New periodic and soliton solutions of nonlinear evolution equations

S.A. El-Wakil a, M.A. Abdou a,b,*, A. Hendi c

a Theoretical Research Group, Physics Department, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
b Faculty of Education for Girls, Physics Department, King Kahlid University, Bisha, Saudi Arabia
c Physics Department, Faculty of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia

Abstract

In this paper, the tanh and sine–cosine methods are used to construct exact periodic and soliton solutions of nonlinear evolution equations arising in mathematical physics. Many new families of exact travelling wave solutions of the generalized Hirota–Satsuma coupled KdV system, generalized-Zakharov equations and (2 + 1)-dimensional Broer–Kaup–Kupershmidt system are successfully obtained. The obtained solutions include solitons, kinks and plane periodic solutions. These solutions may be important of significance for the explanation of some practical physical problems.

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1. Introduction

Nonlinear wave equations have a significant role in several scientific and engineering fields. These equations appear in solid state physics, fluid mechanics, chemical kinetics, plasma physics, population models, nonlinear optics, propagation of fluxons in Josephson junctions, and many others. The pioneer work of Malfiet in [1] introduced the powerful hyperbolic tangent (tanh) method for a reliable treatment of the nonlinear wave equations. The useful tanh method is widely used by many such as in [2–11] and by the references therein. The method introduces a unifying method that one can find exact as well as approximate solutions in a straightforward and systematic way [11–17]. The tanh method has been subjected to many modifications that mainly depend on the Riccati equation and the solutions of well-known equations. The standard tanh method and the proposed modifications all depend on the balance method, where the linear terms of highest order are balanced with the highest order nonlinear terms of the reduced equation.

The rest of this paper is arranged as follows. In Section 2, we simply provide the mathematical framework of the extended tanh and sine–cosine methods. In Section 3, in order to illustrate the method, three nonlinear...
evolution equations are investigated and abundant exact solutions are obtained which include solitons, kinks and plane periodic solutions. Finally, conclusion and discussion are given in Section 4.

In what follows, we highlight the main steps of the sine–cosine algorithm and extended tanh method.

2. The two methods

2.1. The extended tanh method

In this section, we give a brief description of the extended tanh method as follows. For the given a nonlinear evolution equations, say, in two variables

\[ \phi(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \] (1)

We seek the following travelling wave solutions:

\[ u(x, t) = u(\xi), \quad \xi = x \pm ct, \]

which are of important physical significance, \( k \) and \( c \) are constants to be determined later. Then Eq. (1) reduces to nonlinear ordinary differential equations

\[ \psi(u, cu_\xi, u_x, c^2u_{\xi\xi}, u_{\xi\xi}, \ldots) = 0. \] (2)

We introduce the new independent variables:

\[ Y = \tanh(\xi) \quad \text{or} \quad Y = \coth(\xi), \quad \xi = x \pm ct, \] (3)

that leads to the change of derivatives

\[ \frac{d}{d\xi} = (1 - Y^2) \frac{d}{dY}, \]

\[ \frac{d^2}{d\xi^2} = (1 - Y^2) \left[ -2Y \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2} \right], \] (4)

\[ \frac{d^3}{d\xi^3} = (1 - Y^2) \left[ (6Y^2 - 2) \frac{d}{dY} - 6Y(1 - Y^2)^2 \frac{d^2}{dY^2} + (1 - Y^2)^2 \frac{d^3}{dY^3} \right], \]

where the other derivatives can be derived in a similar way. We use new independent variables

\[ Y = \tan(\xi) \quad \text{or} \quad Y = -\cot(\xi), \]

that leads to the change of derivatives

\[ \frac{d}{d\xi} = (1 + Y^2) \frac{d}{dY}, \]

\[ \frac{d^2}{d\xi^2} = (1 + Y^2) \left[ -2Y \frac{d}{dY} + (1 + Y^2)^2 \frac{d^2}{dY^2} \right], \] (5)

\[ \frac{d^3}{d\xi^3} = (1 + Y^2) \left[ (6Y^2 + 2) \frac{d}{dY} - 6Y(1 + Y^2)^2 \frac{d^2}{dY^2} + (1 + Y^2)^2 \frac{d^3}{dY^3} \right]. \]

In the context of tanh function method, many authors [2–7] used the ansatz

\[ u(\xi) = \sum_{i=0}^{M} a_i Y^i(\xi). \]

In order to construct more general, it is reasonable to introduce the following ansatz [17]:

\[ u(\xi) = \sum_{i=0}^{M} a_i Y^i(\xi) + \sum_{i=1}^{M} b_i Y^{-i}(\xi) \] (6)

in which \( a_i \) and \( b_i (i = 0, 1, \ldots, M) \) are all real constants to be determined later, the balancing number \( M \) is a positive integers which can be determined by balancing the highest order derivative terms with highest power.
nonlinear terms in Eq. (2). We substitute anzatz (6) and (4) into Eq. (2) with computerized symbolic computation, equating to zero the coefficients of all power \( Y^{\pm i} \) yields a set of algebraic equations for \( a_i, b_i \) and \( \mu \).

2.2. The sine–cosine algorithm

Wazwaz has summarized the main steps introduced for using sine–cosine method, as following:

Step (1): We introduce the wave variables \( \xi = x - ct \) into the PDE, we get
\[
\phi(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, \ldots) = 0,
\]
where \( u(x, t) \) is travelling wave solution. This enables us to use the following changes:
\[
\frac{\partial}{\partial t} = -c \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial t^2} = c^2 \frac{\partial^2}{\partial \xi^2}, \\
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2}.
\]
One can immediately reduce the nonlinear PDE (7) into a nonlinear ODE:
\[
\psi(u, -cu, u', c^2u'' + u'', \ldots) = 0.
\]

Step (2): By virtue of the technique of solution, we introduce the anstaz
\[
u(x, t) = \lambda \sin^{\beta}(\mu \xi), \quad |\mu \xi| < \frac{\pi}{2},
\]
or
\[
u(x, t) = \lambda \cos^{\beta}(\mu \xi), \quad |\mu \xi| < \frac{\pi}{2\mu},
\]
where \( \lambda, \mu \) and \( \beta \) are parameters are to be determined later, \( \mu \) and \( c \) are the wave number and the wave speed, respectively, we use
\[
u(x, t) = \lambda \sin^{\beta}(\mu \xi), \\
u''(x, t) = \lambda^{\beta} \sin^{\beta}(\mu \xi), \\
(u^\prime)^{\prime\prime} = n\mu^\beta \lambda^{\beta} \sin^{\beta-1}(\mu \xi),
\]
and the derivatives of Eq. (11) becomes
\[
u(x, t) = \lambda \cos^{\beta}(\mu \xi), \\
u''(x, t) = \lambda^{\beta} \cos^{\beta}(\mu \xi), \\
(u^\prime)^{\prime\prime} = n\mu^\beta \lambda^{\beta} \cos^{\beta-1}(\mu \xi)
\]
and so on for the other derivatives.

Step (3): Substituting (9) or (10) into the reduced equation (8), balance the terms of the sine or cosine functions (11) or (12), and solving the resulting system of algebraic equations by using the computerized symbiotic calculations. We collect all terms within the same power in \( \cos^{\beta}(\mu \xi) \) or \( \sin^{\beta}(\mu \xi) \) and set to zero their coefficients to get a system of algebraic equations among the unknowns \( \mu, \beta \) and \( \lambda \).
3. New applications

In the next section, we will demonstrate the two proposed methods on three nonlinear evolution equations of special interest in physics, namely, generalized Hirota–Satsuma coupled KdV system, generalized-Zakharov equations and (2 + 1)-dimensional Broer–Kaup–Kupershmidt system.

3.1. The generalized Hirota–Satsuma coupled KdV system


\[
\begin{align*}
    u_t & = \frac{1}{4} u_{xxx} + 3uu_x + 3(w - v^2)_x, \\
    v_t & = -\frac{1}{2} v_{xxx} - 3uv_x, \\
    w_t & = -\frac{1}{2} w_{xxx} - 3uw_x.
\end{align*}
\]

When \( w = 0 \), Eq. (13) reduces to be the well-known Hirota–Satsuma coupled KdV system. We seek travelling wave solutions for Eq. (13) in the form

\[
\begin{align*}
    u(x, t) &= u(\xi), \quad v(x, t) = v(\xi), \quad w(x, t) = w(\xi), \quad \xi = k(x - ct).
\end{align*}
\]

Substituting Eq. (14) into (13) yields an ODE

\[
\begin{align*}
    -cku' &= \frac{1}{4} k^3 u'' + 3ku' + 3k(w - v^2)', \\
    -ckv' &= -\frac{1}{2} k^3 v'' - 3kuv', \\
    -ckw' &= -\frac{1}{2} k^3 w'' - 3kuv'.
\end{align*}
\]

Let

\[
\begin{align*}
    u &= ax^2 + \beta v + \gamma, \\
    w &= A_0 v + B_0,
\end{align*}
\]

where \( a, \beta, A_0 \) and \( B_0 \) are constants. Inserting Eq. (18) into (16) and (17) integrating once we know that (16) and (17) give rise to the same equation

\[
k^2 v'' = -2ax^3 - 3\beta v^2 + 2(c - 3\gamma)v + c_1,
\]

where \( c_1 \) is an integration constant. Integrating (19) once again we have

\[
k^2 v'^2 = -ax^4 - 2\beta v^3 + 2(c - 3\gamma)v^2 + 2c_1v + c_2,
\]

where \( c_2 \) is an integration constant. By means of Eqs. (18)–(20) we get

\[
k^2 u'' = 2ax^2 v^2 + k^2 (2ax + \beta)v''
\]

\[
\begin{align*}
    & = 2a[-ax^4 - 2\beta v^3 + 2(c - 3\gamma)v^2 + 2c_1v + c_2] + (2ax + \beta)[-2ax^3 - 3\beta v^2 + 2(c - 3\gamma)v + c_1].
\end{align*}
\]

Integrating (15) once we have

\[
\frac{1}{4} k^2 u'' + \frac{3}{2} u'^2 + cu + 3(w - v^2) + c_3 = 0,
\]

where \( c_3 \) is an integration constant. Inserting (18) and (21) into (22) gives

\[
\begin{align*}
    3ax - 3\gamma + \frac{3}{4} \beta^2 - 3 &= 0, \\
    \frac{1}{2} (\alpha c + \beta c + \gamma \beta) + A_0 &= 0, \\
    \frac{1}{4} (2\alpha c_2 + \beta c_1) + \frac{3}{2} \gamma^2 + c_1 + 3B_0 + c_3 &= 0.
\end{align*}
\]
Let
\[ c_1 = \frac{1}{2\alpha}(\beta^3 + 2c\alpha\beta - 6\alpha\beta), \]
\[ v(\xi) = aP(\xi) - \frac{\beta}{2\alpha}. \]  
Therefore from Eq. (23), we have
\[ k^2P''(\xi) - a\left(\frac{3\beta^2}{2\alpha} + 2c - 6\gamma\right)P(\xi) + 2\alpha^2P^3(\xi) = 0, \]  
then Eq. (25) can be written as
\[ AP''(\xi) + BP(\xi) + CP^3(\xi) = 0, \quad A = k^2, \]
\[ B = a\left(\frac{3\beta^2}{2\alpha} + 2c - 6\gamma\right), \quad C = 2\alpha^2 \]  
As described in Section 2, our main goal is to solve Eq. (26) by extended tanh method as follows:
By virtue of the technique of solution, we introduce the ansatz
\[ P(\xi) = \sum_{i=0}^{M} a_iY^i(\xi) + \sum_{i=1}^{M} b_iY^{-i}(\xi), \]  
where the primes denote differentiations, \( a_i \) and \( b_i \) are to be determined later. Balancing \( P^3 \) with \( P'' \) we get \( M = 1 \). Therefore we suppose \( P(\xi) \) in the following form:
\[ P(\xi) = a_0 + a_1Y(\xi) + b_1Y^{-1}(\xi). \]
Substituting Eq. (28) with the aid of Eq. (4) into Eq. (26), and setting each coefficient of \( Y^{\pm 1} \) to zero, yields a set of equations for \( a_0, a_1, b_1, \) and \( \mu \). Solving the system of algebraic equations with the aid of Maple, we obtain
\[ a_1 = 0, \quad a_0 = \sqrt{-\frac{B}{C}}, \quad b_1 = \sqrt{\frac{2B}{C}}, \quad \mu = \sqrt{-\frac{B}{A}}. \]
Substituting Eq. (29) into Eqs. (18) and (24), we obtain the following kink–solitons solutions:
\[ v(\xi) = \left[ \sqrt{-\frac{B}{C}} + \frac{2B}{C} \tanh\left( \sqrt{-\frac{B}{A^3}} - \frac{\beta \alpha}{2} \right) \right]^2, \]
\[ u(\xi) = \alpha \left[ a \left[ \sqrt{-\frac{B}{C}} + \frac{2B}{C} \tanh\left( \sqrt{-\frac{B}{A^3}} - \frac{\beta \alpha}{2} \right) \right]^2 \right] + \beta \left[ a \left[ \sqrt{-\frac{B}{C}} + \frac{2B}{C} \tanh\left( \sqrt{-\frac{B}{A^3}} - \frac{\beta \alpha}{2} \right) \right] \right] + \gamma, \]
\[ w(\xi) = A_0 \left[ a \left[ \sqrt{-\frac{B}{C}} + \frac{2B}{C} \tanh\left( \sqrt{-\frac{B}{A^3}} - \frac{\beta \alpha}{2} \right) \right] \right] + B_0. \]
We now employ the sine–cosine method to solve Eq. (26). Substituting (10) and (12) into (26) gives
\[ -\alpha \lambda (-A\mu^2 \beta^2 \cos^{\beta - 2}(\mu \xi) + A\mu^2 \beta^2 \cos^\beta(\mu \xi) + A\mu^2 \beta \cos^{\beta - 2}(\mu \xi) - B \cos^\beta(\mu \xi) - C\lambda^2 \cos^3(\mu \xi)) = 0. \]
Equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:
\[ (\beta - 1) \neq 1, \quad 3\beta = \beta - 2, \quad -\alpha \lambda (A\mu^2 \beta^2 - B) = 0, \quad -\alpha (-A\mu^2 \beta^2 + A\mu^2 \beta) - \alpha C\lambda^2 = 0. \]
Solving (32) with the aid of Maple, we obtain
\[ \beta = -1, \quad \lambda = \sqrt{\frac{2B}{C}}, \quad \mu = \sqrt{\frac{B}{A}}. \]
The results in Eq. (33) can be easily obtained if we also use the sine method (9). Combining (33) with (10), the following new periodic solutions:

\[ v(\xi) = a \sqrt{\frac{2B}{C}} \sec \left( \sqrt{\frac{B}{A}} \xi \right) - \frac{\beta x}{2}, \]

\[ u(\xi) = \alpha \left[ a \sqrt{\frac{2B}{C}} \sec \left( \sqrt{\frac{B}{A}} \xi \right) - \frac{\beta x}{2} \right]^2 + \beta \left[ a \sqrt{\frac{2B}{C}} \sec \left( \sqrt{\frac{B}{A}} \xi \right) - \frac{\beta x}{2} \right] + \gamma, \]

\[ w(\xi) = A_0 \left[ a \sqrt{\frac{2B}{C}} \sec \left( \sqrt{\frac{B}{A}} \xi \right) - \frac{\beta x}{2} \right] + B_0 \]  

\[ \text{3.2. The generalized-Zakharov equations} \]

The generalized-Zakharov equations for the complex envelope \( \psi(x, t) \) of the high-frequency wave and the real low-frequency field \( v(x, t) \) in the form [18]

\[ i\psi_t + \psi_{xx} - 2\lambda |\psi|^2 \psi + 2\psi v = 0, \]  

\[ v_{tt} - v_{xx} + (|\psi|^2)_{xx} = 0, \]  

where the cubic term in Eq. (35) describes the nonlinear-self interaction in the high frequency subsystem, such a term corresponds to a self-focusing effect in plasma physics. The coefficient \( \lambda \) is a real constant that can be a positive or negative number. Let us assume the travelling wave solution of Eqs. (35) and (36) in the form

\[ \psi(x, t) = e^{i\eta} u(\xi), \quad v = v(\xi), \]

\[ \eta = \alpha x + \beta t, \quad \xi = k(x - 2\alpha t), \]  

where \( u(\xi) \) and \( v(\xi) \) are real functions, the constants \( \alpha, \beta \) and \( k \) are to be determined. Substituting (37) into Eqs. (35) and (36), we have

\[ k^2 u'' + 2uv - (\alpha^2 + \beta)u - 2\lambda u^3 = 0, \]

\[ k^2 (4\alpha^2 - 1) v'' + k^2 (u^2)'' = 0. \]  

In order to simplify ODEs (38) and (39), integrating Eq. (39) once and taking integration constant to zero, and integrating yields

\[ v(\xi) = \frac{u^2}{(1 - 4\alpha^2)} + C \quad \text{if} \quad \alpha^2 \neq \frac{1}{4}, \]  

where \( C \)-integration constant. Inserting Eq. (40) into (38), we have

\[ Au'' + Bu + Cu^3 = 0, \]  

\[ A = k^2, \]

\[ B = [2C - \alpha^2 - \beta], \]

\[ C = 2 \left[ \frac{1}{1 - 4\alpha^2} - \lambda \right]. \]  

Proceeding as before, to solve Eq. (41) using extended tanh method, we introduce the anstaz

\[ u(\xi) = \sum_{i=0}^{M} a_i Y(\xi) + \sum_{i=1}^{M} b_i Y^{-i}(\xi). \]  

Balancing \( u^3 \) with \( u'' \) we get \( M = 1 \). Therefore we suppose \( u(\xi) \) as

\[ u(\xi) = a_0 + a_1 Y(\xi) + b_1 Y^{-1}(\xi). \]
Substituting Eq. (44) with Eq. (4) into Eq. (41), and setting each coefficients of \( Y^{\pm i} \) to zero, yields a set of equations for \( a_0, a_1, b_1, \) and \( \mu \). Solving the system of algebraic equations with the aid of Maple gives
\[
a_0 = 0, \quad a_1 = 0, \quad b_1 = \sqrt{-\frac{A}{B}}, \quad \mu = \frac{1}{2} \sqrt{\frac{2A}{D}}. \tag{45}
\]
Substituting Eq. (45) into Eqs. (44) and (37), we obtain the kink-bell shaped solution
\[
v(\xi) = -\frac{A}{B(1 - 4x^2)} \frac{1}{\tanh^2 \left( \frac{1}{2} \sqrt{\frac{2A}{D}} \xi \right)} + C,
\]
\[
\psi(\xi) = \sqrt{-\frac{A}{B}} \frac{1}{\tanh \left( \frac{1}{2} \sqrt{\frac{2A}{D}} \xi \right)} e^{\eta}, \tag{46}
\]
\[
\eta = \alpha x + \beta t, \quad \zeta = k(x - 2\alpha t).
\]
Similarly as before, the sine–cosine method is used to solve Eq. (41). Substituting (10) and (12) into (41) gives
\[
-\lambda(\lambda - A \cos^2(\mu \zeta) - B \lambda^2 \cos^3(\mu \zeta) - D \beta^2 \mu^2 \cos^{\beta-2}(\mu \zeta) + D \mu^2 \beta^2 \cos^\beta(\mu \zeta)) = 0. \tag{47}
\]
This equation is satisfied only the following system of algebraic equations holds:
\[
(\beta - 1) \neq 1, \quad 3\beta = \beta - 1, \quad A - D\mu^2 \beta^2 = 0, \quad \beta \lambda^2 + D\mu^2 \beta(\beta - 1) = 0. \tag{48}
\]
From (48), we obtain
\[
\beta = -1, \quad \lambda = \sqrt{2A}, \quad \mu = \frac{A}{D}. \tag{49}
\]
It is to be noted that the results in Eq. (49) can be easily obtained if we also use the sine method (9). Combining (49) with (10), we get the following new periodic solutions:
\[
v(\xi) = \frac{2A}{(1 - 4x^2)} \sec^2 \left( \frac{1}{2} \sqrt{\frac{2A}{D}} \xi \right) + C,
\]
\[
\psi(\xi) = \sqrt{2A} \sec \left( \frac{1}{2} \sqrt{\frac{2A}{D}} \xi \right) e^{\eta}, \tag{50}
\]
\[
\eta = \alpha x + \beta t, \xi = k(x - 2\alpha t).
\]

3.3. (2 + 1)-Dimensional Broer–Kaup–Kupershmidt (BKK) system

In this case, the celebrated (2 + 1)-dimensional Broer–Kaup–Kupershmidt system \([11]\) reads
\[
u_{yy} - u_{xyy} + 2(uu_y)_x + 2v_{xx} = 0,
\]
\[
v_t + v_{xx} + 2(uv)_x = 0. \tag{51}
\]

This system has been widely applied in many branches of physics like plasma physics, fluid dynamics, nonlinear optics, etc. So, a good understanding of more solutions of BKK system (51) is very helpful, especially for coastal and civil engineers to apply the nonlinear water model in a harbor and coastal design. The system BKK was used to model nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow water of uniform depth, and can also be derived from the celebrated Kadomtsev–Petviashvili (KP) equation by the symmetry constraint. We consider the wave transformation
\[
u = U(\xi), \quad v = V(\xi), \quad \xi = kx + ly + mt. \tag{52}
\]

Substitute Eq. (52) into (51), we get
\[
lmU'' - k^2 lU'' + 2kl(UU') + 2k^2 V'' = 0, \tag{53}
\]
\[
mV' + k^2 V'' + 2k(UV)' = 0. \tag{54}
\]
Integrating (53) once gives
\[ lmU' - k^2 lU'' + 2kl(UU') + 2k^2 V' = C, \]  
where \( C \) is an integration constant. Integrating Eq. (55) once again, we have
\[ V'(\xi) = \frac{C_1}{2k^2} - \frac{1}{2k} U^2 + \frac{l}{2} U' - \frac{lm}{2k^2} U, \]  
where \( C_1 \) is an integration constant. Substituting (56) into Eqs. (52) and (56) admits to a new solitons solution as follows:
\[ \frac{-3lm}{k} UU' + \left[ \frac{C_1}{k} - \frac{lm}{2k^2} \right] U' - 3lU'U^2 + \frac{lk^2}{2} U'' = 0. \]  
Integrating (57) once yields
\[ \frac{lk^2}{2} U'' - lU^3 - \frac{2lm}{2k} U^2 + \left[ \frac{C_1}{k} - \frac{lm}{2k^2} \right] U = 0. \]  
Then Eq. (58) can be written as
\[ k_1 U'' + k_2 U^3 + k_3 U^2 + k_4 U = 0, \]  
where
\[ k_1 = \frac{lk^2}{2}, \quad k_2 = -l, \quad k_3 = -\frac{2lm}{2k}, \quad k_4 = \left[ \frac{C_1}{k} - \frac{lm}{2k^2} \right]. \]  
Proceeding as before in Section 2, to solve Eq. (59) by means of tanh method, we use the anstaz
\[ U(\xi) = a_0 + a_1 Y(\xi) + b_1 Y^{-1}(\xi), \]  
From Eqs. (61) and (4) with Eq. (59), yields a set of equations for \( a_0, a_1, b_1, \) and \( \mu \). Solving the system of algebraic equations, we obtain
\[ a_0 = 0, \quad a_1 = 0, \quad \mu = \frac{1}{2} \sqrt{\frac{2k_4}{k_1}}, \quad b_1 = \sqrt{\frac{-k_4}{k_2}}. \]  
Substituting Eq. (62) into Eqs. (52) and (56), admits to a new solitons solution as follows:
\[ u(\xi) = \sqrt{-\frac{k_4}{k_2}} \coth \left( \frac{1}{2} \sqrt{\frac{2k_4}{k_1}} \xi \right), \]  
\[ v(\xi) = \frac{C_1}{2k^2} + \frac{lk_4}{kk_2} \coth^2 \left( \frac{1}{2} \sqrt{\frac{2k_4}{k_1}} \xi \right) - \frac{lm}{2k^2} \coth \left( \frac{1}{2} \sqrt{\frac{2k_4}{k_1}} \xi \right) + \frac{l}{2} u'(\xi), \]  
where
\[ \xi = kx + ly + mt. \]  
In the same manner, the sine–cosine method is used to solve Eq. (59). Substituting (10) into (59) yields
\[ -\lambda(-k_4 \cos^2(\mu \xi) - k_3 \lambda \cos^2(\mu \xi) - k_2 \lambda^2 \cos^2(\mu \xi) - k_1 \mu^2 \beta^2 \cos^2(\mu \xi) + k_1 \mu^2 \beta^2 \cos^2(\mu \xi) + k_1 \mu^2 \beta \cos^2(\mu \xi) = 0. \]  
This equation is satisfied only the following system of algebraic equations holds:
\[ (\beta - 1) \neq 1, \quad 3\beta = \beta - 2, \quad k_4 - k_1 \mu^2 \beta^2 = 0, \quad k_2 \lambda^2 - k_1 \beta \mu^2 = 0, \]  
which leads to
\[ \beta = -1, \quad \lambda = \sqrt{-\frac{k_4}{k_2}}, \quad \mu = \sqrt{\frac{k_4}{k_1}}. \]
The results in Eq. (66) can be directly evaluated by using the sine method via Eq. (9). Combining (66) with (10), admits to a new periodic solutions in the form

\[
\begin{align*}
    u(\xi) &= \sqrt{-\frac{k_4}{k_2}} \sec \left( \sqrt{\frac{k_4}{k_1}} \xi \right), \\
    v(\xi) &= \frac{C_1}{2k^2} + \frac{k_4}{kk_2} \sec^2 \left( \sqrt{\frac{k_4}{k_1}} \xi \right) + \frac{l}{2} u'(\xi) - \frac{lm}{2k^2} \sec \left( \sqrt{\frac{k_4}{k_1}} \xi \right), \\
    \xi &= kx + ly + mt.
\end{align*}
\]

(67)

4. Conclusion and discussion

The extended tanh and sine–cosine methods with a computerized symbolic computation are used to construct wide classes of periodic travelling wave solutions of three nonlinear equations arising in nonlinear physics, namely, generalized Hirota–Satsuma coupled KdV system, generalized Zakharov equations and (2 + 1)-dimensional Broer–Kaup–Kupershmidt system.

In this work, we presented a generalized extended tanh method based on the general ansatz (6) in which the exponent of tanh function may take both positive and negative values on the contrary to the solution ansatz (6) where its exponent is only positive values. This, of course, leads to the conclusion that the method of extended tanh method with the ansatz (6) can be used to improve the tanh function method [1–9]. The results revealed remarkable properties of the shapes in that the solutions may came as compactons, solitary pattern, periodic solutions, or solitons. The availability of computer systems like Mathematica or Maple facilitates the tedious algebraic calculations. The methods which we have proposed in this letter is also a standard, direct and computerizable methods, which allow us to do complicated and tedious algebraic calculation.

It is worth noting that the two proposed methods are reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear equations.

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References