On the Sphere Decoding Complexity of STBCs for Asymmetric MIMO Systems

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Abstract—In the landmark paper [1] by Hassibi and Hochwald, it is claimed without proof that the upper triangular matrix $R$ encountered during the sphere decoding of any linear dispersion code is full-ranked whenever the rate of the code is less than the minimum of the number of transmit and receive antennas. In this paper, we show that this claim is true only when the number of receive antennas is at least as much as the number of transmit antennas. We also show that all known families of high rate (rate greater than 1 complex symbol per channel use) multigroup ML decodable codes have rank-deficient $R$ matrix even when the criterion on rate is satisfied, and that this rank-deficiency problem arises only in asymmetric MIMO with number of receive antennas less than the number of transmit antennas. Unlike the codes with full-rank $R$ matrix, the average sphere decoding complexity of the STBCs whose $R$ matrix is rank-deficient is polynomial in the constellation size. We derive the sphere decoding complexity of most of the known high rate multigroup ML decodable codes, and show that for each code, the complexity is a decreasing function of the number of receive antennas.

I. INTRODUCTION

We consider Space-Time Block Codes (STBCs) for an $N$ transmit antenna, $M$ receive antenna, quasi-static MIMO channel with Rayleigh flat fading, $Y = XH + W$, where $X$ is the $T \times N$ codeword matrix, $Y$ is the $T \times M$ received matrix, $H$ is the $N \times M$ channel matrix and the $T \times M$ matrix $W$ is the additive noise at the receiver. The entries of $H$ and $W$ are i.i.d. zero mean, unit variance, circularly symmetric Gaussian random variables. A design $X$ in $K$ real symbols $x_1, \ldots, x_K$, is a matrix $\sum_{i=1}^{K} x_i A_i$ where, the $T \times N$ complex matrices $A_1, \ldots, A_K$, which are called linear dispersion or weight matrices are linearly independent over the real field $\mathbb{R}$. An STBC $C$ is obtained from this design by using a finite signal set $A \subset \mathbb{R}^K$ to encode the real symbols. That is, $C = \{\sum_{i=1}^{K} a_i A_i | (a_1, \ldots, a_K)^T \in A\}$. The rate of this code is $R = \frac{T}{MT}$ complex symbols per channel use (cspcu). The linear independence of the weight matrices in the definition of a design implies that $R \leq N$.

The signal set $A$ is chosen in such a way that the STBC $C$ has full-diversity and large coding gain. In most cases $A$ is chosen to be a subset of $\Theta Z^K$, where $\Theta \in \mathbb{R}^{K \times K}$ is a full-rank matrix. One such instance is when the symbols are partitioned into multiple encoding groups, and each group of symbols is encoded using a rotated integer lattice constellation independently of other groups, such as in Clifford Unitary Weight Designs [2]. Quasi-Orthogonal STBCs [3] and Coordinate Interleaved Orthogonal Designs [4]. In this case, $\Theta$ is an orthogonal matrix. There are also instances when the real symbols are encoded independently using regular PAM constellations of possibly different minimum distances, as done in [5]. In this case $\Theta$ is a diagonal matrix with positive entries.

For a complex matrix $A$, let $\text{vec}(A)$ denote the complex vector obtained by stacking the columns of $A$ one below the other and let $\tilde{\text{vec}}(A) = [\text{vec}(A_{R_1})^T \text{vec}(A_{I_1})^T]^T$. Now, the received matrix $Y$ can be rewritten as $y = \tilde{\text{vec}}(Y) = GX + W$, where $x = [x_1, \ldots, x_K]^T$, $w = \tilde{\text{vec}}(W)$ and the equivalent channel matrix

$$G = [\tilde{\text{vec}}(A_1 H) \, \tilde{\text{vec}}(A_2 H) \, \cdots \, \tilde{\text{vec}}(A_K H)] \in \mathbb{R}^{2MT \times K}.$$ 

Consider the vector of transformed information symbols $s = \Theta^{-1} x$ which takes values from $\mathcal{A}^T = \Theta^{-1} A \subset \mathbb{Z}^K$. The components of $s$ take integer values and hence one can use a sphere decoder to decode $s$ and then obtain the ML estimate of the information vector $x$. The ML decoder is given by

$$\arg \min_{s \in \mathcal{A}} ||y - G\Theta s||_F^2.$$  

Suppose the rank of the equivalent channel matrix is $K' < K$, the conventional sphere decoder [6] can be modified as follows [7]. The $R$ matrix resulting from the QR decomposition of $\Theta$ has the form $R = [R_a \ R_b] \in \mathbb{R}^{K \times K}$, where $R_a$ is a $K' \times K'$ upper triangular full-rank matrix [7]. There is a corresponding partition of $s$ as $[s_a^T \ s_b^T]^T$. If $q$ is the size of the regular PAM constellation used, then for each of the $q^{K-K'}$ values of $s_a$, the conditionally optimal estimate of $s_a$ can be found by first removing the interference from $s_b$, and then using a sphere decoder with the upper triangular matrix $R_a$ to obtain an estimate of $s_a$. Then from the resulting $q^{K-K'}$ estimates of $s$, the optimal vector is chosen. This situation arises whenever $\text{rank}(G) = \text{rank}(\Theta) < K$.

It is claimed in [1] without proof that $R \leq \min\{M, N\}$ or equivalently $R \leq M$ is a sufficient condition for the system of equations defined by (1) to be not underdetermined, i.e., for $\text{rank}(G) = K$. Since $G$ is a $2MT \times K$ matrix, $K \leq 2MT$ (i.e., $R \leq M$) is a necessary condition for $\text{rank}(G) = K$. Hence, throughout this paper we will consider only the case $R \leq \min\{M, N\}$. In the next section we show that the claim made in [1] is true only for $M \geq N$. This observation is the gateway to the new results presented from Section II onwards.

Towards this end, we introduce the notion of singularity of a design as follows (note that since $H$ is a random matrix, $\text{rank}(G)$ is a random variable as well).
Definition 1: Let $X$ be a $T \times N$ design in $K$ real symbols with rate $R$ and let the number of receive antennas $M \geq R$. We say that the design $X$ is singular for $M$ receive antennas if $\text{rank}(G) < K$ with probability 1. Else, we say that $X$ is non-singular for $M$ receive antennas.

For a system where $\text{rank}(G) = K$, the sphere decoder complexity, averaged over noise and channel realizations, is independent of the constellation size $q$ and is roughly polynomial in the number of symbols $K$ \cite{8}. However, if the design is singular for $M$ receive antennas, the average sphere decoding complexity is much higher and is no more independent of the constellation size. The complexity of sphere decoding then grows as $q^{K-K'}$ if $\text{rank}(G) = K'$ with probability 1. In the rest of the paper, by sphere decoding complexity we mean the average sphere decoding complexity and the focus is on the dependence of the complexity on the constellation size $q$.

Example 1: In \cite{9}, a rate $R = \frac{17}{6}$ code for $N = 4$ antennas with $K = 17$ real symbols was given. Define $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $Z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then, the 17 weight matrices $A_1, \ldots, A_{17}$ of the design in \cite{9} are as follows:

\begin{align*}
A_1 &= I_2 \otimes I_2, & A_2 &= iZ \otimes I_2, & A_3 &= ZX \otimes I_2, \\
A_4 &= iX \otimes Z, & A_5 &= X \otimes ZX, & A_6 &= iX \otimes X, \\
A_7 &= I_2 \otimes ZX, & A_8 &= iZ \otimes X, & A_9 &= ZX \otimes X, \\
A_{10} &= iZX \otimes ZX, & A_{11} &= ZX \otimes X, & A_{12} &= iX \otimes I_2, \\
A_{13} &= Z \otimes ZX, & A_{14} &= iI_2 \otimes X, & A_{15} &= iZ \otimes Z, \\
A_{16} &= iI_2 \otimes I_2 \quad \text{and} \quad A_{17} &= iI_2 \otimes Z.
\end{align*}

Let the number of receive antennas be $M = 3$. Then, $M > R$ and the equivalent channel matrix $G$ is of size $24 \times 17$. Now consider the following randomly generated channel matrix $H = \begin{bmatrix} 0.3457 + 0.2299i & 0.2078 - 0.0723i & -0.7558 - 0.6116i \\
0.7316 - 0.5338i & -0.5567 - 0.1707i & -0.5724 + 0.0212i \\
0.5140 + 0.9689i & 0.6282 + 0.2257i & -0.2081 - 0.1166i \\
-0.2146 - 1.2102i & -0.8111 + 0.2212i & 1.0171 + 0.4439i \end{bmatrix}$.

The resulting $G$ matrix has rank only 16 and the structure of the upper triangular matrix $R'$ of size $24 \times 17$ obtained from its QR decomposition is shown in (2) at the top of next page. The non-zero entries of $R'$ are denoted by ‘a’. It is clear that the first 16 columns of $R'$ are linearly independent and the last column lies in the span of the first 16 columns. Removing the last 8 rows of $R'$, which are all zero, we get the $16 \times 17$ real matrix $R$ which is used by the sphere decoder to find the ML estimate of the information vector. In this case, $K' = 16$ and the complexity of sphere decoding this STBC, for this particular channel realization $H$, is of the order of $q^{17-16} = q$. In Section III-A we show that when the randomness of $H$ is taken into account, for this code, $\text{rank}(G) = 16$ with probability 1.

In this paper, we show that all known families of high rate ($R > 1$) multigroup ML decodable codes are non-singular for certain number of receive antennas. An STBC is $g$-group or multigroup ML decodable if its symbols can be partitioned into $g$ groups and each group of symbols can be ML decoded independent of others. Multigroup ML decodable codes with high rate were constructed in \cite{9, 10, 11, 12} and \cite{13}. The contributions and organization of this paper are as follows.

- We introduce the notion of singularity of a design which is a direct indicator of the sphere decoding complexity of an STBC.
- We show that, contrary to the claim made in \cite{1}, $R \leq \min\{M, N\}$ is not a sufficient condition for a design to be non-singular for $M$ receive antennas (Theorem I). We show that the situation of singular designs arises only in the case of asymmetric MIMO with number of receive antennas $M$ less than the number of transmit antennas $N$ (Proposition I).
- We show that all known families of high rate ($R > 1$) multigroup ML decodable codes are singular for certain values of $M$ (Section III see Table I for a summary of results).
- We derive the sphere decoding complexity of almost all known high-rate multigroup ML decodable codes and show that in each case the sphere decoding complexity is a decreasing function of the number of receive antennas $M$ (Section III see Table I). We show that even when a design is singular, multigroup ML decodability helps reduce the sphere decoding complexity. The reduction in complexity is from $O\left(q^{K-K'}\right)$ to $O\left(q^{K-K'}\right)$, where $g$ is the number of ML decoding groups (Section III).

Some related open problems are discussed in Section IV.

**Notation:** For a complex matrix $A$, the transpose, the conjugate and the conjugate-transpose are denoted by $A^T$, $\bar{A}$ and $A^H$ respectively. For a square matrix $A$, $\det(A)$ is the determinant of $A$, $A \otimes B$ is the Kronecker product of matrices $A$ and $B$, $I_n$ is the $n \times n$ identity matrix, $0$ is the all zero matrix of appropriate dimension and $i = \sqrt{-1}$. For square matrices $A_j$, $j = 1, \ldots, d$, $\text{diag}(A_1, \ldots, A_d)$ denotes the square, block-diagonal matrix with $A_1, \ldots, A_d$ on the diagonal, in that order.

**II. BASIC RESULTS ON THE RANK OF THE EQUIVALENT CHANNEL MATRIX**

We now present few results which we will use in the following sections to derive the rank of $G$ for specific STBCs. The following result shows that if any design is singular for $M$ receive antennas, then $M < N$. Thus, the rank-deficiency problem arises only in asymmetric MIMO with $M < N$.

**Proposition 1:** If $M \geq N$, every $T \times N$ design is non-singular for $M$ receive antennas.

**Proof:** Since $\text{vec}(\cdot)$ is an isomorphism of the vector spaces $\mathbb{C}^{TM}$ and $\mathbb{R}^{2MT}$, it is enough to show that $A_1H, \ldots, A_KH$ are linearly independent with probability 1. Suppose $H$ is full-ranked, i.e., $\text{rank}(H) = N$, then there exists a matrix $H' \in \mathbb{C}^{M \times N}$ such that $HH' = I_N$. If $V = \sum_{i=1}^{K} a_i A_i H = 0$, it would mean that $V H' = \sum_{i=1}^{K} a_i A_i = 0$. Since $A_i$ are the weight matrices of a design, they are linearly independent and hence $a_i = 0, i = 1, \ldots, K$. Thus $\text{rank}(G) = K$ if $H$ is full-ranked. Since $H$ is full-ranked.
with probability 1 [14], we have shown that any design is non-singular for $M$ receive antennas if $M \geq N$.

Let $\langle A_1, A_2, \ldots, A_K \rangle$ denote the $\mathbb{R}$-linear subspace of $\mathbb{C}^{T \times N}$ spanned by the matrices $A_1, \ldots, A_K$.

**Proposition 2:** Let $B_1, \ldots, B_K$ be $T \times N$ complex matrices such that $\langle A_1, \ldots, A_K \rangle = \langle B_1, \ldots, B_K \rangle$ and let $H \in \mathbb{C}^{N \times M}$ be any matrix. Then the column spaces of the matrices

\[
G_A(H) = \begin{bmatrix} \vec{v}(A_1H) & \vec{v}(A_2H) & \cdots & \vec{v}(A_KH) \end{bmatrix} \quad \text{and} \quad G_B(H) = \begin{bmatrix} \vec{v}(B_1H) & \vec{v}(B_2H) & \cdots & \vec{v}(B_KH) \end{bmatrix}
\]

are identical. In particular, $\text{rank}(G_A(H)) = \text{rank}(G_B(H))$.

**Proof:** Let $v$ be any vector in the column space of $G_A(H)$. Then, $v = \sum_{i=1}^{K} a_i \vec{v}(A_iH)$, for some choice of real numbers $a_i$, $i = 1, \ldots, K$. Since each of the $A_i \in \langle B_1, \ldots, B_K \rangle$, every $A_i$ can be written as some real linear combination of $B_1, \ldots, B_K$. It follows that every $\vec{v}(A_iH)$ can be written as some real linear combination of $\vec{v}(B_1H), \ldots, \vec{v}(B_KH)$. Hence $v$ belongs to the column space of $G_B(H)$. Similarly we can show that every vector in the column space of $G_B(H)$ belongs to the column space of $G_A(H)$ also. This completes the proof.

The following result shows that if every weight matrix of a given design is multiplied on the left by a constant invertible matrix, then the rank of the equivalent channel matrix is unchanged.

**Proposition 3:** Let $C \in \mathbb{C}^{T \times T}$ be any full-rank matrix, $B_i = CA_i$, $i = 1, \ldots, K$ and $H$ be any $N \times M$ complex matrix. Then we have $\text{rank}(G_A(H)) = \text{rank}(G_B(H))$, where $G_A(H)$ and $G_B(H)$ are as defined in (3) and (4).

**Proof:** Since $\vec{v}(\cdot)$ is an isomorphism of the $\mathbb{R}$-vector spaces $\mathbb{C}^{T \times M}$ and $\mathbb{R}^{2MT}$, it is enough to show that the subspaces $\langle A_1H, \ldots, A_KH \rangle$ and $\langle CA_1H, \ldots, CA_KH \rangle$ have the same dimension. The proof is complete if we show that the vector space homomorphism $\varphi$ from the former subspace to the latter, that sends $V = \sum_{i=1}^{K} a_iA_iH$ to $\sum_{i=1}^{K} a_iCA_iH$ is a one to one map. If $V \in \text{ker}(\varphi)$, then $\varphi(V) = C\sum_{i=1}^{K} a_iA_iH = CV = 0$. Since $C$ is invertible, this means that $V = 0$. This completes the proof.

It is shown in [13] that for any $N \geq 1$, there exists an explicitly constructable set of $N^2$ matrices belonging to $\mathbb{C}^{N \times N}$ that are unitary, Hermitian and linearly independent over $\mathbb{R}$. This set of matrices forms a basis for the space of $N \times N$ Hermitian matrices. Denote by $X_N^\text{Herm}$ the design obtained by using these $N^2$ matrices as weight matrices. For positive integers $n$ and $m$, define the function

\[
f(n, m) = n^2 - (n - m)^2,
\]

where $(a)^+ = \max\{a, 0\}$. We now state the main result of this paper.

**Theorem 1:** The equivalent channel matrix of the design $X_N^\text{Herm}$ for $M$ receive antennas has rank $f(N, M)$ with probability 1.

**Proof:** See Appendix.

The rate of $X_N^\text{Herm}$ is $\frac{N}{2}$ and the rank of the equivalent channel is less than $K = \frac{N^2}{2}$ whenever $M < N$. Thus, this design is singular for all $\frac{N}{2} \leq M < N$.

**Example 2:** Consider the design $X_2^\text{Herm}$ used in an asymmetric MIMO with $M = 2$ receive antennas. In this case, $R = \frac{1}{2} < \min\{M, N\}$ and the equivalent channel matrix $G$...
is of size $12 \times 9$. From Theorem[1] we know that the rank of $G$ is equal to $f(3, 2) = 8$ with probability 1. Hence, this design is singular for 2 receive antennas. The 9 weight matrices of the design $X^\text{Herm}_q$ are as follows

$$
A_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
A_2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}, A_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix},
$$

$$
A_4 = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, A_5 = \begin{bmatrix}
i & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & 1
\end{bmatrix}, A_6 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},
$$

$$
A_7 = \begin{bmatrix}
0 & 0 & i \\
0 & 1 & 0 \\
-i & 0 & 0
\end{bmatrix}, A_8 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \text{ and } A_9 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & i \\
0 & -i & 0
\end{bmatrix}.
$$

The structure of the $12 \times 9$ upper triangular matrix $R'$ obtained from the QR decomposition of $G$ when $H$ equals the following randomly generated matrix

$$
\begin{bmatrix}
-0.5688 & -0.8117i & -0.1723 + 1.8282i \\
0.4926 + 0.0742i & 0.1525 - 0.4716i \\
0.5905 + 0.5107i & -0.8244 + 0.1325i
\end{bmatrix},
$$

is given by

$$
R' = \begin{bmatrix}
a & a & a & a & a & a & a & a & a \\
a & a & a & a & a & a & a & a & a \\
a & a & a & a & a & a & a & a & a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The sphere decoder uses the $8 \times 9$ matrix $R$ obtained from $R'$ by deleting its last 4 rows which are all zero. Hence, for this particular channel realization $H$, the sphere decoding complexity is of the order of $q^{9-8} = q$.

The remaining part of this section is concerned with multi-group ML decodable codes. Suppose the code obtained from a design $X$ with a signal set $A$ is $g$-group ML decodable, for some $g > 1$. The information symbols $\{x_1, \ldots, x_K\}$ can be partitioned into $g$ vectors $x_{I_1}, \ldots, x_{I_g}$ of length $\lambda_1, \ldots, \lambda_g$ respectively such that each symbol vector can be ML decoded independently of other symbol vectors. There is a corresponding partition of the channel matrix into $g$ submatrices $G_1, \ldots, G_g$, such that $Gx = \sum_{k=1}^g G_kx_{I_k}$, where $G_k \in \mathbb{R}^{2MT \times \lambda_k}$, for $k = 1, \ldots, g$. In Theorem 2 of [15] it is shown that for any $k \neq k'$ and any channel realization $H$, every column of $G_k$ is orthogonal to every column of $G_{k'}$. As a direct consequence of this, we have the following proposition.

**Proposition 4:** For any $g$-group ML decodable STBC and any channel realization $H$, $\text{rank}(G) = \sum_{k=1}^g \text{rank}(G_k)$.

**Proof:** Since the column spaces of $G_k$, $k = 1, \ldots, g$ are orthogonal to each other, the dimension of the column space of $G$ is equal to the sum of the dimensions of the column spaces of $G_k$, $k = 1, \ldots, g$.

### III. Sphere decoding complexity of some known families of codes

In this section, we show that all known families of high-rate ($R > 1$) multigroup ML decodable codes are singular for certain number of receive antennas. Using the properties of the rank of the equivalent channel matrix derived in the previous section, we now derive the sphere decoding complexities of these known multigroup ML decodable STBCs. Table I summarizes the results of this section. The table lists the sphere decoding complexity and the minimum number of receive antennas for non-singularity of $X^\text{Herm}_q$, the codes in [9], [11], [12], and the codes in [13] corresponding to even number of ML decoding groups.

#### A. Fast-group-decodable STBC from Ren et. al. [9]

In [9], a rate $R = \frac{12}{8}$ STBC for 4 antennas was constructed. This STBC is 2-group ML decodable with 1 real symbol in the first group and 16 in the second group. Of the 16 symbols in the second group, 5 symbols can be decoded independently of each other conditioned on the values taken by the remaining 11 symbols in the group. The weight matrix corresponding to the only symbol in the first group is $I_4$. Since any two weight matrices from different ML decoding groups are Hurwitz-Radon orthogonal, i.e., satisfy $A_i^H A_j + A_j^H A_i = 0$, all the matrices in the second group must be skew-Hermitian. Let $X_2 = \sum_{j=2}^{17} x_j A_j$ denote the design corresponding to the second group. Then, $\{A_2', \ldots, A_{17}'\}$, where $A_j' = iI_4 \cdot A_j$, are Hermitian, linearly independent over $\mathbb{R}$ and hence form a basis for the space of $4 \times 4$ Hermitian matrices. Let us denote the equivalent channel matrix of the design whose weight matrices are $A_2', \ldots, A_{17}'$ as $G_2$.

If the channel realization is $H$, then the equivalent channel for the first group $G_1 = [\overline{\text{vec}}(H)]$ which is non-zero with probability 1. Thus, the design corresponding to the first group is non-singular for any $M \geq 1$. From Proposition 3 and Theorem 1 the rank of the equivalent channel of the second group is $\text{rank}(G_2) = \text{rank}(G_2') = O(4, M)$ with probability 1. For $M = 3$ receive antennas, we have $R < \min\{N, M\}$, and $\text{rank}(G_2) = 15$. The sphere decoding complexity of the second group is thus $O(q^{16-15}) = O(q)$. Since the two groups can be decoded independently of each other and the first group is single real symbol decodable, the sphere decoding complexity of the code in [9] is $O(q)$ for $M = 3$ antennas. For $M \geq 4$, $f(4, M) = 16$, thus the STBC in [9] is non-singular for $M$ receive antennas and hence its sphere decoding complexity is independent of $q$.

#### B. High-rate 2-group ML decodable codes from Srinath et. al. [12]

A family of 2-group ML decodable STBCs was constructed in [12] for $N = 2^m$, $m > 1$ antennas with rate $R = \frac{4}{2^m} + \frac{4}{2^{m-1}}$ cspsu. This family includes the 4-antenna, rate $\frac{4}{3}$ code of [10] as a special case. The number of symbols in the design is
\[ K = \frac{N^2}{2} + 2. \]

In the rest of this subsection we show that the sphere decoding complexity is \( \mathcal{O} \left( q \left( \frac{N}{2} - M \right)^3 \right) \), which is polynomial in \( q \), and so is large for all \( [R] \leq M < \frac{N^2}{2} \), where \( [a] \) is the smallest integer greater than or equal to \( a \). Note that the sphere decoding complexity is a decreasing function of the number of receive antennas \( M \).

**Derivation of sphere decoding complexity:** The STBCs constructed in [12] have a block diagonal structure. The weight matrices for \( N = 2^m \) antennas are of the form
\[
\begin{bmatrix}
V_1 & 0 \\
0 & V_2
\end{bmatrix},
\]
where \( V_1, V_2 \in \mathbb{C}^{n \times n} \) are unitary and \( n = \frac{N}{2} \). For all the matrices in the first ML decoding group, \( V_1 \) is constant, say \( F_1 \), and for all the matrices in the second group \( V_2 \) is constant, say \( F_2 \). Let \( X_1 \) and \( X_2 \) be the designs corresponding to the first and second group respectively. Each group contains \( n^2 + 1 \) real symbols. We will now derive the rank of the submatrix \( G_1 \) of \( G \) that corresponds to the first ML decoding group. Note that \( G_1 \) is the equivalent channel matrix when using an STBC with design \( X_1 \).

Let the weight matrices of \( X_1 \) be \( A_1, \ldots, A_{n^2+1} \). Consider a new design \( X'_1 \) with weight matrices \( A'_i = CA_i \), \( i = 1, \ldots, n^2+1 \), where \( C = \begin{bmatrix} I_n & 0 \\ 0 & iF_2^H \end{bmatrix} \). Since \( C \) is unitary, from Proposition 3 the rank of \( G_1 \) equals the rank of \( G'_1 \), the equivalent channel matrix of \( X'_1 \). Since the multiplication of all the weight matrices of a design by a unitary matrix does not affect its multigroup ML decodability, for any matrix \( B \) in the design \( X_2 \) and any \( j = 1, \ldots, n^2+1 \), we have
\[ A'_j (CB) + (CB)^H A'_j = 0, \]
where \( A'_j \) and \( CB \) are block diagonal and are of the form
\[
A'_j = \begin{bmatrix} F_1 & 0 \\ 0 & D_j \end{bmatrix} \quad \text{and} \quad CB = \begin{bmatrix} V & 0 \\ 0 & iI_n \end{bmatrix},
\]
for some unitary matrices \( D_j \) and \( V \). Since \( A'_j \) and \( CB \) satisfy (5) and are block diagonal, from Lemma 1 of [12], \( D_j^H iI_n + (iI_n)^H D_j = 0 \). Thus, for \( j = 1, \ldots, n^2+1 \), \( D_j \) is a \( n \times n \) unitary, Hermitian matrix. Next we use Proposition 2 to find the rank of \( G'_1 \).

Note that \( \langle A'_1, \ldots, A'_{n^2+1} \rangle \) is the same as the span of
\[
\begin{bmatrix}
F_1 & 0 & 0 & 0 & 0 \\
0 & 0 & D_1 & \cdots & 0 \\
0 & 0 & 0 & D_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & D_{n^2+1}
\end{bmatrix}.
\]

(6)

Since \( A_1, \ldots, A_{n^2+1} \) are the weight matrices of the design \( X_1 \), they are linearly independent and hence \( A'_1, \ldots, A'_{n^2+1} \) are also linearly independent. Further, the first matrix in (6) is linearly independent of the remaining matrices and hence the dimension of the span of the remaining matrices in (6) is \( n^2 \).

Without loss of generality, let us assume that \( D_1, \ldots, D_{n^2} \) are linearly independent, thus \( \langle D_1, \ldots, D_{n^2} \rangle \) is the space of all \( n \times n \) Hermitian matrices. Then, \( \langle A'_1, \ldots, A'_{n^2+1} \rangle \) equals the space spanned by
\[
\begin{bmatrix}
F_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & D_1 & \cdots \\
0 & 0 & 0 & 0 & D_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(7)

From Proposition 2 it is enough if we concentrate on the design whose weight matrices are given by (7). Let the channel matrix be partitioned as \( H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \), where \( H_1, H_2 \in \mathbb{C}^{n \times M} \).

We need to compute the dimension of the space spanned by the weight matrices multiplied on the right by \( H \) which is
\[
\left\langle \begin{bmatrix} F_1 & H_1 \\ 0 & D_1H_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & D_2H_2 \end{bmatrix} \right\rangle.
\]

With probability 1, \( H_1 \) is non-zero and hence the first matrix is linearly independent of the remaining matrices. From Theorem 1 the dimension of the span of the remaining \( n^2 \) matrices is \( f(n, M) = n^2 - ((n - M)^+) \) with probability 1. Thus \( \text{rank}(G_1) \) equals \( f(n, M) + 1 \) with probability 1.

A similar result can also proved for the second ML decoding group, i.e., for \( \text{rank}(G_2) \). From Proposition 4

<table>
<thead>
<tr>
<th>Code</th>
<th>Transmit Antennas</th>
<th>Delay ( N )</th>
<th>Groups ( g )</th>
<th>Minimum ( M ) for non-singularity</th>
<th>Order of Sphere Decoding Complexity</th>
<th>Exponent of ( q^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_N^{\text{erm}} ) (Theorem 1)</td>
<td>( \geq 1 )</td>
<td>( N )</td>
<td>1</td>
<td>( \frac{N}{2} )</td>
<td>((N - M)^+ )</td>
<td>3</td>
</tr>
<tr>
<td>Ren et. al. [9]</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>( \frac{N}{2} )</td>
<td>((4 - M)^+ )</td>
<td>3</td>
</tr>
<tr>
<td>Ren et. al. [11]</td>
<td>( \geq 1 )</td>
<td>( \geq 2N )</td>
<td>2</td>
<td>( N - \frac{N^2 - 1}{2} )</td>
<td>((N - M)^+ (T - N - M) )</td>
<td>3</td>
</tr>
<tr>
<td>Srinath et. al. [13]</td>
<td>( 2^m, m \geq 2 )</td>
<td>( N )</td>
<td>2</td>
<td>( \frac{N}{2} + \frac{1}{2} )</td>
<td>( \left( \frac{N}{4g^2} \right)^2 )</td>
<td>2</td>
</tr>
<tr>
<td>Natarajan et. al. [13]</td>
<td>( n g 2^{\frac{N}{g-1}}, n \geq 1 )</td>
<td>( 2 \ell, \ell \geq 1 )</td>
<td>( \frac{N}{g^2} + \frac{g^2 - 1}{2N} )</td>
<td>( \frac{N}{g^2} )</td>
<td>( \left( \frac{N}{g^2} - 2\ell \right)M ) ( \left( \frac{N}{g^2} - 2\ell \right)M )</td>
<td>2</td>
</tr>
</tbody>
</table>

\( ^1 \) The size of the real constellation used is denoted by \( q \).
\( ^2 \) For any real number \( a \), \( a^+ \) is defined as \( \max \{a, 0\} \).
\( ^3 \) [13] contains codes for all \( q \geq 1 \) and not just even values of \( q \).
\[ \text{rank}(G) = \text{rank}(G_1) + \text{rank}(G_2) \] which equals
\[ K' = 2 \left( \frac{N^2}{4} - \left( \frac{N}{2} - M \right)^2 + 1 \right). \]

Comparing this with \( K = 2 \left( \frac{N^2}{4} + 1 \right) \), we see that the design given in [12] is non-singular only if \( M \geq \frac{N}{2} \approx 2R \). Hence the code is singular for all \( |R| \leq M < \frac{N}{2} \).

Now, the two groups of symbols can be ML decoded independently of each other and hence the sphere decoding complexity of \( X \) is the sum of the sphere decoding complexities of \( X_1 \) and \( X_2 \). The number of symbols in each group is \( \frac{N}{2} \) and the rank of the equivalent channel matrix of each of the designs \( X_k, k = 1, 2, 3 \), is \( \frac{N}{2} \). Thus, the sphere decoding complexity is \( \mathcal{O} \left( \frac{N^2}{4} \right) \) instead of \( \mathcal{O} \left( K - K' \right) \). Hence, multigroup ML decodability reduces the sphere decoding complexity even if the STBC is singular.

C. Two group ML decodable codes from Ren et. al. [11]

In [11], 2-group ML decodable codes for all \( N \geq 1 \) and all even \( T \geq 2N \) were constructed with rate \( R = N - \frac{N^2}{2} + 1 \). The number of symbols per group is \( \frac{N}{2} = T - N^2 + 1 \). In this subsection, we show that the codes of this family are singular for all \( |R| \leq M < N \) receive antennas and that their sphere decoding complexity is \( \mathcal{O} \left( q^{(N-M)} \cdot (T-N-M) \right) \).

Using Proposition 1 it is clear that these codes are non-singular if and only if \( M \geq N \).

**Derivation of sphere decoding complexity:** The structure and derivation of the sphere decoding complexity of these codes is similar to that of the codes from [12], which was discussed in Section III-B. The weight matrices of the designs in [11] are of the form
\[ \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \] where \( V_1, V_2 \in \mathbb{C}^{\frac{N}{2} \times N} \). For all the matrices in the first group, \( V_1 = \pm F_1 \) for some constant matrix \( F_1 \), and for all the matrices in the second group, \( V_2 = \pm F_2 \) for some constant matrix \( F_2 \). The designs constructed in [11] are such that for each \( k = 1, 2 \), the \( V_k \) submatrices of any two weight matrices belonging to different groups are Hurwitz-Radon orthogonal. We consider the case where \( F_k, k = 1, 2 \) are semi-unitary i.e., \( F_k^H F_k = I_N \). We derive the sphere decoding complexity for only the first group. Using a similar argument the complexity for the second group can be derived, and it is same as that of the first group.

Since \( F_2 \) is semi-unitary, there exists a unitary \( T \times T \) matrix \( \tilde{F}_2 \) such that \( \tilde{F}_2^H F_2 = I_N \). Consider the new design \( X' \) obtained by multiplying all the weight matrices of the original design on the left by \( C = \begin{bmatrix} T & 0 \\ 0 & i \tilde{F}_2^H \end{bmatrix} \). Then, the lower submatrix of all the weight matrices of the second group are of the form \( \pm I_N \). Since the lower submatrix of every matrix in the first group is Hurwitz-Radon orthogonal to \( \pm I_N \), the weight matrices in the first group of \( X' \) have the following structure:
\[ \begin{bmatrix} V_1 & B & E \end{bmatrix}, \] where \( V_1 = \pm F_1 \), \( B \) is an \( N \times N \) Hermitian matrix and \( E \in \mathbb{C}^{\frac{N}{2} \times N} \). Let \( B_1, \ldots, B_N \) be any basis for the space of \( N \times N \) Hermitian matrices over \( \mathbb{R} \), \( L = TN - 2N^2 \) and \( E_1, \ldots, E_L \) be
\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1
\end{bmatrix}, \quad
\begin{bmatrix}
i & 0 & \cdots & 0 \\
0 & i & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & i
\end{bmatrix},
\]
\[ \begin{bmatrix}
0 & B_0 \\
0 & 0
\end{bmatrix}, \text{ for } n = 1, \ldots, N^2, \]
\[ \begin{bmatrix}
0 & 0 \\
E_l
\end{bmatrix}, \text{ for } l = 1, \ldots, L. \]

From dimension count, it is clear that the matrices in [3] form a basis for the space spanned by the weight matrices of the first group of \( X' \). For any non-zero channel realization \( H \in \mathbb{C}^{N \times M} \), the subspaces
\[ \mathcal{V}_1 = \langle F_1 H \rangle, \mathcal{V}_2 = \langle B_n H \rangle, n = 1, \ldots, N^2 \text{ and } \mathcal{V}_3 = \langle E_l H \rangle, l = 1, \ldots, L \] are such that their pairwise intersections contain only the all zero matrix. Thus, the rank of the equivalent channel matrix of the first group of \( X' \) is
\[ \text{rank}(G_1') = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2) + \dim(\mathcal{V}_3). \]

Without probability 1, \( \dim(\mathcal{V}_1) = 1 \) and from Theorem 1 \( \dim(\mathcal{V}_2) = f(N, M) \). It is straightforward to show that \( \dim(\mathcal{V}_3) = (T - 2N^2) \cdot \min\{N, M\} \) with probability 1. From Proposition 3 we know that every design is non-singular for \( M \geq N \). We thus consider only the case \( M < N \). Then from Proposition 3 the rank of the equivalent channel matrix of the first ML decoding group of the design given in [11] is
\[ \text{rank}(G_1) = \text{rank}(G_1') = (T - 2N) \cdot M + f(N, M) + 1. \]

Compare this with the number of symbols in the first group \( \frac{N}{2} = (T - 2N^2) \cdot N + N^2 + 1 \). Thus, for any \( M < N \), \( \text{rank}(G_1) < \frac{N}{2} \) with probability 1.

Using a similar argument it can be shown that the rank of the equivalent channel matrix of the second group also equals \((T - 2N) \cdot M + f(N, M) + 1\) with probability 1. Since, the two groups can be ML decoded independently of each other, the complexity of sphere decoding the STBC proposed in [11]
is $O(q^{(N-M)(T-N-M)})$ for any $M < N$ and the STBC is singular for all $|R| \leq M < N$.

Example 3: Consider the STBC from (11) for $N = 4$ and $T = 2N = 8$. The rate of this code is $R = \frac{12}{6}$ and the number of symbols per decoding group is $\frac{N}{2} = 17$. From the above discussion it is clear that this STBC is singular for $M = 3$. The rank of the equivalent channel matrix of each ML decoding group equals 16 with probability 1 and hence the sphere decoding complexity is $O(q)$.

It is interesting to compare this code with the code from (9) which we have discussed in Section III-A and Example 1. Both codes have the same parameters $N$, $R$ and both have a sphere decoding complexity that is linear in the constellation size $q$. However, the code in (9) is fast-group-decodable and 5 levels can be removed from the sphere decoding search tree of the second decoding group. Hence, after conditioning on the value taken by one of the real symbols (to account for the reduced rank of the equivalent channel matrix), the code in (9) uses a 10-dimensional sphere decoder to decode the second group of 16 symbols. For decoding each ML decoding group, the code from (11) uses a 16 dimensional search tree after conditioning on the value of one of the real symbols. Thus, the sphere decoding complexity of the code from (9) is less than that of (11).

D. Multigroup ML decodable codes from Natarajan et. al [13]

In [13], delay optimal $g$-group ML decodable codes were constructed for all $g > 1$, $N = ng2^{[\frac{n-1}{2}]}$, $n \geq 1$, with rate $R = \frac{N}{g2^{\frac{n-1}{2}}} + \frac{g^2-2}{2N}$. In this subsection we show that the sphere decoding complexity of the codes with even $g$ is of the order of $q\left(\left(2^{-\frac{n-1}{2}}M\right)^{\frac{n-1}{2}}\right)$ and that the codes are non-singular only for $M = \frac{N}{g2^{\frac{n-1}{2}}}$. Also in [13], non-delay optimal codes with $T = gN$, $N = n2^{[\frac{n-1}{2}]}$, $n \geq 1$ were constructed. We show that the sphere decoding complexity of these codes for even values of $g$ is of the order of $q\left(\left(2^{-\frac{n-1}{2}}M\right)^{\frac{n-1}{2}}\right)$.

Simulation results show that the STBCs in [13] for odd values of $g$ are also singular for certain number of receive antennas.

Delay-optimal codes: The number of symbols per constellation is $\frac{N}{g} = n^2 + g - 1$ and let $m = 2^{[\frac{n-1}{2}]}$. We will now derive the rank of the equivalent channel matrix $G_1$ of the design corresponding to the first group. The weight matrices of the first group have a block diagonal structure $diag(D_1, D_2, \ldots, D_g)$, where each $D_j \in \mathbb{C}^{n \times m}$. The first block $D_1$ is one of the $n^2$ matrices of the form $V \otimes U_1$, where $V \in \mathbb{C}^{n \times n}$ is Hermitian, and the remaining $g - 1$ matrices $D_j$, $j = 2, \ldots, g$ are of the form $\pm I_n \otimes U_j$ for some set of $g$ unitary $m \times m$ matrices $U_1, \ldots, U_g$. Let us multiply all the weight matrices of the first group on the right by $C = diag(I_n \otimes U_1^H, \ldots, I_n \otimes U_g^H)$. Clearly, the new set of weight matrices $A_1', \ldots, A_g'$ also have a block diagonal structure $diag(D_1', D_2', \ldots, D_g')$ where $D_1'$ is one of the $n^2$ matrices of the form $V \otimes I_n$, where $V$ is $n \times n$ Hermitian and the remaining $g - 1$ blocks are of the form $\pm I_n \otimes I_m$. It is straightforward to show that there exists a permutation matrix $P$ such that $P \cdot (V \otimes I_m) \cdot P^T = I_{mn} \otimes V$ for any $V \in \mathbb{C}^{n \times n}$. Consider the matrices $A_j' = C'A_j'C'^T$, $j = 1, \ldots, k$, where $C' = diag(P, I_{nm}, \ldots, I_{nm})$. Let $B_1, \ldots, B_n$ be any basis for the space of $n \times n$ Hermitian matrices. From dimension count and the structure of the $A_j'$ matrices, $\langle A_1', \ldots, A_g' \rangle$ equals the span of the matrices $A_1, \ldots, A_g$ which given by $diag(I_m \otimes B_1, 0, \ldots, 0)$, $l = 1, \ldots, n^2$, $diag(I_{nm}, 0, 0, \ldots, 0)$, $l = 1, \ldots, n^2$. Since $C'^T$ is unitary, the statistics of $H$ and $C'H$ are same. Along with Propositions 2 and 3 it is thus clear that $\text{rank}(G_1)$ has the same statistics as the rank of the equivalent channel matrix $G_1$ of the design whose weight matrices are $A_1, \ldots, A_g$.

Let the channel realization be $H = [H_1^T \cdots H_g^T]^T$, where $H_k \in \mathbb{C}^{nm \times M}$ for $j = 1, \ldots, m$. We are interested in the dimension of $\langle A_1H_1, \ldots, A_gH \rangle$ which is the span of

$$\begin{bmatrix} (I_m \otimes B_1) H_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{for } l = 1, \ldots, n^2, \begin{bmatrix} 0 \\ H_2 \\ \vdots \\ H_g \end{bmatrix}.$$  

Thus, with probability 1, $\text{rank}(G_1)$ equals the sum of $g-1$ and the dimension of the span of the first $n^2$ matrices in (9). Let us rewrite $H_1$ as $[H_{1,1} \cdots H_{1,m}]^T$, where $H_{1,j} \in \mathbb{C}^{nm \times M}$ for $j = 1, \ldots, m$. Thus, the dimension of the span of first $n^2$ matrices in (9) is same as that of

$$\begin{bmatrix} B_1 H_{1,1} & B_2 H_{1,1} & B_3 H_{1,1} \\ B_1 H_{1,2} & B_2 H_{1,2} & B_3 H_{1,2} \\ \vdots & \vdots & \vdots \\ B_1 H_{1,m} & B_2 H_{1,m} & B_3 H_{1,m} \end{bmatrix}.$$  

This in turn, is equal to the dimension of the span of the following matrices

$$[B_1 H_{1,1}, B_2 H_{1,1}, \ldots, B_3 H_{1,1}],$$

$$[B_2 H_{1,1}, B_2 H_{1,2}, \ldots, B_3 H_{1,2}],$$

$$\vdots$$

$$[B_n H_{1,1}, B_n H_{1,2}, \ldots, B_n H_{1,m}].$$

From Theorem 3 this dimension equals $f(n,mM)$ with probability 1. Hence,

$$\text{rank}(G_1) = \text{rank}(\hat{G}_1) = f(n,mM) + g - 1$$

with probability 1. Compare this with the number of symbols per group $k_g = n^2 + g - 1$. Thus, the delay optimal designs of [13] for even values of $g$ are singular whenever $|R| \leq \frac{n}{2^{[\frac{n-1}{2}]}}$. Similar results on the rank of the equivalent channel matrix can be proved for other ML decoding groups also. Thus, the sphere decoding complexity of these codes is of the order of $q\left(\left(2^{-\frac{n-1}{2}}M\right)^{\frac{n-1}{2}}\right)$. This design is singular only for $M \geq \frac{n}{2^{[\frac{n-1}{2}]}} \approx 2R$. For $g = 2$, and equal values of $N$, the delay-optimal codes of [13] and the codes
of $[12]$ have equal rate and the same order of sphere decoding complexity.

**Example 4:** Consider the $g = 2$, $N = 6$ delay-optimal code of $[13]$. The underlying design has rate $R = \frac{N}{2g}$ and there are $\frac{N}{2g} = 10$ symbols per ML decoding group. From the ongoing discussion, for $M = 2$ receive antennas rank($G_k$) = 9 with probability 1 for $k = 1, 2$. Hence, this design is singular for $M = 2$ and the sphere decoding complexity is $O(q)$. Let us denote the weight matrices of $X^1_{\text{herm}}$ given in Example 2 by $A_1, \ldots, A_6$. Then, the 20 weight matrices of the $N = 6, g = 2$ code of $[13]$ are given by

$$A_\ell = \begin{bmatrix} iA_\ell' & 0 \\ 0 & I_3 \end{bmatrix}, \ell = 1, \ldots, 9, \quad A_{10} = \begin{bmatrix} iA_1' & 0 \\ 0 & -I_3 \end{bmatrix},$$

$$A_{\ell+10} = \begin{bmatrix} I_3 & 0 \\ 0 & iA_\ell' \end{bmatrix}, \ell = 1, \ldots, 9, \quad A_{20} = \begin{bmatrix} -I_3 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Now consider the $24 \times 20$ $G$ matrix corresponding to the randomly generated channel realization

$$H = \begin{bmatrix} -0.0583 + 1.2105i & 0.0708 + 0.6795i \\ -1.3669 - 0.1373i & -0.3850 + 0.0877i \\ -0.3104 - 1.5120i & 0.2146 + 1.0159i \\ -0.2690 - 0.5937i & -0.4245 - 1.3866i \\ 0.5942 + 0.9578i & 0.3465 - 0.1398i \\ -0.6279 - 0.7584i & 0.5228 - 0.8541i \end{bmatrix}.$$  

The rank of $G$ is only 18 and the structure of the $24 \times 20$ upper triangular matrix $R'$ obtained from the QR decomposition is given in (10) at the top of the next page. The matrix $R'$ is of the form

$$\begin{bmatrix} R_1 & 0_{10 \times 10} \\ 0_{10 \times 10} & R_2 \end{bmatrix},$$

where $R_1, R_2 \in \mathbb{R}^{9 \times 10}$. The $10 \times 10$ all zero submatrix at the upper right corner of $R'$ is due to the 2-group ML decoding property of the code. The two sphere decoders corresponding to the two ML decoding groups use the matrices $R_1$ and $R_2$ respectively. Clearly the rank of $R_1$ and $R_2$ is 9, and hence the sphere decoding complexity is $O(q)$.

**Non delay-optimal codes:** The non delay-optimal code for $N = n^2(\frac{2g}{g})$ has rate $R = \frac{N}{2g^2} + \frac{1}{2} - \frac{1}{2N}$ and number of symbols per group $\frac{N}{2g} = n^2 + g - 1$. The weight matrices of the first group are of the form $[D_1^T, D_2^T, \ldots, D_g^T]^T$, where each $D_j \in \mathbb{C}^{m \times mn}$. The first block $D_1$ is one of the $n^2$ matrices of the form $V \otimes U_1$, where $V \in \mathbb{C}^{m \times n}$ is Hermitian, and the remaining $g - 1$ matrices $D_j, j = 2, \ldots, g$ are of the form $\pm I_{n^2g} \otimes U_j$ for some set of $g$ unitary $m \times m$ matrices $U_1, \ldots, U_g$.

Using an argument similar to the one used with delay-optimal codes, it can be shown that the sphere decoding complexity of the non delay optimal codes is of the order of

$$q\left(\frac{N}{n^2} - \frac{1}{2} + \frac{1}{M}\right),$$

and that the codes are non-singular only for $M \geq \frac{N}{n^2} \approx 2R$. For $g = 2$ and equal values of $N$ and $T$, the non-delay optimal codes of $[13]$ and the codes of $[11]$ have the same rate and sphere decoding complexity.

IV. Discussion

In this paper we have introduced the notion of singularity of designs and showed that all known families of high rate multigroup ML decodable codes are singular for certain numbers of receive antennas. The following facts which were not known before have been shown:

- Though the $N = 4$, $T = 4$ code of $[9]$ and the $N = 4$, $T = 8$ code of $[11]$ have identical rate of $\frac{N}{2g}$ cspcu, the sphere decoding complexity of the code from $[9]$ is less than that of the code from $[11]$.
- For $g = 2$, and equal values of $N$, the delay-optimal codes of $[13]$ and the codes of $[12]$ have equal rate and the same order of sphere decoding complexity.
- For equal values of $N$ and $T$, the codes in $[11]$ and the non-delay optimal codes of $[13]$ have identical rate and sphere decoding complexities.

The results and ideas presented in this paper have brought to light the following important open problems:

- Is there an algebraic criterion that ensures that a code is non-singular? For example, is every code with non vanishing determinants also non singular for all $M \geq R$ receive antennas?
- Do there exist high rate multigroup ML decodable codes that are non-singular?
- Do there exist singular high rate multigroup ML decodable STBCs with lower sphere decoding complexity than that of the known codes?

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Now, let \( H \) be the channel realization, \( H \) ∈ \( \mathbb{C}^{N \times M} \), and let the space \( \mathcal{S} \) be the vector space of matrices \( \mathcal{S} = \{ \mathbf{S} \mid \mathbf{S} \text{ is of rank } 1 \} \). Thus, \( \mathcal{S} \) is equal to \( \ker(\varphi) \), where \( \varphi : \mathcal{S} \rightarrow \mathbb{R} \) is the vector space homomorphism that sends \( \mathbf{S} \) to the real number \( \det(\mathbf{S}) \).

Now, \( \det(\mathbf{S}) \) is either 0 or 1. Suppose, \( \det(\mathbf{S}) = 0 \), then there is no vector \( z \) in \( \mathcal{S} \) such that \( z_1 \) is non-zero. Because if such a \( z \) exists, then the vector \( iz_1^* \cdot z \) belongs to \( \mathcal{S} \) and the imaginary part of its first component is \( |z_1|^2 \neq 0 \), and thus \( \det(\mathbf{S}) = 1 \), which is a contradiction. Since, for all the vectors in \( \mathcal{S} \), the first component is 0, we have

\[
\mathcal{S} = \{ z \mid z^H \mathbf{H} = 0 \} = \{ z \mid z_1^H [H e_1] = 0 \}.
\]

Thus, the dimension of the column space of \( \mathbf{H} \) and \( [H e_1] \) are the same. This means that \( e_1 \) belongs to the column space of \( \mathbf{H} \). Since, \( e_1 \) belongs to the column space of \( \mathbf{H} \) w.p. 0, we conclude that \( \det(\mathbf{S}) = 0 \) w.p. 0. Thus, \( \det(\mathbf{S}) = 1 \) w.p. 1. Further, since \( \mathcal{S} / \ker(\varphi) \) is isomorphic to \( \varphi(\mathcal{S}) \), \( \det(\mathcal{S}) = 2(N - M) - 1 \) w.p. 1.

Let the weight matrices of the design \( X_{2,2}^{\text{Herm}} \) be \( A_1, \ldots, A_{N_2} \) and let the space of of \( N \times N \) Hermitian matrices over \( \mathbb{R} \) be given by \( \mathcal{U} = \{ A_1, \ldots, A_{N_2} \} \). For a given channel realization \( \mathbf{H} \), let \( \rho : \mathcal{U} \rightarrow \mathbb{C}^{N \times M} \) be the \( \mathbb{R} \)-vector
space homomorphism that sends the matrix $A$ to $AH$. Clearly, the rank of the $G$ matrix is equal to the dimension of the subspace $\rho(U)$ over $\mathbb{R}$. Since $\rho(U)$ is isomorphic to $U/\ker(\rho)$ as vector spaces, we have

$$\text{rank}(G) = \text{dim}(U) - \text{dim}(\ker(\rho)) = N^2 - \text{dim}(\ker(\rho)).$$

Thus, it is enough to show that $\dim(\ker(\rho)) = (N - M)^2$ w.p. 1.

Let $A \in \ker(\rho)$ and let $a_1^H, \ldots, a_N^H$ denote the $N$ rows of $A$. Then, $a_1$ satisfies $a_1^H a_1 = 0$, and since the $A$ is Hermitian, the first component of $a_1$ is purely real. Thus, $a_1 \in \ker(\varphi)$ whose dimension is $2(N - M) - 1$ w.p. 1. Given a choice of $a_1$, since $A$ is Hermitian, the first component of $a_2$ equals the second component of $a_1$, and the second component of $a_2$ is purely real. Because of these restrictions and since $a_2^H a_2 = 0$, $a_2$ belongs to coset of the subspace of $S$ whose dimension is $2(N - M) - 1 = 2(N - M) - 3$. Similarly, given a choice for $a_1, \ldots, a_{k-1}$, where $k = 1, \ldots, N$, the first $k - 1$ components of $a_k$ are fixed and the imaginary part of the $k^{th}$ component is zero. Hence, $a_k$ belongs to a coset of a subspace of $S$ with dimension $2(N - M) - 2k - 1$. Thus, the dimension of $\ker(\rho)$ equals

$$2(N - M) - 1 + 2(N - M) - 3 + \cdots + 1 = (N - M)^2,$$

w.p. 1. This completes the proof.