Generalized Silver Codes

K. Pavan Srinath and B. Sundar Rajan,
Dept of ECE, Indian Institute of science,
Bangalore 560012, India
Email:{pavan,bsrajan}@ece.iisc.ernet.in

Abstract

For an \( n_t \) transmit, \( n_r \) receive antenna system \((n_t \times n_r \text{ system})\), a full-rate space time block code (STBC) transmits \( n_{\min} = \min(n_t, n_r) \) complex symbols per channel use. The well known Golden code is an example of a full-rate, full-diversity STBC for 2 transmit antennas. Its ML-decoding complexity is of the order of \( M^{2.5} \) for square \( M\)-QAM. The Silver code for 2 transmit antennas has all the desirable properties of the Golden code except its coding gain, but offers a lower ML-decoding complexity of the order of \( M^2 \). Importantly, the slight loss in coding gain is negligible compared to the advantage it offers in terms of lowering the ML-decoding complexity. For higher number of transmit antennas, the best known codes are the Perfect codes, which are full-rate, full-diversity, information lossless codes (for \( n_r \geq n_t \)) and are known to have a high ML-decoding complexity of the order of \( M^{n_t n_{\min}} \) (for \( n_r < n_t \), the punctured Perfect codes are considered). In this paper, a scheme to obtain full-rate STBCs for \( 2^a \) transmit antennas and any \( n_r \), with reduced ML-decoding complexity of the order of \( M^{n_t(n_{\min} - \frac{3}{4}) - 0.5} \), is presented. The codes constructed are also information lossless for \( n_r \geq n_t \), like the Perfect codes and have higher ergodic capacity than the comparable punctured Perfect codes for \( n_r < n_t \). These codes are referred to as the generalized Silver codes, since they enjoy the same desirable properties as the comparable Perfect codes (except possibly the coding gain) with lower ML-decoding complexity, analogous to the Silver-Golden codes for 2 transmit antennas. Simulation results of the symbol error rates for 4 and 8 transmit antennas show that with a suitably chosen constellation, the generalized Silver codes match the punctured Perfect codes in error performance, while offering lower ML-decoding complexity.

Index Terms

Low ML-decoding complexity, ergodic capacity, full-rate space-time block codes, anticommuting matrices, information losslessness.

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I. INTRODUCTION AND BACKGROUND

Complex orthogonal designs (CODs) [1], [2], although provide linear maximum-Likelihood (ML)-decoding, do not offer a high rate of transmission. A full-rate code for an $n_t \times n_r$ MIMO system transmits $\min(n_t, n_r)$ independent complex symbols per channel use. Among the CODs, only the Alamouti code for 2 transmit antennas is full-rate for a $2 \times 1$ MIMO system. A full-rate STBC can efficiently utilize all the degrees of freedom the channel provides. In general, an increase in the rate tends to result in an increase in the ML-decoding complexity. The Golden code [3] for 2 transmit antennas is an example of a full-rate STBC for any number of receive antennas. Until recently, the ML-decoding complexity of the Golden code was reported to be of the order of $M^4$, where $M$ is the size of the signal constellation. However, it was shown in [4], [5] that the Golden code has a decoding complexity of the order of $M^{2.5}$ for square $M$-QAM. Current research focuses on obtaining high rate codes with reduced ML-decoding complexity (refer to Sec. II for a formal definition). For 2 transmit antennas, the Silver code [6], [7], is a full-rate code with full-diversity and an ML-decoding complexity of the order of $M^2$ for square $M$-QAM. For 4 transmit antennas, Biglieri et. al. proposed a rate-2 STBC which has an ML-decoding complexity of the order of $M^{4.5}$ for square $M$-QAM without full-diversity [8]. It was, however, shown that there was no significant reduction in error performance at low to medium SNR when compared with the previously best known code - the DjABBA code [6]. This code was obtained by multiplexing Quasi-orthogonal designs (QOD) for 4 transmit antennas [9]. In [4], a new full-rate STBC for $4 \times 2$ system with full diversity and an ML-decoding complexity of $M^{4.5}$ was proposed. This code was obtained by multiplexing the coordinate interleaved orthogonal designs (CIODs) for 4 transmit antennas [10]. These results show that codes obtained by multiplexing low complexity STBCs can result in high rate STBCs with reduced ML-decoding complexity and by choosing a suitable constellation, there won’t be any significant degradation in the error performance when compared with the best existing STBCs. Such an approach has also been adopted in [11] to obtain high rate codes from multiplexed orthogonal designs.

In general, it is not known how one can design full-rate STBCs for an arbitrary number of transmit and receive antennas with reduced ML-decoding complexity. It is well known that the ergodic capacity of the MIMO channel with the use of an STBC is at best equal to the ergodic capacity of the MIMO channel without the use of space time coding, in which case the STBC...
is said to be information lossless (see Section II for a formal definition). It is known how to
design information lossless codes [13] for the case where \( n_r \geq n_t \). However, when \( n_r < n_t \)
the only known code in literature which is information lossless is the Alamouti code, which is
information lossless for the \( 2 \times 1 \) system alone. Not much research has been done on designing
codes with a high ergodic capacity for MIMO systems where \( n_r < n_t \). In this paper, we study the
properties of the STBC which enhance the ergodic capacity of the MIMO channel with the use
of space time coding (note that this ergodic capacity can never beat the ergodic capacity of the
MIMO channel without space time coding). Further we design codes which have higher ergodic
capacity at high signal-to-noise ratio (\( SNR \)) than the best existing codes (the Perfect codes with
puncturing [14], [15]) for \( n_r < n_t \), while for \( n_r \geq n_t \), the proposed STBCs are information
lossless, like the comparable Perfect codes. We call these codes the \textit{generalized Silver codes},
since, analogous to the silver code and the Golden code for 2 transmit antennas, the proposed
codes have every desirable property that the Perfect codes have, except the coding gain, but
importantly, have lower ML-decoding complexity than the Perfect codes. The contributions of
the paper are:

1) We analyze the ergodic capacity of MIMO channels with space time codes when \( n_r < n_t \). We relate the entries of the \( R \)-matrix (the upper triangular matrix obtained on QR
decomposition of the equivalent channel matrix) to ergodic capacity at high \( SNR \).

2) We give a scheme to obtain rate-1, 4-group decodable codes (refer Section II for a formal
definition of multi-group decodable codes) for \( n_t = 2^a \) through algebraic methods. The
speciality of the obtained design is that it is amenable for extension for higher number
of receive antennas, resulting in full-rate codes with reduced ML-decoding complexity
for any number of receive antennas, unlike the previous constructions [16]-[18] of rate-1,
4-group decodable codes.

3) Using the rate-1, 4-group decodable codes thus constructed, we propose a scheme to
obtain the generalized Silver codes, which are full-rate codes with reduced ML-decoding
complexity for \( 2^a \) transmit antennas and any number of receive antennas. These codes
are also shown to have higher ergodic capacity than the comparable punctured Perfect
codes for the case \( n_r < n_t \), and lower ML-decoding complexity as well. In terms of error
performance, by choosing the signal constellation carefully, the proposed codes have more
or less the same performance as the corresponding punctured Perfect codes. This is shown through simulation results for 4 and 8 transmit antenna systems.

The paper is organized as follows. In Section II we present the system model and the relevant definitions. The ergodic capacity analysis is presented in Section III and our method to construct rate-1, 4-group decodable codes is proposed in Section IV. The scheme to extend these codes to obtain the generalized Silver codes for higher number of receive antennas is presented in Section V. Simulation results are discussed in Section VI and the concluding remarks are made in Section VII.

Notations: Throughout, bold, lowercase letters are used to denote vectors and bold, uppercase letters are used to denote matrices. Let $X$ be a complex matrix. Then, $X^H$ and $X^T$ denote the Hermitian and the transpose of $X$, respectively and $j$ represents $\sqrt{-1}$. The $(i, j)^{th}$ entry of $X$ is denoted by $X(i, j)$, while $tr(X)$ and $det(X)$ denote the trace and determinant of $X$, respectively. The set of all real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The real and the imaginary part of a complex number $x$ are denoted by $x_I$ and $x_Q$, respectively. $\|X\|$ denotes the Frobenius norm of $X$, while $\|x\|$ denotes the vector norm of a vector $x$, and $I_T$ and $O_T$ denote the $T \times T$ identity matrix and the null matrix, respectively. The Kronecker product is denoted by $\otimes$ and $vec(X)$ denotes the concatenation of the columns of $X$ one below the other. For a complex random variable $X$, $\mathcal{E}[X]$ denotes the mean of $X$ and $\mathcal{E}_X(f(X))$ denotes the mean of $f(X)$, a function of the random variable $X$. The inner product of two vectors $x$ and $y$ is denoted by $\langle x, y \rangle$. Let $S$ denote a set. Then $aS \triangleq \{as|s \in S\}$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets such that $\mathcal{P} \supset \mathcal{Q}$. Then $\mathcal{P} / \mathcal{Q}$ denotes the set of elements of $\mathcal{P}$ excluding the elements of $\mathcal{Q}$. For a complex variable $x$, the $\hat{\cdot}$ operator acting on $x$ is defined as

$$\hat{x} \triangleq \begin{bmatrix} x_I & -x_Q \\ x_Q & x_I \end{bmatrix}.$$ 

The $\hat{\cdot}$ can similarly be applied to any matrix $X \in \mathbb{C}^{n \times m}$ by replacing each entry $x_{ij}$ by $\hat{x}_{ij}$, $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$, resulting in a matrix denoted by $\hat{X} \in \mathbb{R}^{2n \times 2m}$. Given a complex vector $x = [x_1, x_2, \cdots, x_n]^T$, $\hat{x}$ is defined as

$$\hat{x} \triangleq [x_{1I}, x_{1Q}, \cdots, x_{nI}, x_{nQ}]^T.$$
It follows that for $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$ and $C = AB$,

$$
\tilde{C} = \tilde{A}\tilde{B} \\
\tilde{\text{vec}}(C) = (I_p \otimes \tilde{A})\tilde{\text{vec}}(B)
$$

II. System Model

We consider the Rayleigh block fading MIMO channel with full channel state information (CSI) at the receiver but not at the transmitter. For $n_t \times n_r$ MIMO transmission, we have

$$
Y = \sqrt{\frac{\text{SNR}}{n_t}}HS + N, \quad (1)
$$

where $S \in \mathbb{C}^{n_t \times T}$ is the codeword matrix whose average energy is given by $\mathcal{E}(\|S\|^2) = n_tT$, transmitted over $T$ channel uses, $N \in \mathbb{C}^{n_r \times T}$ is a complex white Gaussian noise matrix with i.i.d entries $\sim \mathcal{N}_C(0, 1)$, $H \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix with the entries assumed to be i.i.d circularly symmetric Gaussian random variables $\sim \mathcal{N}_C(0, 1)$, $Y \in \mathbb{C}^{n_r \times T}$ is the received matrix and $\text{SNR}$ is the signal-to-noise ratio at each receive antenna.

**Definition 1:** (Code rate) Code rate is the average number of independent information symbols transmitted per channel use. If there are $k$ independent complex information symbols (or $2k$ real information symbols) in the codeword which are transmitted over $T$ channel uses, then, the code rate is $k/T$ complex symbols per channel use ($2k/T$ real symbols per channel use).

**Definition 2:** (Full-rate STBCs) For an $n_t \times n_r$ MIMO system, if the code rate is $\min(n_t, n_r)$ complex symbols per channel use, then the STBC is said to be full-rate.

Assuming ML-decoding, the ML-decoding metric that is to be minimized over all possible values of codewords $S$ is given by

$$
M(S) = \|Y - \sqrt{\frac{\text{SNR}}{n_t}}HS\|^2.
$$

**Definition 3:** (ML-Decoding complexity) The ML decoding complexity is measured in terms of the maximum number of symbols that need to be jointly decoded in minimizing the ML decoding metric.

For example, if the codeword transmits $k$ independent symbols of which a maximum of $p$ symbols need to be jointly decoded, the ML-decoding complexity is of the order of $M^p$, where
$M$ is the size of the signal constellation. If the code has an ML-decoding complexity of order less than $M^k$, the code is said to admit reduced ML-decoding.

**Definition 4:** (Generator matrix) For any STBC that encodes $2k$ real symbols (or $k$ complex information symbols), the generator matrix $G \in \mathbb{R}^{2Tn_1 \times 2k}$ is defined by [8]

$$\overrightarrow{vec}(S) = Gs,$$

where $S$ is the codeword matrix, $s \triangleq [s_1, s_2, \cdots, s_{2k}]^T$ is the real information symbol vector.

A codeword matrix of an STBC can be expressed in terms of weight matrices (linear dispersion matrices) [19] as

$$S = \sum_{i=1}^{2k} s_i A_i.$$

Here, $A_i, i = 1, 2, \cdots, 2k$ are the complex weight matrices for the STBC and should form a linearly independent set over $\mathbb{R}$. It follows that

$$G = \begin{bmatrix} \overrightarrow{vec}(A_1) & \overrightarrow{vec}(A_2) & \cdots & \overrightarrow{vec}(A_{2k}) \end{bmatrix}.$$

**Definition 5:** (Multi-group decodable STBCs) An STBC is said to be $g$-group decodable [18] if its weight matrices can be separated into $g$ groups $G_1, G_2, \cdots, G_g$ such that

$$A_i A_j^H + A_j A_i^H = O_{n_1}, \quad A_i \in G_l, \quad A_j \in G_p, \quad l, p \in \{1, 2, \cdots, g\}, \quad l \neq p.$$

**Definition 6:** (Punctured Codes) Punctured STBCs are the codes with some of the symbols being zeros, in order to meet the full-rate criterion.

For example, a codeword of the Perfect code for 4 transmit antennas [14] transmits sixteen complex symbols in four channel uses and has a rate of 4 complex symbols per channel use. If this code were to be used for a two receive antenna system, which can only support a rate of two independent complex symbols per channel use, then, eight symbols of the Perfect code can be made zeros, so that the codeword transmits eight complex symbols in four channel uses. These eight symbols correspond to the two layers [14] of the Perfect code.

Equation (1) can be rewritten as

$$\overrightarrow{vec}(Y) = \sqrt{\frac{SNR}{n_t}} H_{eq}s + \overrightarrow{vec}(N),$$
where \( H_{eq} \in \mathbb{R}^{2n_rT \times 2n_{min}T} \), called the equivalent channel matrix, is given by

\[
H_{eq} = (I_T \otimes \tilde{H}) G,
\]

with \( G \in \mathbb{R}^{2n_tT \times 2n_{min}T} \) being the generator matrix as in Definition 4.

**Definition 7: (Ergodic capacity)** The ergodic capacity of an \( n_t \times n_r \) MIMO channel is [20]

\[
C_{n_t \times n_r} = E[H] \left( \log \det \left( I_{n_r} + \frac{SNR}{n_t} HH^H \right) \right).
\]

With the use of an STBC, the ergodic capacity is [21]

\[
C_{STBC} = \frac{1}{2T} E[H] \left( \log \det \left( I_{2n_rT} + \frac{SNR}{n_t} H_{eq} H_{eq}^T \right) \right).
\]

It is known that \( C_{n_t \times n_r} \geq C_{STBC} \). If \( C_{n_t \times n_r} = C_{STBC} \), the STBC is said to be information lossless. If the generator matrix \( G \) is orthogonal (from Definition 4, this case arises only if \( n_r \geq n_t \) and the STBC is full-rate, i.e, \( k = n_tT \)), the STBC is information lossless.

### III. Relationship between weight matrices and ergodic capacity

It has been shown that if the generator matrix is orthogonal, the STBC does not reduce the ergodic capacity of the MIMO channel [13], [21]. For the generator matrix to be orthogonal, a prerequisite is that the number of receive antennas should be at least equal to the number of transmit antennas, because only then will the generator matrix be square. When \( n_r < n_t \), only the Alamouti code has been known to be information lossless for the \( 2 \times 1 \) MIMO channel. Since it is difficult to make an exact analysis of the ergodic capacity when \( n_r < n_t \), we make an approximate analysis in the low and high SNR range.

#### A. Low SNR analysis

Let \( H_{eq} H_{eq}^T = UDU^T \) be the singular value decomposition of \( H_{eq} H_{eq}^T \). Let \( D = \text{diag}[d_1, d_2, \ldots, d_{2Tn_r}] \) and \( H_{eq} = [h_1, h_2, \ldots, h_{2Tn_r}] \). We have,

\[
C_{STBC} = \frac{1}{2T} E[H] \left( \log \det \left( I_{2n_rT} + \frac{SNR}{n_t} UDU^T \right) \right).
\]
\[ \begin{align*}
&= \frac{1}{2T} \mathcal{E}_H \left( \log \det \left( \mathbf{I}_{2n_rT} + \frac{\text{SNR}}{n_t} \mathbf{D} \right) \right) \\
&= \frac{1}{2T} \mathcal{E}_H \left( \log \prod_{i=1}^{2Tn_r} \left( 1 + \frac{\text{SNR}}{n_t} d_i \right) \right) \\
&= \frac{1}{2T} \mathcal{E}_H \left( \sum_{i=1}^{2Tn_r} \log \left( 1 + \frac{\text{SNR}}{n_t} d_i \right) \right) \\
&\approx \frac{1}{2T} \mathcal{E}_H \left( \sum_{i=1}^{2Tn_r} \frac{\text{SNR}}{n_t} d_i \right) \\
&= \frac{\text{SNR}}{2n_T} \mathcal{E}_H \left( \text{tr} \left( \mathbf{H}_{eq} \mathbf{H}_{eq}^T \right) \right) \\
&= \frac{\text{SNR}}{2n_T} \mathcal{E}_H \left( \| \mathbf{H}_{eq} \|^2 \right) \\
&= \frac{\text{SNR}}{2n_T} \mathcal{E}_H \left( \sum_{i=1}^{2Tn_r} \| \mathbf{h}_i \|^2 \right). \quad (2)
\end{align*} \]

Since \( \mathbf{h}_i = \text{vec}(\mathbf{HA}_i) = (\mathbf{I}_T \otimes \tilde{\mathbf{H}}) \text{vec}(\mathbf{A}_i) \), we have

\[ \| \mathbf{h}_i \|^2 = \| \mathbf{HA}_i \|^2 = \text{tr}(\mathbf{HA}_i \mathbf{A}_i^H \mathbf{H}^H). \]

If \( \mathbf{A}_i \mathbf{A}_i^H = \frac{1}{n_r} \mathbf{I}_{n_r}, \forall i = 1, 2, \ldots, 2Tn_r \), then,

\[ \| \mathbf{h}_i \|^2 = \frac{1}{n_r} \| \mathbf{H} \|^2, \forall i = 1, 2, \ldots, 2Tn_r. \]

Hence,

\[ C_{\text{STBC}} \approx \frac{\text{SNR}}{n_t} \mathcal{E}_H(\| \mathbf{H} \|^2). \]

By a similar argument as shown to obtain (2), the ergodic capacity of the \( n_t \times n_r \) MIMO channel at low SNR can be approximated as

\[ C_{n_t \times n_r} \approx \frac{\text{SNR}}{n_t} \mathcal{E}_H(\| \mathbf{H} \|^2). \]

Hence, in the low SNR scenario, if \( \mathbf{A}_i \mathbf{A}_i^H = \frac{1}{n_r} \mathbf{I}_{n_r}, \forall i = 1, 2, \ldots, 2Tn_r \), then, \( C_{\text{STBC}} = C_{n_t \times n_r} \).

So, if the weight matrices of a full-rate STBC are scaled unitary matrices, the capacity of the channel with space time coding is equal to the capacity of the MIMO channel without space time coding at low signal-to-noise ratio.
B. High SNR analysis

For this purpose, we use the QR decomposition of $H_{eq}$, with $Q$ and $R$ having the general form obtained by the Gram–Schmidt process as

$$Q \triangleq [q_1 \ q_2 \ q_3 \cdots q_{2Tn_r}],$$

where $q_i, i = 1, 2, \cdots, 2Tn_r$ are column vectors, and

$$R \triangleq \begin{bmatrix} \|r_1\| \langle q_1, h_2 \rangle \langle q_1, h_3 \rangle \cdots \langle q_1, h_{2Tn_r} \rangle \\ 0 \|r_2\| \langle q_2, h_3 \rangle \cdots \langle q_2, h_{2Tn_r} \rangle \\ 0 0 \|r_3\| \cdots \langle q_3, h_{2Tn_r} \rangle \\ \vdots \vdots \vdots \ddots \vdots \\ 0 0 0 \cdots \|r_{2Tn_r}\| \end{bmatrix}$$

where $r_1 = h_1$, $q_1 = \frac{r_1}{\|r_1\|}$, $r_i = h_i - \sum_{j=1}^{i-1} \langle q_j, h_i \rangle q_j$ and $q_i = \frac{r_i}{\|r_i\|}, i = 2, 3, \cdots, 2Tn_r$. We have,

$$C_{STBC} = \frac{1}{2T} \mathcal{E}_H \left( \log \det \left( I_{2Tn_r} + \frac{SNR}{n_t} H_{eq} H_{eq}^T \right) \right)$$

$$= \frac{1}{2T} \mathcal{E}_H \left( \log \det \left( I_{2Tn_r} + \frac{SNR}{n_t} QRR^T Q^T \right) \right)$$

$$= \frac{1}{2T} \mathcal{E}_H \left( \log \det \left( I_{2Tn_r} + \frac{SNR}{n_t} RR^T \right) \right)$$

$$\approx \frac{1}{2T} \mathcal{E}_H \left( \log \det \left( \frac{SNR}{n_t} RR^T \right) \right)$$

$$= n_r \log \left( \frac{SNR}{n_t} \right) + \frac{1}{2T} \mathcal{E}_H \left( \log \det \left( RR^T \right) \right).$$

Using the well known fact that the determinant of a triangular matrix is the product of its diagonal elements, we have

$$C_{STBC} \approx n_r \log \left( \frac{SNR}{n_t} \right) + \frac{1}{2T} \mathcal{E}_H \left( \log \prod_{i=1}^{2Tn_r} R(i,i) \right)$$

$$= n_r \log \left( \frac{SNR}{n_t} \right) + \frac{1}{2T} \mathcal{E}_H \sum_{i=1}^{2Tn_r} \log R(i,i).$$
From the definition of the $R$-matrix, we have,

$$R(i, i)^2 = \|r_i\|^2 = \langle r_i, r_i \rangle$$

$$= \left\langle \left( h_i - \sum_{j=1}^{i-1} \langle q_j, h_i \rangle q_j \right), \left( h_i - \sum_{j=1}^{i-1} \langle q_j, h_i \rangle q_j \right) \right\rangle$$

$$= \|h_i\|^2 - \sum_{j=1}^{i-1} \langle q_j, h_i \rangle^2.$$  

Hence,

$$C_{STBC} \approx n_r \log \left( \frac{SNR}{n_t} \right) + \frac{1}{2T} \sum_{i=1}^{2Tn_r} \mathcal{E}_H \log \left( \|h_i\|^2 - \sum_{j=1}^{i-1} \langle q_j, h_i \rangle^2 \right). \quad (3)$$

Equation (3) tells us that at high SNR, the entries of the $R$-matrix, i.e, $\langle q_j, h_i \rangle$ dictate the ergodic capacity. If the number of zero entries in the upper block of the $R$-matrix is larger, then the ergodic capacity is expected to be higher. Hence, it is essential that the $R$-matrix has as many zeros as possible. This in turn would also reduce the ML-decoding complexity, since a larger number of symbols would be disentangled from one another. Hence, between two full-rate STBCs, the one with lower ML-decoding complexity can be expected to have higher ergodic capacity. The following theorem tells us when one can have zeros in the upper block of the $R$-matrix.

**Theorem 1:** [4] If $A_iA_j^H + A_jA_i^H = 0_{n_t}$, then, the $i^{th}$ and the $j^{th}$ columns of $H_{eq}$ are orthogonal, irrespective of the channel realization.

From the definition of $R$-matrix, if the $i^{th}$ and the $j^{th}$ columns of $H_{eq}$ are orthogonal, with $i < j$, it is possible, though not guaranteed, that $R(i, j) = 0$. For example, the Alamouti code has its weight matrices such that $A_iA_j^H + A_jA_i^H = 0_2$, $i \neq j$, $i, j \in \{1, 2, 3, 4\}$. Hence, its $R$-matrix is diagonal. It is information lossless mainly due to this property. Hence, to design a good STBC with a high ergodic capacity when $n_r < n_t$, the equivalent channel matrix should have as many columns orthogonal to one another as possible. We would, of course, like all the columns to be orthogonal to one another, but there is a limit to the number, the limit being the maximum number of Hurwitz-Radon matrices [1] for $n_t$ transmit antennas. Except for the Alamouti code, this number is much lesser than $2Tn_r$, which is the number of weight matrices of a full-rate STBC when $n_r < n_t$. If we consider $g$-group decodable codes, columns of $H_{eq}$
can be divided into \( g \) groups such that columns in the same group are not orthogonal to one another, but columns from different groups are orthogonal to one another. At present, the best known low complexity multi-group decodable codes are the rate-1, 4-group decodable codes for any number of transmit antennas [16], [17], [18]. These codes are not full-rate for \( n_r > 1 \). If one were to require a full-rate code, the codes in literature [16], [17], [18] are not suitable for extension for higher number of receive antennas, since their design is obtained by iterative methods. In the next section, we propose a new design methodology to obtain the weight matrices of a rate-1, 4-group decodable code by algebraic methods for 2 transmit antennas. These codes can be extended for higher number of receive antennas to obtain full-rate STBCs with lower ML-decoding complexity than existing designs and hence higher ergodic capacity.

IV. CONSTRUCTION OF RATE-1, 4-GROUP DECODABLE CODES

We make use of the following theorem, presented in [17], to construct rate-1, 4-group decodable codes for \( n = 2^a \) transmit antennas.

Theorem 2: [17] An \( n \times n \) linear dispersion code transmitting \( k \) real symbols is \( g \)-group decodable if the weight matrices satisfy the following conditions:

1) \( A_i^2 = I_n, \quad i \in \{1, 2, \cdots, \frac{k}{g}\} \).

2) \( A_j^2 = -I_n, \quad j \in \{\frac{mk}{g} + 1, m = 1, 2, \cdots, g - 1\} \).

3) \( A_iA_j = A_jA_i, \quad i, j \in \{1, 2, \cdots, \frac{k}{g}\} \).

4) \( A_iA_j = A_jA_i, \quad i \in \{1, 2, \cdots, \frac{k}{g}\}, j \in \{\frac{mk}{g} + 1, m = 1, 2, \cdots, g - 1\} \).

5) \( A_iA_j = -A_jA_i, \quad i, j \in \{\frac{mk}{g} + 1, m = 1, 2, \cdots, g - 1\}, i \neq j \).

6) \( A_m A_{\frac{mk}{g} + i} = A_i A_{\frac{mk}{g} + 1}, \quad m \in \{1, 2, \cdots, g - 1\}, \quad i \in \{1, 2, \cdots, \frac{k}{g}\} \).

Table 1 illustrates the weight matrices of a \( g \)-group decodable code which satisfy the above conditions. The weight matrices in each column belong to the same group.

In order to obtain a rate-1, 4-group decodable STBC for \( 2^a \) transmit antennas, it is sufficient if we have \( 2^{a+1} \) matrices satisfying the conditions in Theorem 2. To obtain these, we make use of the following lemmas.

Lemma 1: Consider \( n \times n \) matrices with complex entries. If \( n = 2^a \) and \( n \times n \) matrices \( F_i, i = 1, 2, \cdots, 2a \) anticommute pairwise, then the set of products \( F_{i_1}F_{i_2}\cdots F_{i_s} \) with \( 1 \leq i_1 < \cdots < i_s \leq 2a \) along with \( I_n \) forms a basis for the \( 2^{2a} \) dimensional space of all \( n \times n \) matrices over \( \mathbb{C} \).
A_1 = I_n & A_{k+1} & \cdots & A_{(g-1)k+1} \\
A_2 & A_{k+2} = A_2 A_{k+1} & \cdots & A_{(g-1)k+2} = A_2 A_{(g-1)k+1} \\
\vdots & \vdots & \cdots & \vdots \\
A_k & A_{2k} = A_k A_{k+1} & \cdots & A_k = A_k A_{(g-1)k+1} \\

**Proof:** Available in [22].

**Lemma 2:** If all the mutually anticommuting \( n \times n \) matrices \( F_i, i = 1, 2, \ldots, 2a \) are unitary and anti-Hermitian, so that they square to \(-I_n\), then the product \( F_1 F_2 \cdots F_s \) with \( 1 \leq i_1 < \cdots < i_s \leq 2a \) squares to \((-1)^{s(s+1)/2} I_n\).

**Proof:** We have

\[
(F_1 F_2 \cdots F_s) (F_1 F_2 \cdots F_s) = (-1)^{s-1}(F_1^2 F_2^2 \cdots F_s^2)(F_1 F_2 \cdots F_s)
\]

\[
= (-1)^{s-1}(-1)^{s-2}(F_1 F_2 \cdots F_s^2)(F_1 F_2 \cdots F_s)
\]

\[
= (-1)^{(s-1)+(s-2)+\cdots+1}(F_1^2 F_2^2 \cdots F_s^2)
\]

\[
= (-1)^{s(s-1)/2}(-1)^s I_n
\]

\[
= (-1)^{s(s+1)/2} I_n,
\]

which proves the lemma.

**Lemma 3:** Let \( F_i, i = 1, 2, \ldots, 2a \) be anticommuting, anti-Hermitian, unitary matrices. Let \( \Omega_1 = \{F_i_1, F_i_2, \cdots, F_i_s\} \) and \( \Omega_2 = \{F_j_1, F_j_2, \cdots, F_j_r\} \) with \( 1 \leq i_1 < \cdots < i_s \leq 2a \) and \( 1 \leq j_1 < \cdots < j_r \leq 2a \). Let \(|\Omega_1 \cap \Omega_2| = p\). Then the product matrix \( F_{i_1} F_{i_2} \cdots F_{i_s} \) commutes with \( F_{j_1} F_{j_2} \cdots F_{j_r} \) if exactly one of the following is satisfied, and anticommutes otherwise.

1) \( r, s \) and \( p \) are all odd.
2) The product \( rs \) is even and \( p \) is even (including 0).

**Proof:** For \( F_{j_k} \in \Omega_1 \cap \Omega_2 \), we note that

\[
(F_1 F_2 \cdots F_s) F_{j_k} = (-1)^{s-1} F_{j_k} (F_1 F_2 \cdots F_s)
\]
and

\[(F_{i_1}F_{i_2}\cdots F_{i_s})F_{j_k} = (-1)^s F_{j_k} (F_{i_1}F_{i_2}\cdots F_{i_s})\]

otherwise. Now,

\[(F_{i_1}F_{i_2}\cdots F_{i_s})(F_{j_1}F_{j_2}\cdots F_{j_r}) = (-1)^{(s-1)(r-1)-ps} (F_{j_1}F_{j_2}\cdots F_{j_r})(F_{i_1}F_{i_2}\cdots F_{i_s})\]

\[= (-1)^{rs-p} (F_{j_1}F_{j_2}\cdots F_{j_r})(F_{i_1}F_{i_2}\cdots F_{i_s}).\]

**Case 1.** Since \(r, s\) and \(p\) are all odd, \((-1)^{rs-p} = 1.\)

**Case 2.** The product \(rs\) is even and \(p\) is even (including 0). Hence \((-1)^{rs-p} = 1.\)

From Theorem 2, to get a rate-1, 4-group decodable STBC, we need 3 pairwise anticommuting, anti-Hermitian matrices which commute with a group of \(2^{a-1}\) Hermitian, pairwise commuting matrices. Once these are identified, the other weight matrices can be easily obtained. From [2], one can obtain \(2a\) pairwise anticommuting, anti-Hermitian matrices and the method to obtain these is presented here for completeness.

Let

\[P_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\]

and \(A \otimes^m \triangleq A \otimes A \otimes A \cdots \otimes A.\)

The \(2a\) anti-Hermitian, pairwise anti-commuting matrices are

\[F_1 = \pm j P_3^{\otimes a},\]

\[F_{2k} = I_2^{\otimes a-k} \otimes P_1 \otimes P_3^{\otimes k-1}, \quad k = 1, \cdots, a,\]

\[F_{2k+1} = I_2^{\otimes a-k} \otimes P_2 \otimes P_3^{\otimes k-1}, \quad k = 1, \cdots, a - 1.\]

Henceforth, \(F_i, i = 1, 2, \cdots, 2a,\) refer to the matrices obtained using the above method.

For a set \(S = \{a_1, a_2, \cdots, a_n\},\) define \(P(S)\) as

\[P(S) \triangleq \{a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}, \lambda_i \in \{0, 1\}\}.\]

We choose \(F_1, F_2\) and \(F_3\) to be the three pairwise anticommuting, anti-Hermitian matrices (to be placed in the top row along with \(I_n\) in Table I). Consider the set \(S = \{jF_4F_5, jF_6F_7,\)
\[ \cdots, jF_{2a-2}F_{2a-1}, F_1F_2F_3 \}, \] the cardinality of which is \( a - 1 \). Using Lemma 2 and Lemma 3, one can note that \( S \) consists of pairwise commuting matrices which are Hermitian. Moreover, it is clear that each of the matrices in the set also commutes with \( F_1, F_2 \) and \( F_3 \). Hence, \( \mathbb{P}(S) \), which has cardinality \( 2^{a-1} \) is also a set with pairwise commuting, Hermitian matrices which also commute with \( F_1, F_2 \) and \( F_3 \). The linear independence of \( \mathbb{P}(S) \) over \( \mathbb{R} \) is easy to see by applying Lemma 1. Hence, we have 3 pairwise anticommuting, anti-Hermitian matrices which commute with a group of \( 2^{a-1} \) Hermitian, pairwise commuting matrices. Having obtained these, the other weight matrices are obtained from Theorem 2. To illustrate with an example, we consider \( n = 8 \) and show below how the weight matrices are obtained for the rate-1, 4-group decodable code.

A. An example - \( n = 8 \)

Let \( F_i, i = 1, 2, \cdots, 6 \) denote the 6 pairwise anticommuting, anti-Hermitian matrices. Choose \( F_1, F_2 \) and \( F_3 \) to be the three anticommuting matrices required for code construction. Let

\[ S = \{ jF_4F_5, F_1F_2F_3 \}, \quad \mathbb{P}(S) = \{ I_8, jF_4F_5, F_1F_2F_3, jF_1F_2F_3F_4F_5 \}. \]

The 16 weight matrices of the rate-1, 4-group decodable code for 8 antennas are as shown in Table II. Each column corresponds to the weight matrices in a group. Note that the product of any two matrices in the first group is some other matrix in the same group.

B. Coding gain calculations

Let \( \Delta(S, S') \triangleq \det(\Delta S \Delta S^H) \), where \( \Delta S \triangleq S - S', S \neq S' \) denotes the codeword difference matrix. Let \( \Delta s_i \triangleq s_i - s'_i, i = 1, 2, \cdots, 2n_t \), where \( s_i \) and \( s'_i \) are the real symbols encoding
codeword matrices $S$ and $S'$, respectively. Hence,

$$\Delta(S, S') = \det \left( \sum_{i=1}^{2n_t} \Delta s_i A_i \sum_{m=1}^{2n_t} \Delta s_m A_m^H \right)$$

$$= \det \left( \sum_{i=1}^{2n_t} \sum_{m=1}^{2n_t} \Delta s_i \Delta s_m A_i A_m^H \right).$$

Note that because of the nature of construction of the weight matrices, we have

$$A_i A_m^H = A_{\frac{p+1}{2}} A_{\frac{m+1}{2}}^H, \quad i, m \in \{1, 2, 3, 4\}, \quad p \in \{1, 2, 3\}.$$ 

Further, since the code is 4-group decodable,

$$\Delta(S, S') = \det \left( \sum_{p=0}^{3} \left( \sum_{i=\frac{p+1}{2}+1}^{\frac{p+1}{2}} \Delta s_i^2 I_{n_t} + 2 \sum_{i=\frac{p+1}{2}+1}^{\frac{p+1}{2}} \sum_{m=i+1}^{n_t} \Delta s_i \Delta s_m A_i A_m^H \right) \right).$$

All the weight matrices in the first group are Hermitian and pairwise commuting and the product of any two such matrices is some other matrix in the same group. It is well known that commuting matrices are simultaneously diagonalizable. Hence,

$$A_i = ED_i E^H, \quad i \in \left\{2, 3, \cdots, \frac{n_t}{2}\right\},$$

where, $D_i$ is a diagonal matrix. Since $A_i$ is Hermitian as well as unitary, the diagonal elements of $D_i$ are $\pm 1$. The following lemma proves that $A_i$ is traceless.

**Lemma 4:** Let $F_i, i = 1, 2, \cdots, 2a$ be $2^a \times 2^a$ unitary, pairwise anticommuting matrices. Then, the product matrix $F_{1}^{\lambda_1} F_{2}^{\lambda_2} F_{2a}^{\lambda_2}, \lambda_i \in \{0, 1\}, i = 1, 2, \cdots, 2a$, with the exception of $I_{2^a}$, is traceless.

**Proof:** It is well known that $tr(AB) = tr(BA)$ for any two matrices $A$ and $B$. Let $A$ and $B$ be two invertible, $n \times n$ anticommuting matrices. Then,

$$AB = -BA.$$

$$ABA^{-1} = -B.$$

$$tr(ABA^{-1}) = -tr(B).$$

$$tr(A^{-1}AB) = -tr(B) \iff tr(B) = -tr(B).$$
\[ \therefore \text{tr}(\mathbf{B}) = 0. \] (4)

Similarly, it can be shown that \( \text{tr}(\mathbf{A}) = 0 \). By applying Lemma 3, it can be seen that any product matrix \( \mathbf{F}_{1}^{\lambda_{1}}\mathbf{F}_{2}^{\lambda_{2}} \cdots \mathbf{F}_{2a}^{\lambda_{2a}} \), excluding \( \mathbf{I}_{2a} \), anticommutes with some other invertible product matrix from the set \( \{ \mathbf{F}_{1}^{\lambda_{1}}\mathbf{F}_{2}^{\lambda_{2}} \cdots \mathbf{F}_{2a}^{\lambda_{2a}}, \lambda_{i} \in \{0, 1\}, i = 1, 2, 3, \cdots, 2a \} \). Hence, from (4), we can say that every product matrix \( \mathbf{F}_{1}^{\lambda_{1}}\mathbf{F}_{2}^{\lambda_{2}} \cdots \mathbf{F}_{2a}^{\lambda_{2a}} \) except \( \mathbf{I}_{2a} \) is traceless.

From the above lemma, \( \mathbf{A}_{i} \) is traceless. Hence, \( \mathbf{D}_{i} \) has an equal number of ‘1’ and ‘-1’.

In fact, because of the nature of construction of the matrices \( \mathbf{F}_{i}, i = 1, 2, \cdots, 2a \), the product matrices \( \mathbf{F}_{i}\mathbf{F}_{i+1} \), for even \( i \), and the product matrix \( \mathbf{F}_{1}\mathbf{F}_{2}\mathbf{F}_{3} \) are always diagonal (easily seen from the definition of \( \mathbf{F}_{i}, i = 1, 2, \cdots, 2a \)). Hence, all the weight matrices of the first group excluding \( \mathbf{A}_{1} = \mathbf{I}_{n_{t}} \) are diagonal with the diagonal elements being \( \pm 1 \). Since these diagonal matrices also commute with \( \mathbf{F}_{2} \) and \( \mathbf{F}_{3} \), the diagonal entries are such that for every odd \( i \), if the \( (i, i) \)th entry is 1(-1), then, the \( (i + 1, i + 1) \)th entry is also 1(-1, resp.). To summarize, the properties of \( \mathbf{A}_{i}, i = 2, \cdots, \frac{n_{t}}{2} \) are listed below.

\[
\mathbf{A}_{i} = \mathbf{A}_{i}^{H},
\]

\[
\mathbf{A}_{i}^{2} = \mathbf{I}_{n_{t}},
\]

\[
\mathbf{A}_{i}(m, n) = 0, \ m \neq n,
\]

\[
\mathbf{A}_{i}(j, j) = \pm 1, \ j = 1, 2, \cdots, n_{t},
\]

\[
\text{tr}(\mathbf{A}_{i}) = 0,
\] (5)

\[
\mathbf{A}_{i}(j, j) = \mathbf{A}_{i}(j + 1, j + 1), \ j = 1, 3, 5, \cdots, n_{t} - 1,
\] (6)

\[
\mathbf{A}_{i}\mathbf{A}_{j} = \mathbf{A}_{k}, \ i, j, k \in \left\{ 1, 2, \cdots, \frac{n_{t}}{2} \right\}.
\] (7)

In view of these properties,

\[
\Delta(S, S') = \det \left( \sum_{p=0}^{3} \left( \sum_{i = \frac{p+1}{n_{t}} + 1}^{\frac{p+1}{n_{t}}} \Delta s_{i}^{2} \mathbf{I}_{n_{t}} + 2 \sum_{i = \frac{p+1}{n_{t}} + 1}^{\frac{p+1}{n_{t}}} \sum_{m = i + 1}^{\frac{p+1}{n_{t}}} \Delta s_{i} \Delta s_{m} \mathbf{D}_{im} \right) \right),
\]

where, \( \mathbf{D}_{im} = \mathbf{A}_{i}\mathbf{A}_{m} = \mathbf{A}_{k} \) for some \( k \in \left\{ 1, 2, \cdots, \frac{n_{t}}{2} \right\}. \) So,
\[
\Delta(S, S') = \prod_{j=1}^{n_t} \sum_{p=0}^{3} \left( \sum_{i=1}^{n_t} d_{ij} \Delta s_{\frac{p+1}{2}} \right)^2,
\]
where, \(d_{ij} = \pm 1\) and \(d_{1j} = 1\). In fact, \(d_{ij} = A_i(j, j)\), \(i = 1, 2, 3, \ldots, \frac{n_t}{2}\). Hence,

\[
\min_{S, S'}(\Delta(S, S')) = \min_{\Delta s_i} \left( \prod_{j=1}^{n_t} \left( \sum_{i=1}^{n_t} d_{ij} \Delta s_i \right)^2 \right),
\]
where \(\min_x(y)\) denotes the minimum value of \(y\) over all possible values of \(x\). From (6),

\[
\min_{S, S'}(\Delta(S, S')) = \min_{\Delta s_i} \left( \prod_{j=1}^{n_t} \left( \sum_{i=1}^{n_t} d_{i(2j-1)} \Delta s_i \right)^4 \right). \quad (8)
\]

We need the minimum determinant to be as high a non-zero number as possible. In this regard, let

\[
W \triangleq \sqrt{\frac{2}{n_t}} [w_{ij}], \quad w_{ij} = d_{i(2j-1)}, \quad i, j = 1, 2, \ldots, \frac{n_t}{2}
\]
and

\[
y_p \triangleq [y_{\frac{n_t}{2}p+1}, y_{\frac{n_t}{2}p+2}, \ldots, y_{\frac{n_t}{2}(p+1)}]^T = W [s_{\frac{n_t}{2}p+1}, s_{\frac{n_t}{2}p+2}, \ldots, s_{\frac{n_t}{2}(p+1)}]^T, \quad p = 0, 1, 2, 3.
\]

**Lemma 5:** \(W\) as defined in (9) is an orthogonal matrix.

**Proof:** From (9), it can be noted that the columns of \(W\) are obtained from the diagonal elements of \(A_i, i = 1, 2, \ldots, \frac{n_t}{2}\). Each element of a column \(i\) of \(W\) corresponds to every odd numbered diagonal element of \(A_i\). Denote the \(i^{th}\) column of \(W\) by \(w_i\). Applying (6), (7) and (5) in that order,

\[
\langle w_i, w_j \rangle = \frac{1}{n_t} tr(A_i A_j) = \frac{1}{n_t} tr(A_k) = \delta_{ij}
\]
where

\[
\delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{otherwise}
\end{cases}
\]

Hence, \(W\) is orthogonal. \(\blacksquare\)
Substituting $y_p$ in (8), we get

$$\min_{S, S'} (\Delta(S, S')) = \min_{y_0} \left( \prod_{j=1}^{n/2} y_j^4 \right).$$

So, the minimum determinant is a power of the minimum product distance in $n_t/2$ real dimensions. If $y_p \in \mathbb{Z}^{n/2}$, the product distance can be maximized by premultiplying $y_p$ with a suitable orthogonal rotation matrix $V$ given in [24]. This operation maximizes the minimum determinant and hence the coding gain. So, the real symbols of the rate-1, 4-group decodable code are encoded by grouping $n_t/2$ real symbols into 4 groups and each group of symbols taking value from a unitarily rotated vector belonging to $\mathbb{Z}^{n/2}$, the rotation matrix being $W^T V$. For 4 transmit antennas,

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0.8507 & -0.5257 \\ 0.5257 & 0.8507 \end{bmatrix},$$

and for 8 transmit antennas,

$$W = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad V = \begin{bmatrix} -0.3664 & -0.7677 & 0.4231 & 0.3121 \\ -0.2264 & -0.4745 & -0.6846 & -0.5050 \\ -0.4745 & 0.2264 & -0.5050 & 0.6846 \\ -0.7677 & 0.3664 & 0.3121 & -0.4231 \end{bmatrix}. $$

If the practically used square QAM constellation of size $M$ is used, encoding is done as follows: the $n_t$ complex symbols in each codeword matrix take values from the $M$-QAM and are split into two groups, one group consisting of the real parts of the $n_t$ symbols and the other group consisting of the imaginary parts. Each group is further divided into two subgroups, each consisting of $n_t/2$ real symbols. So, in all, there are 4 groups consisting of $n_t/2$ real symbols. Denoting the column vectors consisting of the symbols in a group by $y_p$, $p = 0, 1, 2, 3$, let $s_p = W^T V y_p$, where $W$ and $V$ are as explained before. Then the codeword matrix is given by

$$S = \sum_{p=0}^{3} \sum_{i=1}^{n_t/2} s_p^{(i)} A^{n_t p + i},$$

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where, \( s_p^{(i)} \) denotes the \( i^{th} \) entry of \( s_p \). Consequently, the ML-decoding complexity of the code is of the order of \( M^{\frac{n_t - 2}{4}} \). This is because there are four groups consisting of \( n_t/2 \) real symbols each and the symbols in each group can be decoded independently from the symbols in the other groups. In decoding the symbols in the same group jointly, one needs to make a search over \( \sqrt{M^{\frac{n_t}{2}}} = M^{\frac{n_t}{4}} \) possibilities for the symbols, since the real and the imaginary parts of a signal point in a square \( M \)-QAM have only \( \sqrt{M} \) possible values each (the real and the imaginary parts of a signal point of a square \( M \)-QAM take values from a \( \sqrt{M} \)-PAM constellation). However, one need not make an exhaustive search over all the possible \( M^{\frac{n_t}{4}} \) values for the \( n_t/2 \) symbols. For every possible value of the first \( \frac{n_t}{2} - 1 \) real symbols, the last symbol is evaluated by quantization [4]. Hence, the worst case ML-decoding complexity is of the order of \( \sqrt{M^{\frac{n_t}{2}} - 1} = M^{\frac{n_t - 2}{4}} \) only.

V. EXTENSION TO HIGHER NUMBER OF RECEIVE ANTENNAS

When \( n_r = 1 \), a rate-1, 4-group decodable STBC is the best full-rate STBC possible in terms of ML-decoding complexity and as a result, ergodic capacity. However, when \( n_r > 1 \), we need more weight matrices to meet the full-rate criterion. In literature, there does not exist a 4-group decodable STBC with rate greater than 1. So, it is unlikely, though not proven, that there exists a full-rate, multi-group ML-decodable STBC with full-diversity for \( n_r > 1 \). Let \( n_t = 2^a \). We know that if \( F_i, i = 1, 2, \cdots, 2^a \) are pairwise anticommuting, invertible matrices, then, the set \( \mathcal{F} \triangleq \{F_1^{\lambda_1}F_2^{\lambda_2}\cdots F_{2^a}^{\lambda_{2^a}} \mid \lambda_i \in \{0, 1\}, i = 1, 2, \cdots, 2^a \} \) is linearly independent over \( \mathbb{C} \). Hence, the set \( \mathcal{M} = \{\mathcal{F}, j\mathcal{F}\} \) is linearly independent over \( \mathbb{R} \). As a result, the elements of \( \mathcal{M} \) can be used as weight matrices of a full-rate STBC for \( n_r > 1 \). Keeping in view that the ergodic capacity depends on as many non-diagonal entries of the \( \mathbf{R} \)-matrix being zeros, it is important to choose the weight matrices judiciously. The idea is that given a full-rate STBC for \( n_r - 1 \) receive antennas, obtain the additional weight matrices of a full-rate STBC for \( n_r \) receive antennas by using the weight matrices of a rate-1, 4-group decodable STBC such that after the addition of the new weight matrices, the set consisting of the weight matrices of the rate-\( n_r \) code is linearly independent over \( \mathbb{R} \). This is achieved as follows.

1) Obtain a rate-1, 4-group decodable STBC by using the construction detailed in Section IV.

Due to the nature of the construction, the product of any two weight matrices is always some other weight matrix of the code, up to negation. Denote the set of weight matrices by \( \mathcal{G}_1 \).
2) From the set $\mathcal{F}$, choose a matrix that does not belong to $\mathcal{G}_1$ and multiply it with the elements of $\mathcal{G}_1$ to obtain a new set of weight matrices, denoted by $\mathcal{G}_2$. Clearly, the two sets will not have any matrix in common. To see this, let $A \in \mathcal{G}_1$ and $B \in \mathcal{F} \cap (\mathcal{M} / \mathcal{G}_1)$, where $B$ is the matrix chosen to be multiplied with the elements of $\mathcal{G}_1$. Let $BA = C \in \mathcal{G}_1$. Hence, $B = CA^H = \pm CA$ and $CA$ belongs to $\mathcal{G}_1$, up to negation. This contradicts the fact that $B \in \mathcal{F} \cap (\mathcal{M} / \mathcal{G}_1)$. So, $C$ cannot belong to $\mathcal{G}_1$.

The weight matrices of $\mathcal{G}_2$ form a new, rate-1, 4-group decodable STBC. This is because the ML-decoding complexity does not change by multiplying the weight matrices of a code with a unitary matrix. In this case, we have multiplied the elements of $\mathcal{G}_1$ with an element of $\mathcal{F}$, which is a unitary matrix. Now, $\mathcal{G}_1 \cup \mathcal{G}_2$ is the set of weight matrices of a rate-2 code with an ML-decoding complexity of $M^{n_t}M^{2n_t-2} = M^{\frac{3nt-2}{4}}$. This is achieved by decoding the last $n_t$ symbols with a complexity of $M^{n_t}$ and then conditionally decoding the first $n_t$ symbols using the 4-group decodability property as explained in Section IV-B.

3) For increasing $n_r$, repeat as in the second step, obtaining new rate-1, 4-group decodable codes and then appending their weight matrices to obtain a new, rate-$n_r$ code with an ML-decoding complexity of $M^{n_t(n_r-\frac{2}{4})-0.5}$. The new set of weight matrices is $\bigcup_{i=1}^{n_r} \mathcal{G}_i$.

4) When all the elements of $\mathcal{F}$ have been exhausted (this occurs when $n_r = n_t/2$), Step 3 can be continued till $n_r = n_t$ by choosing the matrices that are to be multiplied with the elements of $\mathcal{G}_1$ from $jF \cap (\mathcal{M} / \bigcup_{i=1}^{n_r-1} \mathcal{G}_i)$. Note from Lemma I that this does not spoil the linear independence over $\mathbb{R}$ of the weight matrices.

Note: In case of the Perfect codes for $n_t$ transmit antennas, a layer [14], [15] corresponds to $n_t$ complex symbols. In case of our generalized Silver codes, a layer corresponds to a rate-1, 4-group decodable code, encoding $n_t$ complex symbols. Also, the Silver code for an $n_t \times n_r$ system refers to the STBC containing $n_{\text{min}} = \min(n_t, n_r)$ individual rate-1, 4-group decodable codes, the construction of which has been explained above.

A. An illustration for $n_t = 4$

For $n_t = 4$, let $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ and $\mathbf{F}_4$ be the four anticommuting, anti-Hermitian matrices obtained by the method presented in [2]. Let $\mathcal{F} = \{F_1^{\lambda_1}F_2^{\lambda_2}F_3^{\lambda_3}F_4^{\lambda_4}, \lambda_i \in \{0, 1\}, i = 1, 2, 3, 4\}$. The rate-1, 4-group decodable code has the following 8 weight matrices, with weight matrices in each column belonging to the same group:
\[
\begin{array}{cccc}
I_4 & F_1 & F_2 & F_3 \\
F_1F_2F_3 & -F_2F_3 & F_1F_3 & -F_1F_2
\end{array}
\]

Hence, \( G_1 = \{I_4, F_1, F_2, F_3, F_1F_2F_3, -F_2F_3, F_1F_3, -F_1F_2\} \). Now, we choose a matrix from \( F \) which does not belong to \( G_1 \). One such matrix is \( F_4 \). Pre-multiplying all the elements of \( G_1 \) with \( F_1 \) and applying the anticommuting property, we obtain a new rate-1, 4-group decodable code, whose weight matrices are as follows:

\[
\begin{array}{cccc}
F_4 & -F_1F_4 & -F_2F_4 & -F_3F_4 \\
-F_1F_2F_3F_4 & -F_2F_3F_4 & F_1F_3F_4 & -F_1F_2F_4
\end{array}
\]

Hence, \( G_2 = F_4G_1 = \{F_4, -F_1F_4, -F_2F_4, -F_3F_4, -F_1F_2F_3F_4, -F_2F_3F_4, F_1F_3F_4, -F_1F_2F_4\} \) and \( G_1 \cup G_2 \) is the set of weight matrices of the rate-2 STBC, which is full rate with an ML-decoding complexity of the order of \( M^{4.5} \).

Now, since there are no more elements left in \( F \) (neglecting negation), we can choose elements from \( jF \). To construct a rate-3 code for 3 transmit antennas, we multiply the elements of \( G_1 \) by \( jI_4 \) to obtain the set \( G_3 = jG_1 \). The weight matrices of the rate-3 code constitute the set \( G_1 \cup G_2 \cup G_3 \). Similarly, the weight matrices of a full-rate code for \( n_r \geq 4 \) are the elements of the set \( G_1 \cup G_2 \cup G_3 \cup G_4 \), where \( G_4 = jF_4G_1 = jG_2 \). It is obvious that \( G_1, G_2, G_3 \) and \( G_4 \) represent the weight matrices of four individual rate-1, 4-group decodable codes, respectively.

B. Structure of the R-matrix and ML-decoding complexity

The R-matrix of the Silver code for the \( n_t \times n_r \) system has the following structure, irrespective of the channel realization:

\[
R = \begin{bmatrix}
D & X & \ldots & X \\
O_{2n_t} & D & \ldots & X \\
\vdots & \ddots & \ddots & \vdots \\
O_{2n_t} & O_{2n_t} & \ldots & D
\end{bmatrix}
\]

where \( X \in \mathbb{R}^{2n_t \times 2n_t} \) is a random non-sparse matrix whose entries depend on the channel coefficients and \( D = I_4 \otimes T \), with \( T \in \mathbb{R}^{n_t \times n_t} \) being an upper triangular matrix. The reason for this structure is that the weight matrices of the Silver code for an \( n_t \times n_r \) system are also the weight matrices of \( \min(n_t, n_r) \) separate rate-1, 4-group decodable codes (as illustrated in Sec. [V]). As a result of the structure of \( D \), the R-matrix has a large number of zeros in the upper
block, and hence, compared to other existing codes, the generalized Silver codes are expected to have higher ergodic capacity (for \( n_r < n_t \)) and lower average ML-decoding complexity. The worst case ML-decoding complexity is of the order of \( (M^{n_{t}(n_{min}-1)})(M^{\frac{n_{t}2}{4}}) = M^{n_{t}(n_{min} - \frac{3}{4})-0.5} \), which is because in decoding the symbols, a search is to be made over all possible values of the last \( n_t(n_{min} - 1) \) complex symbols (which requires a complexity of the order of \( M^{n_t(n_{min}-1)} \)), while the remaining \( n_t \) symbols can be conditionally decoded with a complexity of \( M^{n_t} \) only, once the last \( n_t(n_{min} - 1) \) symbols are fixed (a detailed explanation on conditional ML-decoding has been presented in [8], [4]). For \( n_r \geq n_t \), the Silver code is information lossless, because its normalized generator matrix (normalization is done to ensure an appropriate SNR at each receive antenna) is orthogonal. To see this, the generator matrix for \( n_r \geq n_t \) is given as
\[
\mathbf{G} = \frac{1}{\sqrt{n_t}}[\overline{\text{vec}(\mathbf{A}_1)} \overline{\text{vec}(\mathbf{A}_2)} \cdots \overline{\text{vec}(\mathbf{A}_{2n_t^2})}],
\]
where \( \mathbf{A}_i \in \mathcal{M}, i = 1, 2, \cdots, 2n_t^2 \), are the weight matrices obtained as mentioned in Sec. [V] with \( \mathcal{M} = \{\mathcal{F}, j\mathcal{F}\} \), where \( \mathcal{F} = \{\mathbf{F}_{1}^{\lambda_1}\mathbf{F}_2^{\lambda_2} \cdots \mathbf{F}_{2a}^{\lambda_{2a}} \mid \lambda_i \in \{0, 1\}, i = 1, 2, 3, \cdots, 2a\} \). For \( i, j \in \{1, 2, \cdots, 2n_t^2\} \), we have
\[
\langle \overline{\text{vec}(\mathbf{A}_i)}, \overline{\text{vec}(\mathbf{A}_j)} \rangle = \text{real} \left( \text{tr} \left( \mathbf{A}_i^H \mathbf{A}_j \right) \right)
\]
\[
= \pm \text{real} \left( \text{tr} (\mathbf{A}_i \mathbf{A}_j) \right)
\]
\[
= \begin{cases} 
\text{real} \left( \text{tr}(\mathbf{I}_{n_t}) \right) & \text{if } i = j \\
\text{real} \left( \text{tr}(j\mathbf{I}_{n_t}) \right) & \text{if } \mathbf{A}_i = j\mathbf{A}_j \\
\pm \text{real} \left( \text{tr}(\mathbf{A}_k) \right) & \text{otherwise, where } \pm \mathbf{A}_k \in \mathcal{M}/\{\mathbf{I}_{n_t}, j\mathbf{I}_{n_t}\}
\end{cases}
\]
\[
= n_t \delta_{ij}.
\]
Equation (10) holds because \( \mathbf{A}_i, i = 1, \cdots, 2n_t^2 \) are either Hermitian or anti-Hermitian, and (11) follows from Lemma 4.

**C. The Silver code for two transmit antennas**

The Silver code [6], [7] for two antennas, which is well known for being a low complexity, full-rate, full-diversity STBC for \( n_r \geq 2 \), transmits 2 complex symbols per channel use. A
codeword matrix of the Silver code is given as

\[ S = \begin{pmatrix} s_1 + z_1 & -s_2 + z_2 \\ -s_2^* + z_2^* & s_1^* - z_1^* \end{pmatrix}, \]

where,

\[ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 + j & 1 + 2j \\ -1 + 2j & 1 - j \end{pmatrix} \begin{pmatrix} s_3 \\ s_4 \end{pmatrix}. \]

The codeword encodes 4 complex symbols \( s_1, s_2, z_1 \) and \( z_2 \). Clearly, the first four weight matrices are that of the Alamouti code, which is a rate-1, 4-group decodable STBC for 2 transmit antennas. The Silver code’s next 4 weight matrices are obtained by multiplying the first four weight matrices by \( j \) and negating some of the resultant weight matrices. To make the code a full-ranked one, the last 2 complex symbols take values from a different constellation, which is obtained by unitarily rotating the symbol vector \([s_3, s_4]^T \in \mathbb{Z}[j]^{2 \times 1}\). So, \( s_1 \) and \( s_2 \) take values from the regular \( M\)-QAM, while \( z_1 \) and \( z_2 \) take values from a different constellation. The Silver code compares very well with the well known Golden code in error performance, while offering lower ML-decoding complexity of \( M^2 \) for square \( M\)-QAM.

D. Achievability of Full-diversity

The following theorem, stated in [25], guarantees that full-diversity is possible for the generalized Silver codes.

**Theorem 3:** For any given \( n \times n \) square linear design encoding \( k \) real symbols with full-rank weight matrices \( A_i \) and positive integers \( Q_1, \ldots, Q_n \), there exist constellations \( A_i \in \mathbb{R} \), \( i = 1, \ldots, k \) such that

1) \( |A_i| = Q_i \) for \( i = 1, \ldots, k \).

2) The STBC \( S \triangleq \{ S = \sum_{i=1}^k s_i A_i \mid s_i \in A_i, \quad i = 1, 2, \ldots, k \} \) offers full diversity.

Since all the weight matrices of the generalized Silver code are either Hermitian or anti-Hermitian and hence full-ranked, there exist constellations for which the generalized Silver codes have full-diversity. Since this paper deals with the construction of low ML-decoding complexity codes

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1The paper with a more refined version of this theorem, where it is shown that \( A_i \) can be any regular PAM constellation, will be uploaded on arXiv by the authors of [25].
with full-rate, we do not focus on identifying constellations for which the codes provide full-diversity. For the full-rate codes for 1 receive antenna, in Section [IV-B] we have already identified the constellations which not only provide full-diversity, but also maximize the coding gain. For the generalized Silver codes for higher number of receive antennas, each layer, corresponding to a rate-1, 4-group decodable code, is encoded as explained in [IV-B]. In addition, we use a certain scaling factor to be multiplied with a certain subset of weight matrices to enhance the coding gain. The choice of the scaling factor is based on computer search. With the use of the scaling factor, the generalized Silver codes perform very well when compared with the punctured Perfect codes. Although we cannot mathematically prove that our codes have full-diversity with the constellation that we have used for simulation, the simulation plots seem to suggest that our codes have full-diversity, since the error performance of our codes matches that of the comparable punctured Perfect codes, which have been known to have full-diversity.

VI. SIMULATION RESULTS

In all the simulation scenarios in this section, we consider the Rayleigh block fading MIMO channel.

A. 4 Tx

We consider three MIMO systems \(4 \times 2\), \(4 \times 3\) and \(4 \times 4\) systems. The codes are constructed as illustrated in Subsection [V-A]. To enhance the performance of our code for the \(4 \times 2\) system, we have multiplied the weight matrices of \(G_2\) (as defined in Subsection [V-A]) with the scalar \(e^{j\pi/4}\). This is done primarily to enhance the coding gain, which was observed to be the highest when the scalar \(e^{j\pi/4}\) was chosen. It is to be noted that this action does not alter the ML-decoding complexity. Consequently, the weight matrices of the Silver code for the \(4 \times 2\) system can be viewed to be from \(G_1 \cup e^{j\pi/4}G_2\). For the \(4 \times 3\) MIMO system, the weight matrices of the Silver code are from the set \(G_1 \cup e^{j\pi/4}G_2 \cup jG_1\), while the weight matrices of the Silver code for the \(4 \times 4\) system are from the set \(G_1 \cup e^{j\pi/4}G_2 \cup jG_1 \cup je^{j\pi/4}G_2\). Fig. [I] shows the plot of the ergodic capacity for our codes and the punctured Perfect codes [14] for \(4 \times 2\) and \(4 \times 3\) systems. In both the cases, our code has higher ergodic capacity than the punctured Perfect code, as was expected. Regarding error performance, we have chosen 4 QAM for our simulations and encoding is done as explained in Subsection [IV-B].
1) 4 × 2 MIMO

Fig. 2 shows the plots of the symbol error rate (SER) as a function of the SNR at each receive antenna for four codes - the DjABBA code [6], the punctured perfect code, the Silver code for the 4 × 2 system and the EAST code [26]. Since the number of degrees of freedom of the channel is only 2, we use the Perfect code with 2 of its 4 layers punctured. Our code and the EAST code have the best performance. It is to be noted that the curves for the Silver code for the 4 × 2 system and the EAST code coincide. Also, the Silver code for the 4 × 2 system is the same as the one presented in [4], but has been designed using a new, systematic method. The Silver code for the 4 × 2 system and the EAST code have an ML-decoding complexity of the order of $M^{4.5}$ for square QAM constellation.

2) 4 × 3 MIMO

Fig. 3 shows the plots of the SER as a function of the SNR at each receive antenna for two codes - the punctured perfect code (puncturing one of its 4 layers) and the Silver code for the 4 × 3 system. The Silver code for the 4 × 3 system has a marginally better performance than the punctured perfect code in the low to medium SNR range. It has an ML-decoding complexity of the order of $M^{8.5}$ while that of the punctured Perfect code is $M^{11}$ (this reduction from $M^{12}$ to $M^{11}$ is due to the fact that the real and the imaginary parts of the last symbol can be evaluated by quantization, once the remaining symbols have been fixed).

3) 4 × 4 MIMO

Fig. 4 shows the plots of the SER as a function of the SNR at each receive antenna for the Silver code for the 4 × 4 system and the Perfect code. The Silver code for the 4 × 4 system nearly matches the Perfect code in performance at low and medium SNR. More importantly, it has lower ML-decoding complexity of the order of $M^{12.5}$, while that of the Perfect code is $M^{15}$.

B. 8 Tx

To construct the Silver code for the 8 × 2 system, we first construct a rate-1, 4-group decodable STBC as described in Section [IV] and denote the set of obtained weight matrices by $G_1$. Next we multiply the weight matrices of $G_1$ by $F_4$ to obtain a new set of weight matrices which is denoted by $G_2$. The weight matrices of the Silver code for the 8 × 2 system are obtained from

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\( G_1 \cup G_2 \). The Silver code for the \( 8 \times 3 \) system can be obtained by multiplying the matrices of \( G_1 \) with \( F_6 \) and appending the resulting weight matrices to the set \( G_1 \cup G_2 \). The rival code is the punctured perfect code for 8 transmit antennas [15]. The ergodic capacity plots of the two codes are shown in Fig. 5. As expected, our codes achieves higher ergodic capacity, although lower than that of the corresponding MIMO channels.

Fig. 6 shows the symbol error performance of the Silver code for \( 8 \times 2 \) system and the punctured Perfect code [15]. The constellation employed is 4-QAM. Again, to enhance performance by way of increasing the coding gain, we have multiplied the weight matrices of \( G_2 \) with the scalar \( e^{\frac{4\pi}{4}} \), as done for the codes for 4 transmit antennas. The simulation plot suggests that our code has full diversity. The most important aspect of our code is that it has an ML-decoding complexity of \( M^{9.5} \), while that of the comparable punctured Perfect code is \( M^{15} \).

VII. DISCUSSION

In this paper, we analyzed the ergodic capacity of the MIMO channel using space time coding and studied the property of an STBC that allows it to have high ergodic capacity when \( n_r < n_t \). we proposed a scheme to obtain a full-rate STBC for \( 2^a \) transmit antennas and any number of receive antennas with reduced ML-decoding complexity. The STBCs thus obtained have higher ergodic capacity at high SNR than existing STBCs for the case \( n_r < n_t \). It is to be seen if the proposed codes are better suited than existing codes for sub-optimal decoding techniques like lattice reduction aided detection, owing to the fact that more number of symbols are disentangled from one another than in the case of known codes. Also, finding out explicit constellations which can be mathematically proved to guarantee full-diversity and a non-vanishing determinant is an open problem. These are some of the directions for future research.

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Fig. 1. Ergodic capacity Vs SNR for codes for $4 \times 2$ and $4 \times 3$ systems

Fig. 2. SER performance at 4 BPCU for codes for $4 \times 2$ systems
Fig. 3. SER performance at 6 BPCU for codes for $4 \times 3$ systems

Fig. 4. SER performance at 8 BPCU for codes for $4 \times 4$ systems
Fig. 5. Ergodic capacity Vs SNR for codes for $8 \times 2$ and $8 \times 3$ systems

Fig. 6. SER performance at 4 BPCU for codes for $8 \times 2$ systems