathcal{X}_P = \left\{ \sum_{i=1}^{9} (x_{i1} A_{i1} + x_{iQ} A_{iQ}) \mid x_{i1}, x_{iQ} \in \mathcal{A}_{M^2-HEX}, i = 1, 2, \ldots, 9, \right\},
\]
where \( \mathcal{A}_{M^2-HEX} \) is an \( M^2 \)-HEX constellation given by
\[
\mathcal{A}_{M^2-HEX} = \left\{ a + \omega b \mid a, b \in \mathcal{A}_{M-PAM}, \right\},
\]
where
\[
\mathcal{A}_{i1} = A_{i1}, \mathcal{A}_{iQ} = -\frac{1}{2} A_{i1} + \frac{\sqrt{2}}{2} A_{iQ} \}
\]
\[
i = 1, 2, \ldots, 9.
\]

In general, any LSTBC \( \mathcal{X}_L \) with a generator matrix \( \mathbf{G} \) and a 2\( k \)-dimensional constellation \( \mathcal{A} \) that is a subset of a \( 2k \)-dimensional real lattice \( \mathcal{L} \) can be alternatively viewed to have \( \mathbf{G} \mathbf{G}^T \) as its generator matrix and a \( 2k \)-dimensional constellation \( \mathcal{A} \) that is a subset of \( \mathbb{R}^{2k \times 2k} \), where \( \mathbf{G} \mathbf{G}^T \) is the generator matrix of \( \mathcal{L} \).

In the following lemma, we prove that for an LSTBC scheme to be DMT-optimal, the symbol rate of its LSTBCs has to be at least equal to \( n_{min} \).

**Lemma 1:** An LSTBC scheme whose LSTBCs have a symbol rate less than \( n_{min} = \min(n_1, n_2) \) is not DMT-optimal.

**Proof:** With the system model given by (34), from (2), we have
\[
\mathbb{E}_s \left[ \text{tr} \left( \mathbf{G} \mathbf{s} \mathbf{s}^T \mathbf{G}^T \right) \right] \leq T \text{SNR}. \]
Hence, \( \text{tr} \left( \mathbf{G} \mathbf{G}^T \right) \leq T \text{SNR} \), where \( \mathbb{E}_s \left[ \mathbf{ss}^T \right] \in \mathbb{R}^{2k \times 2k} \). Since \( \mathbf{G} \) is fixed for an LSTBC, we assume that \( \text{tr}(\mathbf{Q}) = \alpha \text{SNR} \) for some finite positive constant \( \alpha \) with the overall constraint \( \text{tr}(\mathbf{G} \mathbf{G}^T) \leq T \text{SNR} \) being satisfied. Now, the ergodic capacity \( C \) of the equivalent channel is given by (5)\(^6\)
\[
C = \text{tr}(\mathbf{G} \mathbf{G}^T) \leq T \text{SNR} \]
\[
C(\mathbf{Q}) = \frac{1}{2T} \mathbb{E}_H \left[ \log_2 \det \left( I_{2Tn_r} + \mathbf{H} \mathbf{Q} \mathbf{G}^T \mathbf{H}^T \right) \right]
\]
where \( \mathbf{H} = \mathbf{I}_T \otimes \mathbf{H} \) and \( \mathbf{s} \) is jointly Gaussian with zero mean and covariance matrix \( \mathbf{Q} \) with \( \text{tr}(\mathbf{Q}) = \alpha \text{SNR} \). Now, \( (\alpha \text{SNR}) \mathbf{I}_{2k} - \mathbf{Q} \) is positive semidefinite and so \( \mathbf{H} (\alpha \text{SNR}) \mathbf{I}_{2k} - \mathbf{Q} \)\( \mathbf{G}^T \mathbf{H}^T \) is also positive semidefinite. Hence,
\[
\mathbf{I}_{2Tn_r} + (\alpha \text{SNR}) \mathbf{H} \mathbf{G}^T \mathbf{H}^T \geq \mathbf{I}_{2Tn_r} + \mathbf{H} \mathbf{Q} \mathbf{G}^T \mathbf{H}^T,
\]
where \( \mathbf{A} \succeq \mathbf{B} \) denotes that \( \mathbf{A} - \mathbf{B} \) is positive semidefinite. Using the inequality \( \det(A) \geq \det(B) \) when \( A \succeq B \) (32), we have
\[
C \leq \frac{1}{2T} \mathbb{E}_H \left[ \log_2 \det \left( \mathbf{I}_{2Tn_r} + (\alpha \text{SNR}) \mathbf{H} \mathbf{G}^T \mathbf{H}^T \right) \right]
\]
\(^6\)since \( \mathbf{Q} \) is symmetric and positive semidefinite with \( \text{tr}(\mathbf{Q}) = \alpha \text{SNR} \), each eigenvalue of \( \mathbf{Q} \) is at most equal to \( \alpha \text{SNR} \). With \( \mathbf{Q} = \mathbf{U} \mathbf{P} \mathbf{U}^T \), where \( \mathbf{U} \) is an orthonormal matrix and \( \mathbf{P} \) is a diagonal matrix with the diagonal entries being the eigenvalues of \( \mathbf{Q} \), it is clear that \( (\alpha \text{SNR}) \mathbf{I}_{2k} - \mathbf{Q} \) is positive semidefinite.
that of definite matrices, and (44) is obtained upon the singular value decomposition of $G^T G$, resulting in $G^T G = U D U^T$. We note that $G$ is full-ranked since $\det(G) = 1$. Since the ergodic capacity itself is at most $\log_2 \det(G)$, the latter result relies on STBC-minimal property. The usefulness of our result is due to Jensen’s inequality and the fact that $\log(1 + x) \leq x$ for $x > 0$.

Corollary 2: Consider a rate-$n_t$, minimum delay LSTBC-scheme $X' \in \mathcal{X}(SNR)$ with the NVD property, where $X'(SNR) = \{\mu X | X \in \mathcal{X}_U(SNR)\}$ with $\mu^2 \leq SNR^{(1 - \frac{1}{n_t})}$ and $X_U(SNR)$

\[
X_U(SNR) = \left\{ \begin{array}{l}
\sum_{i=1}^{n_t} (s_{it} A_{iti} + s_{iq} A_{iQ}) \left| s_{iti}, s_{iq} \in \mathcal{A}_{M-PAM}, i = 1, 2, \ldots, n_{\text{min}} \right., \\
M \leq SNR^\frac{1}{n_t}
\end{array} \right\},
\]

is DMT-optimal for the quasi-static Rayleigh faded $n_t \times n_r$ MIMO channel with CSIR if it has the non-vanishing determinant property.

The proof follows from the application of Theorem 2. Notice the difference between the above result and the result from Theorem 2 of [10]. The latter result relies on STBC-schemes that are based on rate-$n_t$ LSTBCs, irrespective of the values of $n_r$, while our result only requires that the symbol rate of the LSTBC be $\min(n_t, n_r)$ complex symbols per channel use which, together with NVD, guarantees DMT-optimality of the LSTBC-scheme. The usefulness of our result for asymmetric MIMO systems is discussed in the following section.
A. Full-diversity QOSTBC-scheme for the $4 \times 1$ MIMO system

The QOSTBC of [13], which is a rate-$1$ LSTBC, is given by

$$X_Q = \left\{ \begin{array}{c}
\{ x_1, x_2, x_3, x_4 | x_1, x_2 \in \mathcal{A}_{M^2-QAM}^I, x_3, x_4 \in e^{j\pi/4}\mathcal{A}_{M^2-QAM} \} \\
\{ x_1, x_2, x_3, x_4 | x_1, x_2 \in \mathcal{A}_{M^2-QAM}^Q, x_3, x_4 \in e^{j\pi/4}\mathcal{A}_{M^2-QAM} \}
\end{array} \right\},$$

(45)

where the $M^2$-QAM constellation, denoted by $\mathcal{A}_{M^2-QAM}$, is given by $\mathcal{A}_{M^2-QAM} \triangleq \{ a + j b | a, b \in \mathcal{A}_{M-PAM} \}$, and $e^{j\pi/4}\mathcal{A}_{M^2-QAM} \triangleq \{ e^{j\pi/4}(a + j b), a, b \in \mathcal{A}_{M-PAM} \}$ is the $\pi/4$ radian rotated $M^2$-QAM. The QOSTBC has a minimum determinant of 256 [13], irrespective of the value of $M$. Expressing (45) as

$$X_Q = \left\{ \sum_{i=1}^{4} (x_i A_{iI} + x_i A_{iQ}) | x_1, x_2 \in \mathcal{A}_{M^2-QAM}^I, x_3, x_4 \in e^{j\pi/4}\mathcal{A}_{M^2-QAM} \right\},$$

we note that $X_Q$ can also be written as

$$X_Q = \left\{ \sum_{i=1}^{4} (s_i A'_{iI} + s_i A'_{iQ}) | s_i, s_Q \in \mathcal{A}_{M-PAM} \right\},$$

where

$$A'_{iI} = A_{iI}, \quad A'_{iQ} = A_{iQ}, \quad i = 1, 2, \quad A'_{iI} = \frac{1}{\sqrt{2}}(A_{iI} + A_{iQ}), \quad A'_{iQ} = \frac{1}{\sqrt{2}}(-A_{iI} + A_{iQ}). \quad i = 3, 4.$$
with respect to $M$,

$$X_M = \left\{ \sum_{i=1}^{4} (s_{iL}A_{iL} + s_{iQ}A_{iQ}) \mid s_{iL}, s_{iQ} \in \mathbb{Z} \right\}$$

is such that for any non-zero matrix $X$ of $X_M$,

$$\det (XX^H) \geq 1.$$

Hence, the QOSTBC-scheme has the NVD property and is DMT-optimal for the $4 \times 1$ MIMO system.

\section*{B. Schemes based on CIOD for the $2 \times 1$ and $4 \times 1$ MIMO systems}

The STBC from CIOD \cite{13} for 4 transmit antennas, denoted by $X_C$, and given by (46) at the top of the page, is a rate-1 LSTBC with symbol-by-symbol ML-decodability. $X_C$ has a minimum determinant of 10.24 when its symbols $x_i$, $i = 1, 2, 3, 4$ take values from a $\tan^{-1}(2)/2$ radian rotated $M^2$-QAM constellation, irrespective of the value of $M$. Expressing (46) as

$$X_C = \left\{ \sum_{i=1}^{4} (x_{iL}A_{iL} + x_{iQ}A_{iQ}) \mid x_i \in e^{j\theta}A_{M^2-QAM}, \begin{array}{l} i = 1, 2, 3, 4, \\ \theta = \frac{1}{2} \tan^{-1}(2) \end{array} \right\}, \quad (47)$$

we note that (47) can be alternatively written as

$$X_C = \left\{ \sum_{i=1}^{4} (s_{iL}A_{iL} + s_{iQ}A_{iQ}) \mid s_{iL}, s_{iQ} \in \mathcal{A}_{M-PAM} \right\},$$

where

$$A_{iL} = \cos \theta A_{iL} + \sin \theta A_{iQ}, \quad A_{iQ} = -\sin \theta A_{iL} + \cos \theta A_{iQ}. \quad \theta = \frac{1}{2} \tan^{-1}(2).$$

Since $X_C$ has a minimum determinant of 10.24 independent of the value of $M$, any non-zero matrix $X$ of

$$X_M = \left\{ \sum_{i=1}^{4} (s_{iL}A_{iL} + s_{iQ}A_{iQ}) \mid s_{iL}, s_{iQ} \in \mathbb{Z} \right\},$$

is such that

$$\det (XX^H) \geq 0.04.$$

Hence, the CIOD based STBC-scheme has the NVD property and is DMT-optimal for a $4 \times 1$ MIMO system. Using the same analysis, one can show that the STBC-scheme based on the CIOD for 2 transmit antennas is DMT-optimal for the $2 \times 1$ MIMO system.

\section*{C. Four-group decodable STBC-schemes for $n_t \times 1$ MIMO systems}

For the special case of $n_t$ being a power of 2, rate-1, 4-group decodable STBCs have been extensively studied in literature \cite{13}.

For all these STBCs, the 2$n_t$ real symbols taking values from PM constellations can be separated into four equal groups such that the symbols of each group can be decoded independently of the symbols of all the other groups. For all these STBCs, the minimum determinant, irrespective of the size of the signal constellation, is given by \cite{13}

$$\min_{\Delta X \neq 0} (\Delta X \Delta X^H) = d_{p_{\text{min}}}^4$$

where $d_{p_{\text{min}}}$ is the minimum product distance in $n_t/2$ real dimensions, which has been shown to be a constant bounded away from 0 in \cite{13}. Hence, from Corollary 1, LSTBC-schemes consisting of these 4-group decodable STBCs are DMT-optimal for $n_t \times 1$ MIMO systems, with $n_t$ being a power of 2.

\section*{D. Fast-decodable STBCs}

In \cite{13} a rate-2, LSTBC was constructed for the $4 \times 2$ MIMO system and this code was conjectured to have a minimum determinant of 10.24 when the real symbols take values from regular $M$-PAM without regards to the value of $M$. An interesting property of this LSTBC is that it allows fast-decoding, meaning which, for ML-decoding the 16 real symbols (or 8 complex symbols) of the STBC using sphere decoding, it suffices to use a 9 real-dimensional sphere decoder instead of a 16 real-dimensional one. We conjecture that the LSTBC-scheme based on this fast-decodable STBC has the NVD property and hence is DMT-optimal for the $4 \times 2$ MIMO system.

Several rate-$n_r$ fast-decodable STBCs have been constructed in \cite{12} for various asymmetric MIMO configurations - for example, for $4 \times 2, 6 \times 2, 6 \times 3, 8 \times 2, 8 \times 3, 8 \times 4$ MIMO systems. For an $n_t \times n_r$ asymmetric MIMO system, these STBCs transmit a total of $n_t n_r$ complex symbols in $n_t$ channel uses and with regards to ML-decoding, only an $n_t n_r - \frac{n_r}{2}$ complex-dimensional sphere decoder is required, as against an $n_t n_r$ complex-dimensional sphere decoder required for decoding general rate-$n_r$ LSTBCs. These STBCs are constructed from division algebra and STBC-schemes based on these STBCs have the NVD property \cite{12}. Hence, for an $n_t \times n_r$ asymmetric MIMO system, LSTBC-schemes consisting of these rate-$n_r$ fast-decodable STBCs are DMT-optimal. The DMT curves for some well known DMT-optimal LSTBC-schemes are shown in Fig. 1 Fig. 2 Fig. 3 and Fig. 4. In all the figures, the perfect code-scheme refers to the LSTBC-scheme that is based on rate-$n_t$ perfect codes \cite{9, 10} and this scheme is known to be DMT-optimal for arbitrary number of receive antennas \cite{4}. The DMT-curves of the LSTBC-schemes that are based on rate-$n_r$ LSTBCs coincide with that of the rate-$n_t$ perfect code-scheme.
\( \tilde{O} \triangleq \left\{ |h_{ij}|^2 \mid \sum_{i=1}^{n_t} \log_2 \left( 1 + \frac{SNR}{n_r} \sum_{j=1}^{n_s} |h_{ij}|^2 \right) > r \log_2 SNR + o(\log_2 SNR) \right\} \geq \frac{1}{n_r} \sum_{i=1}^{n_r} \sum_{j=1}^{n_s} \log_2 (1 + SNR|h_{ij}|^2) \) \quad (48)

VI. CONCLUDING REMARKS

In this paper, we have presented an enhanced sufficient criterion for DMT-optimality of STBC-schemes using which we have established the DMT optimality of several low ML-max schemes for certain asymmetric MIMO systems. However, obtaining a necessary and sufficient condition for DMT-optimality of STBC-schemes is still an open problem. Further, obtaining low ML-decoding complexity STBC-schemes with NVD for arbitrary number of transmit antennas is another possible direction of research.

APPENDIX A

EVALUATION OF P(\( \tilde{O} \))

We have
\[
P(\tilde{O}) = \int_{\mathcal{O}} p(H) dH
\]
\[
= \int_{\mathcal{O}} \prod_{i=1}^{n_r} \prod_{j=1}^{n_t} p(h_{ij}) d(h_{ij})
\]
\[
= \int_{\mathcal{O}} \prod_{i=1}^{n_r} \prod_{j=1}^{n_t} p(|h_{ij}|^2) d(|h_{ij}|^2),
\]
where (49) is because of the independence of the entries of \( H \) and (50) is by change of variables, with \( \tilde{O} \) defined as shown in (48) at the top of the page. It is well known that \( p(|h_{ij}|^2) = e^{-|h_{ij}|^2} \) for Rayleigh fading. Let \( |h_{ij}|^2 = SNR^{-\alpha_{ij}} \). Now, \( p(\alpha_{ij}) = (\log_2 SNR)^{-|SNR^{-\alpha_{ij}}|} SNR^{-\alpha_{ij}} \). Defining the vector \( \alpha \) as \( \alpha = [\alpha_{ij}]_{i=1}^{n_r} \), \( j=1 \ldots n_t \), we have
\[
P(\tilde{O}) = \kappa \int_{\mathcal{O}} e^{-\sum_{i,j} SNR^{-\alpha_{ij}} \sum_{i,j} \alpha_{ij} d\alpha},
\]
where \( \kappa = (\log_2 SNR)^{n_r n_t} \) and
\[
\tilde{O} = \left\{ \alpha \mid \sum_{i,j} \log_2 \left( 1 + \sum_{j} SNR^{-\alpha_{ij}} \right) > r \log_2 SNR + o(\log_2 SNR), \right. \]
\[
\left. \sum_{i,j} \log_2 \left( 1 + SNR^{-\alpha_{ij}} \right) \leq \frac{1}{n_r} \sum_{i=1}^{n_r} \sum_{j=1}^{n_t} \log_2 (1 + SNR|h_{ij}|^2) \right\} \quad (51)
\]
where \( \max\{} \) denotes “the largest element of”. Note that in (51), the integrand is exponentially decaying with \( SNR \) when any one of the \( \alpha_{ij} \) is negative, unlike a polynomial decay when all the \( \alpha_{ij} \) are non-negative. Hence, using the concept developed in [2],
\[
P(\tilde{O}) \triangleq SNR^{-f(\alpha^*)},
\]
where
\[
f(\alpha^*) = \inf_{\alpha \in \mathbb{R}^+_{n_r \times n_t}} \left\{ \sum_{i=1}^{n_r} \sum_{j=1}^{n_t} \alpha_{ij} \right\}
\]
with \( \mathbb{R}^+ \) representing the set of non-negative real numbers. It is easy to check that the infimum occurs when all but two of \( \alpha_{ij} = 1 - \frac{r}{n_r} \), while the other two are \( 1 - \frac{r}{n_r} - \delta \) and \( 1 - \frac{r}{n_r} - \delta \) respectively, where \( \delta \to 0^+ \). Hence,
\[
P(\tilde{O}) \triangleq SNR^{-n_t(n_s-n_r)}.
\]

APPENDIX B

We prove here that \( O_1 = O_1 \) almost surely. As before, the rows of \( H \) are denoted by \( h_i, i = 1, 2, \ldots n_r \). Let \( |h_{ij}|^2 = SNR^{-\alpha_{ij}} \) and \( u \triangleq [u_1, u_2, \ldots u_n]^T \) be a complex column vector independent of \( h_i \), with either \( |u_j|^2 = SNR^0 \) or \( u_j = 0, j = 1, 2, \ldots n_t \). Defining the indicators \( I_1, I_2, \ldots I_n \) as
\[
I_j = \begin{cases} 1, & \text{if } |u_j|^2 = SNR^0 \\ 0, & \text{otherwise} \end{cases} \quad j = 1, \ldots n_t,
\]
we have
\[
|h_i u|^2 = \sum_{j=1}^{n_t} h_{ij} u_j \sum_{k=1}^{n_r} h_{ik}^* u_k^* = \sum_{j=1}^{n_t} |h_{ij}|^2 |u_j|^2 + 2 \sum_{j=1}^{n_t} \sum_{k=j+1}^{n_r} Re(h_{ij}h_{ik}^* u_j u_k^*) \geq SNR^{-\beta} \] almost surely,
where \( Re(\cdot) \) denotes “the real part of” and
\[
\beta = \min\{\alpha_{ij} \mid I_j \neq 0, j = 1, 2, \ldots n_t\}.
\]
Note that (52) is due to the fact that the \( h_{ij} \)’s are independent random variables. Now, denoting the \( i^{th} \) row of \( HU_i \) by \( h_i(l) \), let \( |h_{ij}(l)|^2 \) \( \geq SNR^{-\beta_{ij}} \). It is to be noted that since \( U_i \) is unitary, each row and column of \( U_i \) has at least one non-zero entry. Denoting the positions of these non-zero entries in the \( i^{th} \) column by \( \eta_i, i = 1, 2, \ldots n_r, \eta_i \neq \eta_j \) for \( i \neq j \) (note that \( \eta_1, \ldots, \eta_n \) = [1, 2, \ldots n_t]P, where \( P \) is some permutation matrix of size \( n_t \times n_t \)), from (52), we have
\[
\beta_{ij} \leq \alpha_{in_j} \] almost surely.

Hence,
\[
\sum_{j=1}^{n_t} \log_2 (1 + SNR|h_{ij}(l)|^2) \geq \sum_{j=1}^{n_t} \log_2 (1 + SNR^{1-\beta_{ij}}) \]
\[
= \sum_{j=1}^{n_t} \log_2 (1 + SNR^{1-\alpha_{ij}}) \]
\[
= \sum_{j=1}^{n_t} \log_2 (1 + SNR|h_{ij}|^2) \]
almost surely and this is true for all $i, 1, 2, \ldots, n_r$. Hence, almost surely
\[
\sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}(l)|^2 \right) \geq \sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}|^2 \right).
\]
So, $\sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}|^2 \right) > n_r \log_2 SNR + o(\log_2 SNR) \Rightarrow \sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}|^2 \right) \geq n_r \log_2 SNR + o(\log_2 SNR)$ almost surely. Since $U_l$ is unitary, it can be similarly proven that $\sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}(l)|^2 \right) > n_r \log_2 SNR + o(\log_2 SNR) \Rightarrow \sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}(l)|^2 \right) \geq n_r \log_2 SNR + o(\log_2 SNR)$ almost surely. Hence,
\[
\sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}|^2 \right) > n_r \log_2 SNR + o(\log_2 SNR)
\]
\[
\sum_{i,j} \log_2 \left( 1 + SNR|h_{ij}(l)|^2 \right) \Leftrightarrow n_r \log_2 SNR + o(\log_2 SNR)
\]
almost surely and so, $O_l = O'_l$ almost surely.

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