K₆-Minors in triangulations and complete quadrangulations

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Abstract

In this paper, we shall prove that a projective-planar (resp., toroidal) triangulation $G$ has $K_6$ as a minor if and only if $G$ has no quadrangulation isomorphic to $K_4$ (resp., $K_5$) as a subgraph. As an application of the theorems, we can prove that Hadwiger’s conjecture is true for projective-planar and toroidal triangulations.

Keywords: triangulation, minor, quadrangulation, complete graph, torus, projective plane

1 Introduction

A triangulation on a closed surface is a simple graph embedded in the surface so that each face is triangular. A quadrangulation on a surface is a simple graph embedded in the surface with each face quadrilateral. An $H$-quadrangulation is a quadrangulation isomorphic to $H$ as a graph. For a graph $G$ on a surface and a vertex $v$, the link of $v$ is the boundary closed walk of the union of all faces incident to $v$ in $G$. Clearly, the link of each vertex in a triangulation must be a cycle. For a graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. A $k$-cycle means a cycle of length exactly $k$. For a graph $G$ and $S \subset V(G)$, let $[S]$ denote the subgraph of $G$ induced by $S$.

When we use symbols with subscripts, we take the subscripts with suitable modulus.

Contraction of an edge $e$ (or contracting $e$) in a graph $G$ on a surface as a local transformation in $G$ removing $e$ and identifying the endpoints of $e$. (When we apply a contraction of an edge in a triangulation, we have to replace two pairs of multiple edges by two single edges respectively after the contraction.) The inverse operation of a contraction of $e$ is called a vertex-splitting of $[e]$, where $[e]$ is the vertex obtained from $e$ by its contraction. A graph $H$ is called a minor of a graph $G$ if $H$ can be obtained from $G$ by a series of contractions and deletions of edges. We say that $G$ has an $H$-minor if $G$ has $H$ as a minor.
In this paper, we consider the following problem: For a given triangulation $G$ on a surface $F^2$, which complete graph $K_n$ is contained in $G$ as a minor?

Obviously, every triangulation $G$ on any surface has a $K_4$-minor. (Observe that each vertex of $G$ has degree at least 3. The graph consisting of a vertex and its link includes a subdivision of $K_4$.) For $K_5$-minors in graphs on surfaces, the following is well-known:

**THEOREM 1 (Kuratowski-Wagner [12])** Every planar graph has no $K_5$ as a minor.

Though no triangulation on the sphere has a $K_5$-minor by Theorem 1, we can easily prove the following theorem, whose proof is put in the next section.

**PROPOSITION 2** Every triangulation on any nonspherical surface has $K_5$ as a minor.

However, it is known that for

$$m = \left\lfloor \frac{7 + \sqrt{49 - 24\epsilon(F^2)}}{2} \right\rfloor$$

(where $\epsilon(F^2)$ is the Euler characteristic of $F^2$), $G$ has no $K_n$-minor for any $n > m$, since $K_n$ does not embed in $F^2$.

In this paper, we shall characterize triangulations on the projective plane and the torus with $K_6$-minors by the existence of a complete quadrangulation (i.e., a complete graph quadrangulating the surfaces). Our main theorems are the following:

**THEOREM 3** Let $G$ be a triangulation on the projective plane. Then $G$ has $K_6$ as a minor if and only if $G$ has no $K_4$-quadrangulation as a subgraph.

**THEOREM 4** Let $G$ be a triangulation on the torus. Then $G$ has $K_6$ as a minor if and only if $G$ has no $K_5$-quadrangulation as a subgraph.

Note that only the projective plane and the torus admit a quadrangulation by $K_m$ with $m \leq 5$, and hence such theorems for $K_6$-minors do not hold in any other surfaces. Moreover, every nonspherical surface admits a triangulation with no $K_6$-minor. (Such an example can be constructed from several copies of a toroidal or a projective planar triangulation with no $K_6$-minor, by the operation in Lemma 9.)

The following is an immediate consequence from Theorems 3 and 4, since any triangulation containing a complete quadrangulation as a subgraph must have a separating 4-cycle.

**COROLLARY 5** Every 5-connected triangulation on the projective plane and the torus has $K_6$ as a minor. ■

Fijavž and Mohar [4] proved that every 5-connected graph $G$ on the projective plane with face-width at least 3 has a $K_6$-minor. (The *face-width* of $G$ is the minimum number of intersecting points of $G$ and $\gamma$, where $\gamma$ ranges over all simple closed curves not bounding 2-cells on the surface. Hence every triangulation on a nonspherical surface has face-width at least 3.) This result is stronger than Corollary 5 since they do not restrict a graph to be a triangulation, but our proof is much shorter and Theorem 3 characterizes triangulations with $K_6$-minors.

A *k-coloring* of a graph $G$ is a map $c : V(G) \to \{1, \ldots, k\}$ such that for any $xy \in E(G)$, $c(x) \neq c(y)$. We say that a graph $G$ is *k-colorable* if $G$ admits a $k$-coloring. The following is a well-known conjecture by Hadwiger:
Hence, every triangulation on any nonspherical surface has a this order and since the choice of any vertex $v$, we can find an edge $v_i v_{i+1}$ with deg$(v_i) = k \geq 4$. Let $C = v_1 \cdots v_k$ be the link of $v$. Observe that for any $i$, a contraction of $v_i v_j$ in $G$ yields multiple edges. Therefore, for each $i$, there is an edge $v_i v_j$ for some $j \notin \{i - 1, i, i + 1\}$. Among all such pairs of $\{i, j\}$, choose $i, j$ so that $|j - i|$ is minimum, where we may suppose that $1 \leq i \leq j \leq k$. Now consider a contraction of an edge $v_i v_{i+1}$. Since the contraction of $v_i v_{i+1}$ must yield multiple edges, we can find an edge $v_{i+1} v_{j'}$ with $j' \notin \{i, i + 1, i + 2\}$. By the choice of $i, j$, we have $j' \notin \{i, i + 1, \ldots, j\}$. (For otherwise, we would have $|j' - (i + 1)| < |j - i|$, contrary to the choice of $i, j$.) Therefore, since $v_i, v_{i+1}, v_j, v_{j'}$ are distinct four vertices lying on $C$ in this order and since $v_i v_j, v_{i+1} v_{j'} \in E(G)$, we can find a subdivision of $K_5$ in $\{(v) \cup V(C)\}$. Hence, every triangulation on any nonspherical surface has a $K_5$-minor.

The above is an elementary proof of Proposition 2 using a property of irreducible triangulations, but some earlier results are known to imply the existence of $K_5$-minors in a wider class of graphs than triangulations on nonspherical surfaces: Wagner’s decomposition theorem [12] implies that every graph on $n$ vertices having at least $3n - 5$ edges contains a $K_5$-minor, see Corollary 7.3.5 in [2]. (A more recent result of Mader [8] states that even a subdivision of $K_5$ exists in such a graph.) By Euler formula, every triangulation on any nonspherical surface with $n$ vertices has at least $3n - 5$ edges.

It was proved in [1] that the projective plane admits precisely two irreducible triangulations, denoted by $P1$ and $P2$, shown in Figure 1. Note that $P1$ is isomorphic to $K_6$ as a graph, and that $P2$ has a $K_4$-quadrangulation as a subgraph. Moreover, the torus admits precisely 21 irreducible triangulations, denoted by $T1, \ldots, T21$ [6]. The complete lists of the irreducible triangulations on the sphere [10] and the Klein bottle [7, 11] have also been determined.

**Lemma 7** Each of $T1, \ldots, T20$ has $K_6$ as a minor, and $T21$ has a $K_4$-quadrangulation as a subgraph.
Proof. We have checked that $T_1, T_5, T_{12}, T_{19}$ include $K_6$ as a subgraph, and $T_2, T_3, T_4, T_6, T_7, T_8, T_9, T_{10}, T_{11}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{20}$ have $K_6$ as a minor. The details are added in Appendix. On the other hand, $T_{21}$ has a $K_5$-quadrangulation, as shown in Figure 2.

3 Lemmas

Let $G$ be a graph and let $A, B$ be disjoint subgraphs of $G$. We say that $A$ and $B$ are adjacent if $ab \in E(G)$ for some $a \in V(A)$ and $b \in V(B)$. The following is obvious from the definition of a minor.

**Lemma 8** A graph $G$ has a $K_n$-minor if and only if $G$ can be decomposed into $n$ disjoint connected subgraphs $A_1, \ldots, A_n$ so that for any distinct $i, j$, $A_i$ and $A_j$ are adjacent.

**Lemma 9** Let $m \leq 4$. For $i = 1, 2$, let $H_i$ be a graph which includes $K_m$ with $m$ vertices $v^i_1, \ldots, v^i_m$ as a subgraph. Let $G$ be the graph obtained from $H_1$ and $H_2$ by identifying $v^1_j$ and $v^2_j$ for $j = 1, \ldots, m$. If each of $H_1$ and $H_2$ has no $K_6$-minor, then neither does $G$.

Proof. Suppose that $G$ contains a $K_6$-minor as well as a complete separator $S$ with $|S| \leq 4$, i.e., $S \subset V(G)$ such that $G - S$ is disconnected and that $[S]$ is complete. By Lemma 8, $G$
can be decomposed into six pairwise adjacent disjoint connected subgraphs. We call each of the connected graphs a bag of the $K_6$-minor.

Since $|S| \leq 4$, at most four bags of the $K_6$-minor intersect $S$. (Such a bag is called an S-bag.) Since any two of the remaining (at least two) bags must be adjacent in $G$, all of such bags are contained in the same component $T$ of $G - S$, where a bag contained in $T$ is called a T-bag. Let $H_1$ be the subgraph of $G$ induced by $S \cup \partial(T)$.

We shall prove that $H_1$ has six pairwise adjacent bags. Note that an S-bag in $G$ might become smaller by the removal of all vertices of $H$, since any planar graph has no $K_6$-minor. Hence we regard it as an S-bag in $H_1$. Therefore, all of $S$-bags are pairwise adjacent in $H_1$ since $S$ is complete. Moreover, any two T-bags are adjacent in $T$ and any T-bag is adjacent to all S-bags in $H_1$, since $G$ has a $K_6$-minor. Therefore, the proof completes. \[\Box\]

A plane triangulation $G$ means a 2-connected plane graph with each inner face triangular, where the outer cycle might not be triangular. Let $\partial G$ denote the outer cycle of $G$, and let $\bar{G} = G - V(\partial G)$. We use these notations throughout this paper.

Let us prove the necessity of Theorems 3 and 4, using Lemma 9.

**Proposition 10** If a triangulation $G$ has a $K_m$-quadrangulation $H_0$ with $m \in \{4, 5\}$ as a subgraph, then $G$ has no $K_6$-minor.

**Proof.** Let $f_1, \ldots, f_k$ be all quadrilateral faces of $H_0$. For $i = 1, \ldots, k$, let $H_i$ denote the subgraph of $G$ induced by the vertices contained in the interior and the boundary of $f_i$. Then each $H_i$ is a simple graph obtained from a plane triangulation with outer 4-cycle $f_i = x_1x_2x_3x_4$ by adding two edges $x_1x_3$ and $x_2x_4$. Observe that $H_0$ has no $K_6$-minor since $H_0$ is a complete graph with at most five vertices. Moreover, for $i = 1, \ldots, k$, $H_i$ has no $K_6$-minor since $H_i$ with one edge, say $x_1x_3$, removed is a plane triangulation and since any planar graph has no $K_5$-minor by Theorem 1. Hence, by Lemma 9, $H_0 \cup H_1$ has no $K_6$-minor, since $[V(H_0) \cap V(H_1)] = K_4$. Similarly, $(H_0 \cup H_1) \cup H_2$ has no $K_6$-minor. Repeating this, we conclude that $(H_0 \cup \cdots \cup H_{k-1}) \cup H_k = G$ has no $K_6$-minor. \[\Box\]

**Lemma 11** Let $G$ be a plane triangulation whose outer cycle $C$ is a 4-cycle. A 4-coloring of $C$ using precisely four distinct colors extends to a 5-coloring of $G$.

**Proof.** We prove that $G$ has a 5-coloring such that the four vertices of $C$ are colored by four distinct colors. Let $C = v_1v_2v_3v_4$. By the planarity of $G$, we have $v_1v_3 \notin E(G)$ or $v_2v_4 \notin E(G)$, say the former. Let $G' = G \cup \{v_1v_3\}$. Since $G'$ is planar, $G'$ has a 4-coloring $c$ such that $c(v_1), c(v_2)$ and $c(v_3)$ are all distinct. If $c(v_2) \neq c(v_4)$, then $c$ is a required 5-coloring. Otherwise, color $v_4$ by a fifth color. Then, we can get a required 5-coloring. \[\Box\]

Let $G$ be a plane triangulation. For distinct $x, y \in V(\partial G)$, a path $P$ in $G$ is called an internal $(x, y)$-path if $V(P) \cap V(\partial G) = \{x, y\}$ and $E(P) \cap E(\partial G) = \emptyset$. In particular, if an internal $(x, y)$-path has length exactly one, then it is called a diagonal of $G$.

**Lemma 12** Let $G$ be a plane triangulation with outer cycle $C$. Then, the following holds:

(i) If $C$ has no diagonal, then $\bar{G} = G - V(C)$ is connected or empty. Moreover, each vertex on $C$ is incident to $\bar{G}$ if $\bar{G}$ is non-empty.
(ii) Let \( x, y \in V(C) \) with \( xy \notin E(C) \). Then \( G \) has an internal \((x, y)\)-path if and only if there is no diagonal \( pq \) for any \( p, q \in V(C) - \{x, y\} \) such that \( x, p, y, q \) appear on \( C \) in this cyclic order.

**Proof.** (i) We use induction on \(|V(G)|\). If \(|V(G)| = 3\), then the lemma clearly holds since \( G \) is empty. Hence we suppose that \(|V(G)| \geq 4\). Let \( v \in V(C) \) and let \( P = v_1 \cdots v_k \) be the path of \( G \) consisting of the neighbors of \( v \), where \( v_1, v_k \in V(C) \). Since \( G \) has no diagonal, we have \( k \geq 3 \). Let \( G' = G - v \), which is a plane triangulation since \( G \) has no diagonal. Observe that \( G' \) might have diagonals incident to \( v_2, \ldots, v_{k-1} \). Let \( B_1, \ldots, B_m \) be the plane triangulations contained in \( G' \) sharing these diagonals one another, where \( m \geq 1 \). By induction hypothesis, \( B_i \) is connected or empty. Moreover, every \( B_i \) has some \( v_j \) on its boundary for \( j \in \{2, \ldots, k-1\} \), and hence the vertex \( v_j \) is incident to \( B_i \). Therefore, \( G \) is connected, since \( V(G) = \{v_2, \ldots, v_{k-1}\} \cup V(B_1) \cup \cdots \cup V(B_m) \). Clearly, every vertex on \( C \) is incident to \( G \), since \( C \) has no diagonal.

(ii) The sufficiency is obvious and hence we prove the necessity. Let \( G \) be a plane triangulation with \( x, y \in V(C) \). Since \( xy \notin E(C) \), the length of \( C \) is at least 4. If \( xy \) is a diagonal in \( G \), then we are done, and hence we suppose \( xy \notin E(G) \). If \( G \) has no diagonal, then the assertion immediately follows from (i). Even if \( G \) has several diagonals, we can get a plane triangulation \( G' \) with \( x, y \in V(\partial G') \) and no diagonals from \( G \) by cutting along diagonals, since \( G \) has no diagonal separating \( x \) and \( y \). Then apply (i) to \( G' \) to get an internal \((x, y)\)-path, which is a required one in \( G \).\( \blacksquare \)

## 4 Proof of the theorems

In this section, we shall prove Theorems 3 and 4.

**Proof of Theorem 3.** Since the necessity is proved by Proposition 10, we prove only the sufficiency. In order to get a contradiction, we assume that there exists a triangulation \( G \) with no \( K_6 \)-minor and no \( K_4 \)-quadrangulation as a subgraph. Since the projective plane admits only two irreducible triangulations \( P1 \) and \( P2 \), \( G \) is contractible to one of them. Since \( P1 \) is isomorphic to \( K_6 \) as a graph and since \( G \) has no \( K_6 \)-minor, \( G \) must be contractible to \( P2 \). Let \( G = G_0, G_1, \ldots, G_m = P2 \) be a sequence of triangulations such that \( G_{i+1} \) is obtained from \( G_i \) by a single contraction of an edge, for \( i = 0, \ldots, m-1 \). Since \( P2 \) has a \( K_4 \)-quadrangulation, there exists \( k \) such that \( G_{k+1} \) has a \( K_4 \)-quadrangulation \( H \) as a subgraph but \( G_k \) does not. Then \( G_k \) includes a subgraph \( H' \) obtained from \( K_4 \) by a single vertex-splitting. Since \( K_4 \) is 3-regular, \( H' \) is isomorphic to a \( K_4 \) with one edge subdivided by a single vertex \( x \), as shown in Figure 3, in which a contraction of \( xy \) transforms \( H' \) into \( H = K_4 \).

Let \( F_1, F_2, F_3 \) be the plane subgraphs of \( G_{k+1} \) contained in the 2-cell regions and the boundary 4-cycles of the three faces of \( H \) as in Figure 3, and let \( F_i' \) be the one of \( G_k \) corresponding to \( F_i \), for \( i = 1, 2, 3 \). Observe that each \( F_i \) has no diagonal. (For otherwise, \( G \) would have a pair of nonhomotopic parallel edges, contrary to the simpleness of \( G \).) Since \( F_1 = F_1' \) and since \( F_1' \) has no diagonal, \( F_1' \) has an internal \((r, x)\)-path, by Lemma 12 (ii). Observe that \( qy \notin E(F_2') \) and \( py \notin E(F_3') \) since each of these forms a pair of nonhomotopic parallel edges in \( G_{k+1} \) after the contraction of \( xy \). Here note that \( F_2' \) and \( F_3' \) might have edges \( py \) and \( qy \), respectively, but both of them do not appear simultaneously,
since $G_k$ has no $K_4$-quadrangulation as a subgraph. Hence, by the symmetry, we may suppose that $py \notin E(F'_2)$. Observe that $F'_2$ has no diagonal since $G$ has no pair of nonhomotopic parallel edges and no $K_4$-quadrangulation. Then, by Lemma 12 (i), we can contract $F'_2$ into a single vertex adjacent to all vertices of $\partial F'_2$. On the other hand, $F'_3$ can have only a diagonal $qy$. Then, by Lemma 12 (ii), we first take an internal $(p, y)$-path $P$ and second take an internal $(q, y)$-path $P'$ in $F'_3$ so that $P'$ and $P$ share as many vertices as possible, and then we suppose $P$ and $P'$ share a path between $y$ and $z$, where possibly $y = z$. Therefore, contracting each edge the path, we can get a subdivision of $K_6$ in $G_k$, contrary to the choice of $G_k$. ■

![Figure 3: $H, H'$ and a $K_6$-minor of $G$](image)

**Proof of Theorem 4.** By Proposition 10, we prove only the sufficiency. For a contradiction, we assume that there exists a triangulation $G$ with no $K_6$-minor and no $K_5$-quadrangulation as a subgraph. By Lemma 7, $G$ must be contractible to $T21$. Let $G = G_0, G_1, \ldots, G_m = T21$ be a sequence of toroidal triangulations such that $G_{i+1}$ is obtained from $G_i$ by contracting a single edge, for $i = 0, \ldots, m - 1$. Since $T21$ has a $K_5$-quadrangulation, there exists $k$ such that $G_{k+1}$ has a $K_5$-quadrangulation $H$ as a subgraph but $G_k$ does not. Then $G_k$ has a subgraph $H'$ obtained from $H$ by a single vertex-splitting. In particular, let $x$ and $y$ be the vertices of $H'$ obtained from a vertex of $H$, denoted by $[xy]$, by a single vertex-splitting. There are two possibilities:

(i) $\deg_{H'}(x) = 2$ and $\deg_{H'}(y) = 4$, and

(ii) $\deg_{H'}(x) = \deg_{H'}(y) = 3$,

which are shown in the left of Figures 4 and 5, respectively. Let $F_1, F_2, F_3, F_4$ be the plane subgraphs of $G_{k+1}$ contained in the 2-cell regions and the boundaries of the four faces of $H$ incident to $[xy]$, as in the left of Figures 4 and 5, and let $F'_i$ be the one of $G_k$ corresponding to $F_i$, for $i = 1, 2, 3, 4$.

(i) Observe that each $\partial F'_i$ has no diagonal since $G_{k+1}$ is simple. Since $F_3 = F'_3$, $F'_3$ has an internal $(s, y)$-path $P_1$, by Lemma 12 (i). Observe that $rx \notin E(F'_1)$ and $px \notin E(F'_2)$ since each of these forms a pair of nonhomotopic parallel edges in $G_{k+1}$ after the contraction of $xy$. Note that $s$ and $y$ are adjacent in neither $F'_1$ nor $F'_2$ since $G_k$ has no $K_5$-quadrangulation as a subgraph. Moreover, since $G$ is simple, $F'_1$ and $F'_2$ can have only diagonals $px$ and $qx$, respectively. Similarly to the projective-planar case, using Lemma 12 (ii), we can take an internal $(r, x)$-path $P_2$ in $F'_1$. Then take an internal $(p, x)$-path $P_3$.
in $F'_2$, and next an internal $(q, x)$-path $P_4$ in $F'_2$ so that $P_3$ and $P_4$ share as many vertices as possible (and then we suppose they share a path between $x$ and $z$, where possibly $y = z$). Then $H' \cup P_1 \cup P_2 \cup P_3 \cup P_4$ has a $K_6$-minor in $G_k$, contrary to the choice of $G_k$. See the right of Figure 4.

(ii) We may suppose that $sy, px \in E(F'_1)$ and $rx, qy \notin E(F'_3)$, for otherwise, the case has already dealt in (i). Moreover, we also have $rx \notin E(F'_1)$ and $sy \notin E(F'_3)$ (for otherwise, then $G_{k+1}$ would have a pair of nonhomotopic parallel edges). Consequently, each region $F'_i$ cannot have any diagonal, and hence, by Lemma 12 (ii), we can take required paths freely, as in the right of Figure 5, and get a $K_6$-minor in $G_k$, a contradiction. \hfill \blacksquare

The following proposition asserts that Hadwiger’s conjecture is true for triangulations on the projective plane and the torus.

**Proposition 13** Let $G$ be a triangulation on the projective plane and the torus. If $G$ has no $K_k$ as a minor, then $G$ is $(k - 1)$-colorable.

**Proof.** As described in the introduction, every triangulation on a nonspherical surface has a $K_m$-minor for all $m \leq 5$, and furthermore, every projective-planar (resp., toroidal) triangulation has no $K_7$-minor (resp., $K_8$-minor). On the other hand, every projective-planar (resp., toroidal) triangulation is known to be 6-colorable (resp., 7-colorable).

By Theorem 3, if a triangulation $G$ on the projective plane has no $K_6$-minor, then $G$ has a $K_4$-quadrangulation $H$ as a subgraph. By Lemma 11, a 4-coloring of $H$ extends to a 5-coloring of $G$, and hence $G$ is 5-colorable.
Now let $G$ be a triangulation on the torus. If $G$ has no $K_6$-minor, then $G$ has a $K_5$-quadrangulation $H$ as a subgraph, by Theorem 4. Similar to the above, we can conclude that $G$ is 5-colorable. Moreover, it is proved in [3] that if $G$ is not 6-colorable, then $G$ has $K_7$ as a subgraph. Hence, if $G$ has no $K_7$-minor, then $G$ is 6-colorable. ■

5 Appendix

Figures 6 shows a subgraph isomorphic to $K_6$ in $T_1, T_5, T_{12}$ and $T_{19}$, and Figures 7 shows $K_6$-minors in all others, except $T_{21}$. In each of the figures, vertices surrounded by a single polygon corresponds to a single vertex of $K_6$ after the contractions of them.

Figure 6: $K_6$-subgraphs in irreducible triangulations on the torus

References


Figure 7: $K_6$-minors in irreducible triangulations on the torus