N-Flips in Even Triangulations on the Sphere
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Abstract
A triangulation is said to be even if each vertex has even degree. For even triangulations, define the N-flip and the $P_2$-flip as two deformations preserving the number of vertices. We shall prove that any two even triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of N- and $P_2$-flips.

Keywords: even triangulation, tripartite graph, sphere

1 Introduction
A triangulation $G$ on a closed surface $F^2$ is a fixed embedding of some simple graph on $F^2$ such that each face of $G$ is bounded by a 3-cycle. (A $k$-cycle means a cycle of length $k$.) For a vertex $v$ of $G$, the link of $v$ is the boundary walk of the 2-cell region formed by the faces of $G$ incident to $v$. For a triangulation $G$, the link of $v$ is a cycle for any $v \in V(G)$.

Let $G$ be a graph. A $k$-coloring of $G$ is a map $c : V(G) \to \{1, \ldots, k\}$ such that for any $xy \in E(G)$, $c(x) \neq c(y)$. We say that $G$ is $k$-colorable if $G$ admits a $k$-coloring, and that $G$ is $k$-chromatic if $G$ is $k$-colorable but not $(k - 1)$-colorable.

A triangulation $G$ is said to be even if each vertex of $G$ has even degree. (Even triangulations are sometimes called Eulerian triangulations, for example, in [2, 5].) It is well-known that every even triangulation on the sphere has a unique vertex 3-coloring. Thus, $V(G)$ can uniquely be decomposed into $V_R(G) \cup V_B(G) \cup V_Y(G)$, where these classes are referred as red, blue, and yellow vertices of $G$, respectively. (Such a decomposition of $V(G)$ is called the tripartition of $G$.) In this paper, we deal with even triangulations on the sphere with fixed 3-colorings of the vertices by red, blue and yellow. We say that an edge with red and blue endpoints is an rb-edge. Similarly, we can define a by-edges and an ry-edges as one with blue and yellow endpoints and one with red and yellow endpoints, respectively. It is well-known that the smallest even triangulation on the sphere is the octahedron, which is the complete tripartite graph $K_{2,2,2}$ as an abstract graph.

Suppose that an even triangulation $G$ has a hexagonal region $v_1v_2v_3v_4v_5v_6$ with diagonals $v_1v_3, v_3v_6$ and $v_4v_6$ and no inner vertices. The $N$-flip of the path $v_1v_3v_6v_4$ is to replace

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the diagonals $v_1v_3$, $v_3v_6$ and $v_4v_6$ with $v_1v_5$, $v_2v_5$ and $v_2v_4$ in the hexagonal region. (See Figure 1. The $N$-flip is to flip the path forming “N” with another N, fixing its endpoints.) If the graph obtained from $G$ by an $N$-flip is not simple, then we don’t apply it. Clearly, an $N$-flip transforms an even triangulation into an even triangulation, and preserves the tripartition of the graph. Two even triangulations $G$ and $G'$ are said to be $N$-equivalent and denoted by $G \sim_N G'$ if $G$ and $G'$ can be transformed into each other by a sequence of $N$-flips. Note that we deal with unlabeled triangulations.

Our main theorem is as follows:

**THEOREM 1** Two even triangulations $G$ and $G'$ on the sphere are $N$-equivalent if $|V_R(G)| = |V_R(G')|$, $|V_B(G)| = |V_B(G')|$ and $|V_Y(G)| = |V_Y(G')|$.

Consider another operation called the “$P_2$-flip”. Let $G$ be an even triangulation on the sphere and let $v$ be a vertex of $G$ with link $v_1 \cdots v_k$. Put two vertices $x$ and $y$ on $vv_1$ and join them to $v_2$ and $v_k$, and let $G'$ be the resulting graph. The $P_2$-flip of $\{x, y\}$ in $G'$ is to move the inserted vertices $x$ and $y$ to the edge $vv_2$ to join them to $v_1$ and $v_3$, as shown in Figure 2. This operation always preserves the simpleness of the graph since $G$ is simple. Clearly, a $P_2$-flip does not preserve the tripartition of the graph. Using $P_2$-flips in addition to $N$-flips, we can change the size of partite sets of even triangulations. The following is our second theorem:

**Figure 1: $N$-flip**

**Figure 2: $P_2$-flip**
**Theorem 2** Any two even triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of $N$- and $P_2$-flips.

This research on even triangulations was motivated by the earlier results on diagonal flips in triangulations and quadrangulations. These have been considered on all surfaces and form a large stream of research with many papers. (See the survey papers [4] and [3] for triangulations and quadrangulations, respectively.) Theorems 1 and 2 are expected as the starting points of the researches on even triangulations on surfaces. So, in order to extend the theorems in a further research, we will specify lemmas which hold on any surface.

## 2 Proof of Theorems

Let $G$ be an even triangulation and let $v$ be its vertex of degree 4 with link $v_1v_2v_3v_4$. The **contraction** of $v$ at $\{v_1, v_3\}$ is to identify three vertices $v, v_1$ and $v_3$ and replace two pairs of triple edges by single edges, respectively, as shown in Figure 3. If this operation yields multiple edges or loops, then we don’t apply it. Note that we define the contraction only for a vertex of degree 4, and that the contraction transforms an even triangulation $G$ into an even triangulation with $|V(G)| - 2$ vertices.

![Figure 3: Contraction of $v$ at $\{v_1, v_3\}$](image)

Let $G$ be an even triangulation and let $f$ be a face of $G$ bounded by $v_1v_2v_3$. The **addition of an octahedron** to $f$ is to put a 3-cycle $a_1a_2a_3$ into $f$ and add edges $a_iv_j$ for each distinct $i, j \in \{1, 2, 3\}$. The inverse operation is called the **removal of an octahedron**. The **double octahedron** is an even triangulation on the sphere obtained from the octahedron by a single addition of an octahedron. (See Figure 4.) It is easy to see that the double octahedron is invariant by $N$-flips, that is, every $N$-flip transforms the double octahedron into itself. Let $O$ and $\tilde{O}$ denote the octahedron and the double octahedron.

**Lemma 3** Every even triangulation on the sphere with an even number of vertices can be transformed into the octahedron by a sequence of contractions of vertices of degree 4 and $N$-flips. One with an odd number of vertices can be transformed into the double octahedron by the two operations.
Proof. Let $G$ be an even triangulation on the sphere with $n$ vertices. In [1], it was shown that $G$ can be transformed into the octahedron by a sequence of a contraction of a vertex of degree 4 and a removal of an octahedron, preserving the simpleness of the graphs. Therefore it suffices to prove that if $G$ can be transformed into an even triangulation $G'$ by a single removal of an octahedron, then $G$ can be transformed into an even triangulation with a fewer number of vertices by a sequence of a contraction of a vertex of degree 4 and an $N$-flip, unless $G$ is the double octahedron.

Suppose that $G'$ is obtained from $G$ by removing three vertices $a_1, a_2$ and $a_3$ forming a facial 3-cycle from a triangular region bounded by $v_1v_2v_3$, where $a_iv_j \in E(G)$ for any distinct $i, j \in \{1, 2, 3\}$. Since $G$ is not the octahedron, we have $\deg_G(v_i) \geq 6$ for any $i \in \{1, 2, 3\}$.

We first suppose that $\deg_G(v_i) \geq 8$ for some $i$, say $\deg_G(v_1) \geq 8$. Let $v_2a_3a_2v_3b_1b_2 \cdots b_{2k}$ be the link of $v_1$ in $G$. Note that since $\deg_G(v_1) \geq 8$, we have $k \geq 2$. Observe that by the planarity, we have $v_2b_2 \notin E(G)$ or $v_3b_{2k-1} \notin E(G)$. Therefore, by the symmetry, we may suppose that $v_2b_2 \notin E(G)$. Now apply the $N$-flip of the path $a_2v_3v_1b_1$ in $G$, and let $\tilde{G}$ be the resulting graph. (See Figure 5.) Since for any $i$ and $j$, $a_i$ and $b_j$ are separated by the 3-cycle $v_1v_2v_3$ in $G$, they are not adjacent in $G$. Moreover,
v_2, v_3, b_1, \ldots, b_{2k} are all distinct since they lie on the link of v_1 in G. Hence \( \hat{G} \) is simple. Moreover, since \( v_2b_2 \notin E(\hat{G}) \), we can apply the contraction of \( a_3 \) at \( \{v_2, a_2\} \) in \( \hat{G} \) without making loops or multiple edges. Hence we get an even triangulation with \( |V(G)| - 2 \) vertices.

Secondly we consider the case when \( \deg(v_1) = \deg(v_2) = \deg(v_3) = 6 \). This means that \( v_1, v_2, v_3 \) and their neighbors in \( G' \) induce the octahedron. Therefore, unless \( G \) is the double octahedron, all neighbors of \( v_1, v_2 \) and \( v_3 \) in \( G' \) have degree at least 6. Now apply the \( N \)-flip of \( a_2v_3v_1b_1 \) in \( G \) similarly to the above case. (See Figure 5 again, where \( b_2 = b_{2k} \).) In the resulting graph, \( v_1, a_2 \) and \( a_3 \) are vertices of degree 4 and form a facial 3-cycle. Moreover, \( b_2 (= b_{2k}) \), which is one of the neighbors of \( v_1, a_2 \) and \( a_3 \), has degree at least 8 in the current graph. This case has already been dealt in the former case.

Let \( G \) be an even triangulation and let \( e = rb \) be an \( rb \)-edge of \( G \) shared by two faces \( rby_1 \) and \( rby_2 \), where \( r \in V_R(G), b \in V_B(G) \) and \( y_1, y_2 \in V_Y(G) \). Subdivide \( e \) by two vertices \( u \) and \( v \) to form a path \( ruvb \), and add edges \( vy_i, uy_i \) for \( i = 1, 2 \), as shown in Figure 6. Clearly, we have \( u \in V_B(G) \) and \( v \in V_R(G) \). We call this operation a 2-subdivision of \( e \) in \( G \). In the resulting graph, \( \{u, v\} \) is called a 2-subdividing pair of vertices.

![Figure 6: 2-subdivision of xy](image)

We shall prove the following two lemmas for even triangulations on all surfaces. Though every non-spherical surface admits non 3-colorable even triangulations, we restrict even triangulations to be 3-chromatic in the lemmas. Therefore we suppose that the vertices of a 3-chromatic even triangulation are colored by red, blue and yellow.

**Lemma 4** Let \( K \) be a 3-chromatic even triangulation on any surface \( F^2 \), and let \( e \) and \( e' \) be two \( rb \)-edges of \( G \). Let \( G \) (resp., \( G' \)) be the even triangulation on \( F^2 \) obtained from \( K \) by a 2-subdivision of \( e \) (resp., \( e' \)). Then \( G \) and \( G' \) are \( N \)-equivalent.

**Proof.** Since \( K - V_Y(G) \) is clearly connected, it suffices to prove the lemma when \( e \) and \( e' \) are two \( rb \)-edges of \( K - V_Y(G) \) which are consecutive with respect to the rotation of some vertex \( r \in V_R(G) \). Let \( b_1y_1 \cdots b_ky_k \) be the link of \( r \) in \( G \), where \( k \geq 2 \). Suppose that \( G \) is obtained from \( K \) by subdividing \( rb_1 \) by two vertices \( u \) and \( v \) to form a path \( ruvb_1 \). If \( \deg_K(r) = 4 \), then in \( G \), \( \{u, r\} \) can be regarded as a 2-subdividing pair of vertices on \( vb_2 \). Therefore, we have \( G = G' \), and hence we may suppose that \( \deg_K(r) \geq 6 \). Then \( G \)
can be transformed into $G'$ by two $N$-flips, as shown in Figure 7. Note that each of the three graphs in Figure 4 is simple since all $b_1, y_1, \ldots, b_k, y_k$ are distinct in $G$. ■

Let $G$ be an even triangulation on a closed surface $F^2$. Let $G + rb(1)$ denote an even triangulation on $F^2$ obtained from $G$ by a 2-subdivision of an $rb$-edge $e$ arbitrarily chosen in $G$. Note that $G + rb(1)$ expresses various even triangulations, depending on the choice of $e$. However, Lemma 4 guarantees that they are all $N$-equivalent, and hence $G + rb(1)$ expresses a unique even triangulation, up to $N$-equivalence. Similarly, we denote an even triangulation obtained from $G$ by adding $p, q$ and $r$ 2-subdividing pairs to an $rb$-edge, a by-edge, and an $ry$-edge, respectively, by $G + rb(p) + by(q) + ry(r)$, which has precisely $|V_R(G)| + p + r$ red vertices, $|V_B(G)| + p + q$ blue vertices and $|V_Y(G)| + q + r$ yellow vertices, and hence has $|V(G)| + 2p + 2q + 2r$ vertices in total.

**Lemma 5** Let $G$ be a 3-chromatic even triangulation on any surface $F^2$ and let $r \in V_R(G)$ be a vertex of degree 4 with link $b_1 y_1 b_2 y_2$, where $b_i \in V_B(G)$ and $y_i \in V_Y(G)$ for $i = 1, 2$. If $G$ can be transformed into an even triangulation $H$ by the contraction of $r$ at $\{b_1, b_2\}$, then $G$ is $N$-equivalent to $H + rb(1)$.

Proof. We prove the lemma by induction on the degree of $b_1$ in $G$. If $\deg_G(b_1) = 4$, then $\{r, b_1\}$ can be regarded as a 2-subdividing pair of vertices lying on an edge of $H$, and hence $G = H + rb(2)$. Hence we suppose that $b_1$ has degree at least 6. Let $r y_1 p_1 q_1 p_2 q_2 \cdots p_k y_2$ be the link of $b_1$, where $p_1, \ldots, p_k \in V_R(G)$ and $q_1, \ldots, q_{k-1} \in V_Y(G)$. Apply the $N$-flip to the path $r y_1 b_1 p_1$, and let $G'$ be the resulting graph. (See Figure 8.) Since the contraction of $r$ at $\{b_1, b_2\}$ does not break the simpleness of the graph, each $p_i$ and each $q_i$ are not adjacent to $b_2$ in $G$, and hence $G'$ is simple. Moreover, $r$ has degree 4 in $G'$, and we can apply the contraction of $r$ at $\{b_1, b_2\}$ in $G'$ since the graph arisen is nothing but $H$. Since $\deg_G(b_1) = \deg_G(b_1) - 2$, we have $G' \sim_N H + rb(1)$, by induction hypothesis. Since $G \sim_N G'$, we have $G \sim_N H + rb(1)$. ■

Now we have prepared all to prove Theorem 1.

**Proof of Theorem 1**. Let $G$ be an even triangulation on the sphere. Then, by Lemma 3, there exists a sequence of even triangulations $G = G_0, G_1, \ldots, G_k$ such that $G_{i+1}$ is obtained from $G_i$ by a single $N$-flip or a single contraction of a vertex of degree 4, for
**Proof Theorem 2.** It is easy to see that $O + rb(p) + by(q) + ry(r)$ with $q \geq 1$ can be transformed into $O + rb(p + 1) + by(q - 1) + ry(r)$ by $N$- and $P_3$-flips. Hence every even triangulation on the sphere with an even number of vertices can be transformed into $O + rb(m)$, where $|V(G)| = 2m + 6$. Similarly, one with an odd number of vertices can be transformed into $\tilde{O} + rb(m)$, where $|V(G)| = 2m + 9$. Thus, the theorem holds.
3 Remarks

In this section, we consider a possibility to extend our theorems to non-spherical surfaces. Observe that an \(N\)-flip and a \(P_2\)-flip preserve the 3-chromaticity of even triangulations. Since every non-spherical surface \(F^2\) admits non-3-colorable even triangulations, there exists a pair of even triangulations on \(F^2\) which have the same number of vertices but cannot be transformed into each other by the two operations. For example, Figure 9 shows 3-colorable and non-3-colorable even triangulations on the projective plane with the same number of vertices. (Each octagon expresses the projective plane, where each pair of antipodal points have to be identified.)

![Figure 9: Two even triangulations on the projective plane](image)

What can we say when we restrict even triangulations on a non-spherical surface \(F^2\) to be 3-chromatic? If we use the method in this paper, we have to determine the minimal set of even triangulations on \(F^2\) with respect to the contractions of vertices of degree 4 and \(N\)-flips, as in Lemma 3. It seems that \(F^2\) admits many minimal even triangulations, and that it is not so easy to determine them even for the projective plane. Furthermore, an even triangulation on \(F^2\) might not have a vertex of degree 4 if the Euler characteristic is non-positive. (It is easily obtained from Euler’s formula.) Therefore, we need more ideas to extend the theorems to other surfaces.

References


