Improved interval estimation for the two-parameter Birnbaum–Saunders distribution

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Abstract

An improved interval estimation for the two-parameter Birnbaum–Saunders distribution is discussed. The proposed method is based on the recently developed higher-order likelihood-based asymptotic procedure. The probability coverages of confidence intervals are based on the proposed method and those procedures discussed in Ng et al. (Comput. Statist. Data Anal., (2003)) are evaluated using Monte Carlo simulations for small and moderate sample sizes. Two real life examples and some concluding remarks are also presented.

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1. Introduction

Birnbaum and Saunders (1969a) proposed the two-parameter Birnbaum–Saunders distribution with density

\[ f(t; \alpha, \beta) = \frac{1}{2\sqrt{2\pi}\alpha \beta} \left[ \left( \frac{\beta}{t} \right)^{1/2} + \left( \frac{\beta}{t} \right)^{3/2} \right] \times \exp \left[ -\frac{1}{2\alpha^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right] \quad t, \alpha, \beta > 0, \]  

(1)
as a failure time distribution for fatigue failure caused under cyclic loading. The parameters \( \alpha \) and \( \beta \) are the shape and the scale parameters, respectively. In their derivation, it was assumed that the failure is due to the development and growth of a dominant crack. Desmond (1985) derived (1) based on a biological model and he also strengthened the physical justification for the use of this distribution by relaxing the assumptions made by Birnbaum and Saunders (1969a). Mann et al. (1974, p. 155) showed that (1) is unimodal and although the hazard rate is not an increasing function of \( t \), the average hazard rate is nearly a non-decreasing function of \( t \). A review of developments of (1) can be found in Johnson et al. (1995).

Birnbaum and Saunders (1969b) showed that to obtain the maximum likelihood estimators (MLEs), one needs to solve a non-linear equation in \( \beta \). Moreover, the exact joint distribution of the MLEs are unavailable. Engelhardt et al. (1981) obtained the asymptotic joint distribution of the MLEs and showed that the MLEs are asymptotically independent. Hence, confidence intervals for \( \alpha \) and \( \beta \) can be constructed based on this asymptotic joint distribution.

Since the conventional moment estimators for \( \alpha \) and \( \beta \) may not always exist and even if they do, they may not be unique (Ng et al., 2003, Section 4), Ng et al. (2003) proposed a modified moment estimators (MMEs) for \( \alpha \) and \( \beta \) which exists uniquely. The asymptotic distributions of the MMEs are derived and are used to construct confidence intervals for the unknown parameters. Simulation results in Ng et al. (2003) revealed that MLEs and MMEs are biased. They proposed two methods to reduce the bias and obtained two sets of estimations: (UMLE, UMME) and (JMLE, JMME). The first set of estimation is based on a standard bias reduction technique, whereas the second set of estimation is based on the Jackknife technique. Ng et al. (2003) used Monte Carlo simulations to compare the performance of all these estimators and concluded that even though the Jackknife estimates gives good probability coverages for small and moderate sample sizes, they were not recommended for large sample sizes due to computational intensity. On the other hand, the probability coverages for UMLEs and UMMEs were satisfactory for moderate sample size but not for small sample size. Due to the simplicity in calculations, the UMLEs and UMMEs were recommended by Ng et al. (2003).

Unfortunately, those expressions for the intervals of UMLE, MME and UMME for \( \beta \) are reported incorrectly in Ng et al. (2003). Furthermore, there is no guarantee that the upper bounds of those intervals are always positive. Details of these problems are recorded in Appendix. Hence a proper and more accurate inference method is needed for small sample size inference.

In this paper, a likelihood based higher order asymptotic method is proposed in Section 2 for constructing confidence intervals for the unknown parameters. This method, in theory, has accuracy of \( O(n^{-3/2}) \) and therefore is extremely accurate even for very small sample sizes. The method is then applied to the Birnbaum–Saunders distribution in Section 3. Monte Carlo simulations and examples are given in Section 4 to illustrate the accuracy, in terms of the probability coverages, of the proposed method. Some concluding remarks are recorded in Section 5.
2. Likelihood based inference

Consider \( t = (t_1, \ldots, t_n) \) be an independent sample from a distribution with log-likelihood function \( \ell(\theta) = \ell(\theta; t) \), where \( \theta = (\alpha, \beta) \) with \( \beta \) being the scalar parameter of interest and \( \alpha \) being the nuisance parameter. Two widely used methods for inference concerning \( \beta \) are based on the asymptotic distribution of the MLEs and of the signed log-likelihood ratio statistic, respectively. But both of them can be quite inaccurate when the sample size is small. The advantage of using the MLE for inference is because of the simplicity in calculations. However, the signed log-likelihood ratio statistic generally gives better coverage probabilities than the MLE (see Doganaksoy and Schmee, 1993).

In literature, various likelihood-based inference procedures have been proposed which, in theory, have higher order of accuracy, and are extremely accurate even when the sample size is very small. Reid (1996) gave a detail overview of this development. One of the higher order likelihood based method is given by Barndorff-Nielsen (1986, 1991) and is known as the modified signed log-likelihood ratio statistic.

\[
r^*(\beta) = r(\beta) + r(\beta)^{-1} \log \left\{ \frac{q(\beta)}{r(\beta)} \right\},
\]

where \( r(\beta) \) is the signed log-likelihood ratio statistic as defined by

\[
r(\beta) = \text{sgn}(\hat{\beta} - \beta) \{ 2[\ell(\hat{\theta}) - \ell(\hat{\theta}_\beta)] \}^{1/2},
\]

where \( \hat{\theta} = (\hat{\alpha}, \hat{\beta}) \) and \( \hat{\theta}_\beta = (\hat{\alpha}_\beta, \beta) \) are the overall MLE of \( \theta \) and constrained MLE of \( \theta \) for a given \( \beta \), respectively, and \( q(\beta) \) is a statistic which can be expressed in various forms depending on the type of available information. A general definition of \( q(\beta) \) is defined in Fraser et al. (1999) which takes the form

\[
q(\beta) = \frac{\left| \ell_{,V}(\hat{\theta}) - \ell_{,V}(\hat{\theta}_\beta) \ell_{,xV}(\hat{\theta}_\beta) \right|}{\left| \ell_{,xV}(\hat{\theta}) \right|} \left\{ \frac{|j_{\beta\theta}(\hat{\theta})|}{|j_{xx}(\hat{\theta}_\beta)|} \right\}^{1/2},
\]

where

\[
j_{\beta\theta}(\theta) = \begin{pmatrix} j_{xx}(\theta) & j_{x\beta}(\theta) \\ j_{x\beta}(\theta) & j_{\beta\beta}(\theta) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 \ell(\theta)}{\partial \alpha^2} & \frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell(\theta)}{\partial \beta^2} \end{pmatrix}
\]

is the observed Fisher information matrix and \( j_{xx}(\theta) \) is the observed nuisance information matrix. A new quantity in (4) is the likelihood gradient \( \ell_{,V}(\theta) \), where

\[
V = (v_1, v_2) = - \left( \frac{\partial z(t; \theta)}{\partial t} \right)^{-1} \left( \frac{\partial z(t; \theta)}{\partial \theta} \right) \bigg|_{\hat{\theta}}
\]
is a vector array with two vectors obtained from a vector pivotal quantity \( z(t; \theta) = (z_1(t; \theta), \ldots, z_n(t; \theta)) \). Hence the likelihood gradient is

\[
\ell_{V}(\theta) = \left\{ \frac{d}{dv_1} \ell(\theta; t), \frac{d}{dv_2} \ell(\theta; t) \right\}^	op,
\]

where \( (d/dv_k)\ell(\theta; t) = \sum_{i=1}^{n} \ell_{t_i}(\theta; t) v_{ki} \) \( (k = 1, 2) \) is the directional derivative of the log-likelihood function taken in the direction \( v_k = (v_{k1}, \ldots, v_{kn}) \) on the data space with gradient \( \ell_{t_i}(\theta; t) = \langle \partial/\partial t_i \rangle \ell(\theta; t), i = 1, \ldots, n \). Furthermore,

\[
\ell_{0,V}(\theta) = \frac{\partial \ell_{V}(\theta)}{\partial \theta}.
\]

Note that Fraser et al. (1999) showed that \( r^*(\beta) \) is asymptotically distributed as \( N(0, 1) \) and with order of accuracy \( O(n^{-3/2}) \). A \( 100(1 - \gamma)\% \) confidence interval for \( \beta \) based on \( r^*(\beta) \) is

\[
\{ \beta : |r^*(\beta)| \leq z_{\gamma/2} \}.
\]

3. Inference on parameters of the Birnbaum–Saunders distribution

Let \( t = (t_1, \ldots, t_n) \) be an independent sample from the Birnbaum–Saunders distribution with density given in (1). The log-likelihood function for \( \theta = (\alpha, \beta) \) can be written as

\[
\ell(\theta) = -n \log(\alpha \beta) + \sum_{i=1}^{n} \log(\beta \alpha^{-1} + \beta \alpha^{-1} \alpha^{2} t^{-1/2} + \beta \alpha^{-3/2} t^{-3/2}) - \frac{1}{2 \alpha^2} \sum_{i=1}^{n} \left( \frac{t_i}{\beta} - 1 \right) \left( \frac{t_i}{\beta} + \frac{1}{\beta} \right) \left( \frac{t_i}{\beta} - 2 \right).
\]

Let

\[
u = \frac{1}{n} \sum_{i=1}^{n} t_i, \quad v = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_i} \right)^{-1}, \quad \text{and} \quad K(x) = \left( \frac{1}{n} \sum_{i=1}^{n} \frac{1}{x + t_i} \right)^{-1}.
\]

Then the first two derivatives of the log-likelihood function with respect to the two parameters are

\[
\frac{\partial \ell(\theta)}{\partial \alpha} = -\frac{n}{\alpha} + \frac{n}{\alpha^3} \left( \frac{u}{\beta} + \frac{\beta}{v} - 2 \right),
\]

\[
\frac{\partial \ell(\theta)}{\partial \beta} = -\frac{n}{\beta} + \frac{n}{\beta^2} \left( \frac{1}{\beta} + \frac{2}{K(\beta)} \right) + \frac{n}{2 \alpha^2} \left( \frac{u}{\beta^2} - \frac{1}{v} \right),
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \alpha^2} = \frac{n}{\alpha^2} - \frac{3n}{\alpha^4} \left( \frac{u}{\beta} + \frac{\beta}{v} - 2 \right),
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \beta^2} = \frac{n}{\beta^2} + \frac{2n}{\beta^3} \left( \frac{1}{\beta} + \frac{2}{K(\beta)} \right) + \frac{n}{\alpha^2} \left( \frac{u}{\beta^2} - \frac{1}{v} \right),
\]

\[
\frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \beta} = -\frac{n}{\alpha \beta} + \frac{2n}{\alpha \beta^2} \left( \frac{1}{\beta} + \frac{2}{K(\beta)} \right) + \frac{3n}{\alpha^2 \beta} \left( \frac{u}{\beta^2} - \frac{1}{v} \right).
\]
\[
\frac{\partial^2 \ell(\theta)}{\partial \alpha \partial \beta} = \frac{n}{\alpha^3} \left( -\frac{u}{\beta^2} + \frac{1}{v} \right), \\
\frac{\partial^2 \ell(\theta)}{\partial \beta^2} = \frac{n}{\beta^2} - \frac{n}{2} \left( \frac{1}{\beta^2} + \frac{2K'(\beta)}{K^2(\beta)} \right) - \frac{n}{\alpha^2 \beta^2}.
\]

The overall MLEs are obtained by solving \( \left( \frac{\partial \ell(\theta)}{\partial \alpha} \right) = 0 \) and \( \left( \frac{\partial \ell(\theta)}{\partial \beta} \right) = 0 \), simultaneously. With restriction of the parameter space to positive, the MLE of \( \beta \) can be obtained as the unique positive root of the non-linear equation
\[
\hat{\beta}^2 + v[u + K(\hat{\beta})] - \hat{\beta}[2v + K(\hat{\beta})] = 0.
\]
Once \( \hat{\beta} \) is obtained, the MLE of \( \alpha \) can be obtained explicitly by
\[
\hat{\alpha} = \left( \frac{u}{\hat{\beta}} + \hat{\beta} - 2 \right)^{1/2}.
\]

Engelhardt et al. (1981) obtained the asymptotic joint distribution for \( (\hat{\alpha}, \hat{\beta}) \) and showed that \( (\hat{\alpha}, \hat{\beta}) \) are asymptotically independent.

### 3.1. Inference concerning \( \beta \)

For a given \( \beta \), the constrained MLE of \( \alpha \) is
\[
\hat{\alpha}_\beta = \left( \frac{u}{\hat{\beta}} + \hat{\beta} - 2 \right)^{1/2}.
\]

Hence, from (3) the signed log-likelihood ratio statistic for \( \beta \), \( r(\beta) \) can be obtained. Note that
\[
z_i = z_i(t_i; \theta) = \frac{1}{\alpha} \left[ \left( \frac{t_i}{\beta} \right)^{1/2} - \left( \frac{t_i}{\beta} \right)^{-1/2} \right]
\]
is distributed as \( \text{N}(0,1) \). Thus \( z = (z_1, \ldots, z_n) \) is a vector pivotal quantity. Moreover, for \( i \neq j \),
\[
\frac{\partial z_i}{\partial t_j} = 0
\]
and
\[
\frac{\partial z_i}{\partial t_i} = \frac{1}{2\alpha} t_i^{-3/2} \beta^{-1/2}(t_i + \beta).
\]

Furthermore, we have
\[
\frac{\partial z_i}{\partial \alpha} = -\frac{1}{2\alpha} t_i^{-1/2} \beta^{-1/2}(t_i - \beta),
\]
\[
\frac{\partial z_i}{\partial \beta} = -\frac{1}{2\alpha} t_i^{-1/2} \beta^{-3/2}(t_i + \beta).
\]
and the \( i \)th components of vector \( v_1 \) and \( v_2 \) are

\[
v_{1i} = \frac{2t_i(t_i - \hat{\beta})}{t_i + \hat{\beta}} \quad \text{and} \quad v_{2i} = \frac{t_i}{\hat{\beta}}.
\]

Therefore,

\[
\ell_{d_i}(\theta) = \frac{\partial \ell(\theta)}{\partial t_i} = -\frac{1 + 3\beta t_i^{-1}}{2(t_i + \hat{\beta})} - \frac{1}{2\alpha^2} \left( \frac{1}{\hat{\beta}} - \frac{\beta}{t_i} \right),
\]

\[
\ell_{x_d}(\theta) = \frac{\partial \ell_{d_i}(\theta)}{\partial x} = \frac{1}{\alpha^2} \left( \frac{1}{\hat{\beta}} - \frac{\beta}{t_i^2} \right),
\]

\[
\ell_{\beta_d}(\theta) = \frac{\partial \ell_{d_i}(\theta)}{\partial \beta} = -(t_i + \beta)^{-2} + \frac{1}{2\alpha^2} \left( \frac{1}{\beta^2} + \frac{1}{t_i^2} \right).
\]

Thus, we have

\[
\ell_{\theta_d}(\theta) = \left( \sum_{i=1}^{n} \ell_{d_i}(\theta)v_{1i} \right)
\]

\[
\ell_{\theta_d}(\theta) = \left( \sum_{i=1}^{n} \ell_{d_i}(\theta)v_{2i} \right)
\]

\[
\ell_{\theta_d}(\theta) = (\ell_{x_d}(\theta), \ell_{\beta_d}(\theta)) = \left( \sum_{i=1}^{n} \ell_{x_d}(\theta)v_{1i}, \sum_{i=1}^{n} \ell_{\beta_d}(\theta)v_{1i} \right).
\]

Finally \( q(\beta) \) can be obtained from (4) and hence inference concerning \( \beta \) can be obtained from \( r^*(\beta) \) which is defined in (2).

3.2. Inference concerning \( x \)

For a given \( x \), the constrained MLE of \( \beta \) is same as the overall MLE. That is \( \hat{\beta}_x = \hat{\beta} \) and \( \hat{\theta}_x = (x, \hat{\beta}_x) = (x, \hat{\beta}) \). The observed nuisance information matrix is \( j_\beta(\theta) = -(\hat{\beta}^2 \ell(x, \beta) / \hat{\beta}^2) \). All other quantities are given in Section 3.1. Hence \( q(x) \) can be obtained from (4) with \( x \) being the parameter of interest and \( \beta \) being the nuisance parameter. Thus \( r^*(x) \) can then be obtained from (2).

4. Numerical examples

4.1. Simulation study

To study the accuracy of the proposed method, we performed the following Monte Carlo simulation study. We took the sample size as \( n = 5, 10, 20 \), and the shape
Table 1

<table>
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<th>n</th>
<th>x</th>
<th>r</th>
<th>r^*</th>
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<th>MME</th>
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parameter as \( x = 0.1, 0.5, 1.0, 2.0 \). Without loss of generality, the scale parameter \( \beta \) was kept fixed at 1.0.

Notice that if \( T \) has Birnbaum–Saunders distribution with parameters \( x \) and \( \beta \), then

\[
X = \frac{1}{2} \left[ \left( \frac{T}{\beta} \right)^{1/2} - \left( \frac{T}{\beta} \right)^{-1/2} \right]
\]

has a normal distribution with mean 0 and variance \( \frac{1}{4} x^2 \). Hence random data that follow Birnbaum–Saunders distribution can be generated from the normal distribution based on this relationship.

All the results were based on 10,000 Monte Carlo runs. We computed the MLE, MME, UMLE, UMME, \( r \) and \( r^* \) for each run, and then obtained the 90% and 95% coverage probability. The results obtained are reported in Tables 1–4.

From our simulation results, the coverage probabilities for \( r \), MLE and MME are not satisfactory for both \( x \) and \( \beta \). For sample sizes 10 and 20, the coverage probabilities of UMLE and UMME are satisfactory for \( x \), but they are not satisfactory for \( \beta \). Furthermore, for sample size 5, their coverage probabilities are much worse. In particular, the coverage probabilities of MLE and MME for \( \beta \) when \( n = 5 \) encountered the problem of negative upper bounds (see the appendix for more discussion on this problem). However, \( r^* \) gives excellent probability coverages for both \( x \) and \( \beta \) for any sample size being considered.

**Example 1.** The data set is given by Birnbaum and Saunders (1969b) on the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles/s (cps). The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data are present in Ng et al. (2003). The results of the confidence intervals for \( x \) and \( \beta \) based on the MLEs, MMEs, UMLEs, UMMEs, \( r \) and
Table 2
Probability coverages of 95% confidence intervals for \( x \)

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Table 3
Probability coverages of 90% confidence intervals for \( \beta \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x )</th>
<th>( r )</th>
<th>( r^* )</th>
<th>MLE</th>
<th>MME</th>
<th>UMLE</th>
<th>UMME</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.1</td>
<td>0.8465</td>
<td>0.8879</td>
<td>0.7736</td>
<td>0.7737</td>
<td>0.8490</td>
<td>0.8490</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8266</td>
<td>0.8899</td>
<td>0.7731</td>
<td>0.7730</td>
<td>0.8466</td>
<td>0.8464</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.8268</td>
<td>0.8892</td>
<td>0.7692</td>
<td>0.7695</td>
<td>0.8332</td>
<td>0.8332</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.8160</td>
<td>0.8902</td>
<td>0.5969(^a)</td>
<td>0.5695(^a)</td>
<td>0.8021</td>
<td>0.8080</td>
</tr>
<tr>
<td>10</td>
<td>0.1</td>
<td>0.8666</td>
<td>0.8911</td>
<td>0.8373</td>
<td>0.8374</td>
<td>0.8751</td>
<td>0.8752</td>
</tr>
<tr>
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<td>0.5</td>
<td>0.8661</td>
<td>0.8946</td>
<td>0.8399</td>
<td>0.8400</td>
<td>0.8755</td>
<td>0.8756</td>
</tr>
<tr>
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<td>0.8652</td>
<td>0.8943</td>
<td>0.8392</td>
<td>0.8401</td>
<td>0.8674</td>
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<tr>
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<td>2.0</td>
<td>0.8627</td>
<td>0.8956</td>
<td>0.8334</td>
<td>0.8370</td>
<td>0.8509</td>
<td>0.8542</td>
</tr>
<tr>
<td>20</td>
<td>0.1</td>
<td>0.8854</td>
<td>0.8930</td>
<td>0.8684</td>
<td>0.8683</td>
<td>0.8858</td>
<td>0.8858</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8852</td>
<td>0.8970</td>
<td>0.8696</td>
<td>0.8697</td>
<td>0.8882</td>
<td>0.8883</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.8819</td>
<td>0.8950</td>
<td>0.8660</td>
<td>0.8663</td>
<td>0.8804</td>
<td>0.8810</td>
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<tr>
<td></td>
<td>2.0</td>
<td>0.8775</td>
<td>0.8939</td>
<td>0.8585</td>
<td>0.8641</td>
<td>0.8701</td>
<td>0.8720</td>
</tr>
</tbody>
</table>

\(^a\) Due to negative upper bounds.

\( r^* \) are presented in Table 5. Due to large sample size, all six methods give very similar 90% and 95% confidence intervals.

**Example 2.** This example is from McCool (1974) on the fatigue life in hours of 10 bearings of a certain type. The data are

<table>
<thead>
<tr>
<th>152.7</th>
<th>172.0</th>
<th>172.5</th>
<th>173.3</th>
<th>193.0</th>
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</thead>
<tbody>
<tr>
<td>204.7</td>
<td>216.5</td>
<td>234.9</td>
<td>262.6</td>
<td>422.6</td>
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</table>
Table 4
Probability coverages of 95% confidence intervals for $\beta$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha$</th>
<th>$r$</th>
<th>$r^*$</th>
<th>MLE</th>
<th>MME</th>
<th>UMLE</th>
<th>UMME</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.1</td>
<td>0.9105</td>
<td>0.9416</td>
<td>0.8346</td>
<td>0.8345</td>
<td>0.8970</td>
<td>0.8970</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8957</td>
<td>0.9424</td>
<td>0.8338</td>
<td>0.8339</td>
<td>0.8912</td>
<td>0.8914</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.8956</td>
<td>0.9433</td>
<td>0.7937</td>
<td>0.7973</td>
<td>0.8732</td>
<td>0.8783</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.8868</td>
<td>0.9457</td>
<td>0.3264</td>
<td>0.2984a</td>
<td>0.8534</td>
<td>0.8563</td>
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<tr>
<td>10</td>
<td>0.1</td>
<td>0.9257</td>
<td>0.9443</td>
<td>0.8982</td>
<td>0.8982</td>
<td>0.9241</td>
<td>0.9241</td>
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<tr>
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<td>0.5</td>
<td>0.9254</td>
<td>0.9466</td>
<td>0.8964</td>
<td>0.8966</td>
<td>0.9224</td>
<td>0.9227</td>
</tr>
<tr>
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<td>1.0</td>
<td>0.9244</td>
<td>0.9460</td>
<td>0.8888</td>
<td>0.8901</td>
<td>0.9138</td>
<td>0.9147</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.9200</td>
<td>0.9448</td>
<td>0.8803</td>
<td>0.8792</td>
<td>0.9010</td>
<td>0.9035</td>
</tr>
<tr>
<td>20</td>
<td>0.1</td>
<td>0.9409</td>
<td>0.9461</td>
<td>0.8888</td>
<td>0.8901</td>
<td>0.9138</td>
<td>0.9147</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.9401</td>
<td>0.9496</td>
<td>0.9256</td>
<td>0.9258</td>
<td>0.9374</td>
<td>0.9374</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.9407</td>
<td>0.9497</td>
<td>0.9213</td>
<td>0.9212</td>
<td>0.9324</td>
<td>0.9330</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.9365</td>
<td>0.9474</td>
<td>0.9085</td>
<td>0.9112</td>
<td>0.9221</td>
<td>0.9240</td>
</tr>
</tbody>
</table>

Note: *Due to negative upper bounds.

Table 5
Interval estimates of $\alpha$ and $\beta$ for Example 1

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\alpha$</th>
<th></th>
<th></th>
<th>$\beta$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90% CI</td>
<td>95% CI</td>
<td></td>
<td>90% CI</td>
<td>95% CI</td>
<td></td>
</tr>
<tr>
<td>MLE</td>
<td>(0.1527, 0.1927)</td>
<td>(0.1497, 0.1976)</td>
<td>(128.2552, 135.5861)</td>
<td>(127.5944, 136.3325)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MME</td>
<td>(0.1527, 0.1927)</td>
<td>(0.1497, 0.1976)</td>
<td>(128.2556, 135.5866)</td>
<td>(127.5948, 136.3330)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UMLE</td>
<td>(0.1541, 0.1949)</td>
<td>(0.1511, 0.1999)</td>
<td>(128.2116, 135.6143)</td>
<td>(127.5448, 136.3685)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UMME</td>
<td>(0.1541, 0.1949)</td>
<td>(0.1511, 0.1999)</td>
<td>(128.2121, 135.6148)</td>
<td>(127.5452, 136.3690)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r$</td>
<td>(0.1525, 0.1922)</td>
<td>(0.1493, 0.1969)</td>
<td>(128.1823, 135.5578)</td>
<td>(127.4856, 136.2995)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r^*$</td>
<td>(0.1536, 0.1940)</td>
<td>(0.1504, 0.1988)</td>
<td>(128.1552, 135.5862)</td>
<td>(127.4532, 136.3326)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results of the confidence intervals for $\alpha$ and $\beta$ based on the MLE, MME, UMLE, UMME, $r$ and $r^*$ are presented in Table 6. In this example, the sample size is small. Hence the six methods give relatively different 90% and 95% confidence intervals. Notice that the confidence intervals based on MLE and MME are closer to those of $r$. Also the confidence intervals based on UMLE and UMME are close to those of $r^*$ but some differences exist for the upper bounds of the confidence intervals for $\alpha$ and the lower bounds of the confidence intervals for $\beta$. Based on the simulation results we have previously, we will suggest to use $r^*$.

5. Discussion

A simple and accurate method to approximate confidence intervals for the parameters of the Birnbaum–Saunders distribution is proposed. From our Monte Carlo simulations,
Table 6
Interval estimates of $\theta_T$ and $\theta_F$ for Example 2

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\theta_T$</th>
<th>$\theta_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>90% CI</td>
<td>95% CI</td>
</tr>
<tr>
<td>MLE</td>
<td>(0.2065, 0.4468)</td>
<td>(0.1964, 0.5029)</td>
</tr>
<tr>
<td>MME</td>
<td>(0.2065, 0.4468)</td>
<td>(0.1964, 0.5029)</td>
</tr>
<tr>
<td>UMLE</td>
<td>(0.2228, 0.5308)</td>
<td>(0.2111, 0.6118)</td>
</tr>
<tr>
<td>UMME</td>
<td>(0.2228, 0.5308)</td>
<td>(0.2111, 0.6118)</td>
</tr>
<tr>
<td>$r$</td>
<td>(0.2035, 0.4294)</td>
<td>(0.1927, 0.4718)</td>
</tr>
<tr>
<td>$r^*$</td>
<td>(0.2168, 0.4892)</td>
<td>(0.2045, 0.5432)</td>
</tr>
</tbody>
</table>

$r^*$ outperformed MLE, MME, UMLE and UMME in terms of coverage probabilities. Moreover, $r^*$ is very simple to use.

Acknowledgement

The first author’s research was supported in part by National Cancer Center support grant CA21765 and American Lebanses Syrian Associated Charities (ALSAC).

Appendix.

The interval expressions of $\theta_F$ for UMLE, MME and UMME in Ng et al. (2003) are reported incorrectly. The correct 100$(1 - \gamma)$% confidence intervals for $\theta_F$ based on the MLE, MME, UMLE and UMME are

\[
\begin{align*}
\hat{\beta} \left( \hat{\theta}_2 \sqrt{\frac{h_1(\hat{\theta})}{n} + 1} \right)^{-1}, & \quad \hat{\beta} \left( \hat{\theta}_2 \sqrt{\frac{h_1(\hat{\theta})}{n} + 1} \right)^{-1}, \\
\hat{\beta} \left( \hat{\theta}_2 \sqrt{\frac{h_2(\hat{\theta})}{n} + 1} \right)^{-1}, & \quad \hat{\beta} \left( \hat{\theta}_2 \sqrt{\frac{h_2(\hat{\theta})}{n} + 1} \right)^{-1}, \\
\hat{\beta}^* \left( \sqrt{\frac{n}{n_1(\hat{\theta}^*)} \frac{4\hat{\theta}_2}{4n + \hat{\theta}_2^2} + 1} \right)^{-1}, & \quad \hat{\beta}^* \left( \sqrt{\frac{n}{n_1(\hat{\theta}^*)} \frac{4\hat{\theta}_2}{4n + \hat{\theta}_2^2} + 1} \right)^{-1}, \\
\hat{\beta}^* \left( \frac{4\hat{\theta}_2}{4n + \hat{\theta}_2^2} \sqrt{n}h_2(\hat{\theta}^*) + 1 \right)^{-1}, & \quad \hat{\beta}^* \left( \frac{4\hat{\theta}_2}{4n + \hat{\theta}_2^2} \sqrt{n}h_2(\hat{\theta}^*) + 1 \right)^{-1},
\end{align*}
\]

respectively, where the function $h_1$ and $h_2$ are given in Ng et al. (2003). We notice that the upper bounds of those intervals may be negative for small sample size and some $\theta$ values. To illustrate our argument, let us consider the following: for the intervals
based on MLE, MME, UMLE and UMME, the upper bounds is positive if only if their boundary functions $h_1(\hat{\alpha}), \left(1/\hat{\alpha}^2 h_2(\hat{\alpha})\right), h_1(\hat{\alpha}^*)\left(1 + (\hat{\alpha}^*)^2/4n\right)^2$ and $(1+(\hat{\alpha}^*/4n))^2$ are greater than $z^2_{\alpha}/n$. For sample size $n = 5, 10$ and $20$, we list the values of $z^2_{\alpha}/n$ in following table:

<table>
<thead>
<tr>
<th></th>
<th>90% CI</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$z^2_{\alpha}/n$</td>
<td>$z^2_{\alpha}/n$</td>
</tr>
<tr>
<td>5</td>
<td>0.54</td>
<td>0.768</td>
</tr>
<tr>
<td>10</td>
<td>0.27</td>
<td>0.384</td>
</tr>
<tr>
<td>20</td>
<td>0.135</td>
<td>0.192</td>
</tr>
</tbody>
</table>

We plotted the boundary functions for $n = 5$ in Figs. 1–4. From Fig. 1, we observe that for $\hat{\alpha} > 1.856$, the boundary function is less than $z^2_{\alpha}/n$ which gives the upper
bound of confidence interval based on MLE is negative. From Fig. 2, observe that for $\hat{\alpha} > 2.099$, the boundary function is less than $z_{0.025}^2/n$ which gives the upper bound of confidence interval based on MME is negative. From Figs. 3 and 4, the upper bounds for both UMLE and UMME is finite as the bound functions have minimum values 0.837 and 0.8048, respectively.

**References**