Polychromatic 4-coloring of cubic bipartite plane graphs

Elad Horev*, Matthew J. Katz*, Roi Krakovski* and Atsuhiro Nakamoto†

Abstract

It is proved that the vertices of a cubic bipartite plane graph can be colored with four colors such that each face meets all four colors. This is tight, since any such graph contains at least six faces of size four.

Keywords: cubic bipartite plane graph, polychromatic coloring, Eulerian triangulation

1 Introduction

Let $G$ be a plane graph. A subset $S \subseteq V(G)$ of the vertices of $G$ is called a face-cover of $G$, if for any face $f$, there is $s \in S$ incident to $f$. Let $\lambda(G) = \min\{|S| : S$ is a face-cover of $G\}$, which is called the face-cover number of $G$.

Let $\rho$ be a $k$-coloring (not necessarily proper) of a plane graph $G$. (A coloring $\rho$ is proper if no edge has two endpoints with the same color.) We say that a face $f$ of $G$ is polychromatic under $\rho$ if all $k$ colors appear in the boundary closed walk of $f$. The coloring $\rho$ is polychromatic if it is a $k$-coloring of $G$ such that every face of $G$ is polychromatic.

Figure 1 shows a polychromatic proper 4-coloring of a cube, where the numbers 1, 2, 3 and 4 stand for colors assigned to the vertices.

![Figure 1: Polychromatic proper 4-coloring of a cube](attachment:image.png)

The polychromatic number of $G$, denoted $p(G)$, is the maximum integer $k$ such that $G$ admits a polychromatic $k$-coloring. We denote by $g(G)$ the length of the shortest face of $G$. Clearly $p(G) \leq g(G)$. Observe that if $G$ has a polychromatic coloring $\rho : V(G) \rightarrow \{1, \ldots, k\}$, then the set $\rho^{-1}(i)$ is a face-cover of $G$ for any $i \in \{1, \ldots, k\}$. Hence, we have the face-cover number $\lambda(G) \leq \left\lfloor \frac{n}{k} \right\rfloor$ and in particular $\lambda(G) \leq \left\lfloor \frac{n}{p(G)} \right\rfloor$.

*Department of Computer Science, Ben-Gurion University, Beer-Sheva 84105, Israel. Email: [horevel, katz, roikr]@cs.bgu.ac.il

†Department of Mathematics, Yokohama National University, Yokohama 240-8501, Japan. Email: nakamot@ynu.ac.jp
An example of the importance and usefulness of the concept of polychromatic coloring is Fisk’s proof [4] of the classical art gallery theorem, first proved by Chvátal [3]. Given a polygon \( P \) with \( n \) vertices, a subset of its vertices is a \textit{guarding set} for \( P \) if for any point \( p \in P \) there exists a guard (located on a vertex of the polygon) in the set that sees \( p \). Chvátal proved that \( \lfloor \frac{n}{3} \rfloor \) guards are always sufficient and sometimes necessary to guard a simple \( n \)-gon. Fisk presented an elegant proof to this theorem. First he showed that any simple polygon can be triangulated, by adding edges only, into a plane graph in which every bounded face is a triangle. Then, he proved that the resulting graph has a proper 3-coloring. Thus, each of the triangular faces has three colors on its boundary. Now, by picking the vertices associated with the least frequent color, and observing that the triangular faces are all convex, one obtains a guarding set of size at most \( \lfloor \frac{n}{3} \rfloor \) for our original polygon.

In 1969, Lovász [7] proved that every plane graph, \( G \), admits a polychromatic 2-coloring. The proof is short and simple and goes as follows: Triangulate the graph \( G \) by adding edges, resulting in a new graph \( H \) where each face has length 3. The dual graph \( H^* \) of \( H \) (the dual graph, \( X^* \), of a plane graph, \( X \), is the graph with vertex set \( F(X) \), such that there is an edge between two vertices in \( X^* \) if, and only if, the faces of \( X \) corresponding to these two vertices share an edge in common in \( X \)) is then 3-regular and 2-edge connected (that is, for every \( e_1, e_2 \in E(H^*) \), \( H^* \setminus \{e_1, e_2\} \) is connected). By Petersen’s Theorem (see, e.g., [10]), there exists a perfect matching \( M \) in \( H^* \). After deleting the edges of \( H \) corresponding to those of \( M \), the remaining graph \( H' \) has only faces of length 4, and hence is bipartite by a well-known result of MacLane (see [9]). Thus, there is a proper 2-coloring of \( H' \), which is a polychromatic 2-coloring of \( H \) and hence also of \( G \), as required. (Mohar and Škrekovski [8] provided a short alternative proof for the result of Lovász, based on the Four-Color Theorem.)

There are plane graphs that do not admit a polychromatic 3-coloring, as witnessed by \( W_k \) (the wheel with \( k \) spokes) for any odd \( k \geq 3 \). In general, for a graph \( G \), it is \( \text{NP} \)-hard to determine if \( p(G) \geq 3 \) [1]. In [6], it was proved that if \( G \) is of maximum degree at most three, and is not isomorphic to \( K_4 \) or a subdivision of \( K_4 \) with five vertices, then \( G \) admits a polychromatic 3-coloring. It is known that every 2-connected plane bipartite graph \( G \) admits a proper polychromatic 3-coloring (since \( G \) can be extended to a Eulerian plane triangulation by adding only edges [5]).

A general result regarding polychromatic coloring was obtained in [1]. It was proved that

\[
p(G) \geq \left\lfloor \frac{3g(G) - 5}{4} \right\rfloor,
\]

for any plane graph \( G \), and that this result is essentially tight, as there are plane graphs \( G \) for which \( p(G) \leq \left\lfloor \frac{3g(G)+1}{4} \right\rfloor \), for any choice of \( g(G) \).

Inequality (1) is nontrivial only for graphs, \( G \), with \( g(G) \geq 6 \). Consequently, there is an interest in studying \( p(G) \) in case that \( 3 \leq g(G) \leq 5 \) (see [1] for further applications). In this paper, such graphs are considered and in particular, the following is the main result of this paper.

**THEOREM 1** Let \( G \) be a cubic bipartite plane graph. Then, \( G \) has a polychromatic proper 4-coloring.

By Euler’s formula, it can be easily seen that a cubic bipartite plane graph has at least six faces of length four. Hence, Theorem 1 is best possible in term of the number of colors. The focus on cubic bipartite plane graphs is justified, as can be seen in Figure 2.

Theorem 1 asserts that every cubic bipartite plane graph with \( n \) vertices satisfies \( \lambda(G) \leq \lfloor \frac{n}{4} \rfloor \). This result is also tight in the sense that there exists an infinite family of graphs, \( G \), for which \( \lambda(G) = \lfloor \frac{n}{4} \rfloor \). The family of bipartite prism graphs of order divisible by four is such an example.
Figure 2: Three plane graphs $G$ for which $p(G) < 4$. (a) A cubic non-bipartite graph; (b) A non-cubic bipartite graph of maximum degree three; (c) A bipartite graph of minimum degree three but with a degree four vertex.

2 Polychromatic 4-coloring of graphs in $G_3$

For $l = 1, 2, 3$, let $G_l$ be the set of all simple $l$-connected cubic bipartite plane graphs. By definition, $G_1 \supseteq G_2 \supseteq G_3$. The graph operations 2-bridging and a hexagon addition, allowing the generation of larger 3-connected cubic bipartite plane graphs from smaller ones, are depicted in Figure 3. We then have the following.

![Figure 3: 2-bridging and a hexagon addition](image)

**Proposition 2** Every $G \in G_3$ can be obtained from a cube, through $G_3$, by a sequence of a 2-bridging and a hexagon addition.

**Proof.** The proposition immediately follows from the two facts:

(i) For any $G \in G_3$, the dual $G^*$ of $G$ is a simple Eulerian plane triangulation. In particular, the dual of a cube is an octahedron.

(ii) Every simple Eulerian plane triangulation can be obtained from an octahedron, through simple Eulerian plane triangulations, by a sequence of a vertex-splitting and an octahedron addition shown in Figure 4 [2].
Observe that the dual operations of a vertex-splitting and an octahedron addition are a 2-bridging and a hexagon addition, respectively. Hence the proposition follows.

\[\text{Figure 4: Vertex-splitting and octahedron addition}\]

The following is a key ingredient in the proof of Theorem 1.

**Theorem 3** Every \( G \in \mathcal{G}_3 \) has a polychromatic proper 4-coloring.

**Proof.** We proceed by induction on \(|V(G)|\). If \( G \) is isomorphic to the cube, then the claim follows as is easily seen by Figure 1. Hence, by Proposition 2, we may assume that there exists \( G' \in \mathcal{G}_3 \) so that \( G \) can be obtained from \( G' \) by either a 2-bridging and a hexagon addition. By the induction hypothesis, \( G' \) has a polychromatic proper 4-coloring \( \rho' : V(G') \to \{1, 2, 3, 4\} \). We will show how to extend \( \rho' \) to a polychromatic proper 4-coloring, \( \rho \), of \( G \).

**Case 1.** \( G \) is obtained from \( G' \) by a hexagon addition.

Let \( v \) be a degree-three vertex of \( G' \) with neighbors \( a, b, c \), and we suppose that a hexagon addition is applied to \( v \) to get \( G \), where we label the vertices and faces of \( G \) as in Figure 3. We suppose that \( \rho'(v) = 1 \), and we let \( \rho(w) = \rho'(w) \) for any \( w \in V(G) - \{u, x, y, z, p, q, r\} \). It remains to assign color to the vertices \( u, x, y, z, p, q, r \).

Note that for \( \rho \) to be polychromatic, existence of the faces \( f_1, f_2, f_3 \) forces that \( \rho(x) = \rho(r) \), \( \rho(y) = \rho(q) \) and \( \rho(z) = \rho(p) \) and these three colors must be all distinct. Moreover, since each of the new three faces of \( G \) incident to \( \{a, b\} \), and \( \{b, c\} \) and \( \{a, c\} \) respectively must be incident to a vertex colored “1”, under \( \rho \), at least one of \( x, y, z \) must be colored “1”.

If \( \rho(a) = \rho(b) = \rho(c) \) (say \( = 2 \)), then we let \( \rho(u) = 2 \), and \( \rho(x) = \rho(r) = 1 \), \( \rho(y) = \rho(q) = 3 \) and \( \rho(z) = \rho(p) = 4 \). If exactly two of \( \rho(a), \rho(b), \rho(c) \) are the same color (say \( \rho(a) = \rho(b) = 2 \) and \( \rho(c) = 3 \)), then we let \( \rho(u) = 4 \), and \( \rho(x) = \rho(r) = 1 \), \( \rho(y) = \rho(q) = 3 \) and \( \rho(z) = \rho(p) = 2 \). Finally, if \( \rho(a), \rho(b), \rho(c) \) are all distinct (say \( \rho(a) = 2, \rho(b) = 3, \rho(c) = 4 \)), then we let \( \rho(u) = 4 \), and \( \rho(x) = \rho(r) = 3 \), \( \rho(y) = \rho(q) = 4 \) and \( \rho(z) = \rho(p) = 1 \). Hence, \( \rho \) is a polychromatic proper 4-coloring of \( G \).

**Case 2.** \( G \) is obtained from \( G' \) by a 2-bridging.

Suppose that the 2-bridging is applied to edges \( ab \) and \( cd \) in \( G' \) (as in Figure 3). Let \( f \) be a face of \( G' \) incident to both \( ab \) and \( cd \). Since \( G' \) is 2-connected, the boundary of \( f \) forms a cycle \( C \) of \( G \). Let \( P_{ac} \) and \( P_{bd} \) be the disjoint paths contained in \( C \) between \( a \) and \( c \), and \( b \) and \( d \), respectively. By the definition of 2-bridging, both \( P_{ac} \) and \( P_{bd} \) have odd length. Note that under \( \rho' \), the set \( V(P_{ac}) \cup V(P_{bd}) \) meets all four colors.

Let \( f_1, f_2 \) and \( f_3 \) be faces of \( G \), as shown in the top of Figure 3. Let \( \rho(v) = \rho'(v) \) for any \( v \in V(G) - \{x, y, z, w\} \). Under \( \rho \), all faces of \( G \) except \( f_1, f_2, f_3 \) are polychromatic. It remains
to assign color to the vertices $x, y, z$ and $w$. Note that for $\rho$ to be polychromatic, these four vertices must be assigned distinct colors, and $x, z$ (resp., $y, w$) must be assigned all colors which do not appear in $P_{ac}$ (resp., $P_{bd}$) under $\rho'$.

(i) Suppose that $\rho(a) = \rho(c) (= 1$, say). Since the length of $P_{ac}$ is odd and at least three in this case and since $\rho'$ is proper, $P_{ac}$ contains at least three colors including “1”, say $\{1, 2, 3\}$. Observe that $\rho(b) \neq 1$ and $\rho(d) \neq 1$ since $ab, cd \in E(G')$, and that if $\rho(b) = \rho(d)$, then the length of $P_{bd}$ is odd and at least three. Hence, if we let $\alpha = \rho(b)$, then we can choose a color $\beta$ appearing in $P_{bd}$ which is distinct from “1” and $\alpha$. In particular, if $P_{bd}$ contains “4” but $\alpha \neq 4$, then we let $\beta = 4$. Now let $\rho(x) = \alpha$ and $\rho(z) = \beta$. Then $f_1$ has a polychromatic proper coloring, since $\alpha, \beta, “1”$ are all distinct. (Note that even if $P_{ac}$ does not contain “4”, then either $\alpha = 4$ or $\beta = 4$.) Let $\rho(w) = 1$ and $\rho(y) = \gamma$, where $\gamma$ is the color distinct from “1”, $\alpha, \beta$. Then all four colors appear in $f_2$. Moreover, all four colors appear in $f_3$ without color collisions, since $\rho(d) \neq 1$ and $\rho(b) = \alpha \neq \gamma$ and since the two colors except $\alpha, \beta$ are used to color $y$ and $w$.

(ii) By symmetry, suppose that $\rho(a) \neq \rho(c)$ and $\rho(b) \neq \rho(d)$. Let $L_{xz}$ (resp., $L_{yw}$) denote the set of colors which do not appear in the coloring $\rho$ of $P_{ac}$ (resp., $P_{bd}$). Observe that $|L_{xz}| \leq 2$ and $|L_{yw}| \leq 2$. Since each of $f_1$ and $f_3$ needs to meet all 4 colors under $\rho$, all colors in $L_{xz}$ (resp., $L_{yw}$) must be used for $x, z$ (resp., $y, w$) in $\rho$. Observe that $L_{xz} \cap L_{yw} = \emptyset$ since $f$ is polychromatic in $\rho'$. Therefore, we can partition $\{1, 2, 3, 4\}$ into two disjoint 2-element subsets $S$ and $S'$ so that $S \supseteq L_{xz}$ and $S' \supseteq L_{yw}$. Use $S$ to color $\{x, z\}$ in $\rho$. Since $\rho(a) \neq \rho(c)$ and $|S| = 2$, we can always avoid color collisions in the edges $ax$ and $cz$ in $G$. The same holds for coloring $\{y, w\}$ by $S'$. So $\rho$ is a proper 4-coloring such that all of $f_1, f_2$ and $f_3$ are polychromatic.

Hence, in Cases 1 and 2, we get a polychromatic proper 4-coloring $\rho$ of $G$. ■

3 Proof of the theorem

In this section, we prove Theorem 1. Prior to the proof, we have the following easy proposition.

PROPOSITION 4 Every connected cubic bipartite plane graph is 2-connected, that is, $G_1 = G_2$.

Proof. For suppose not, and let $G \in G_1 - G_2$. Let $v \in V(G)$ be a cut vertex of $G$. Since $G$ is cubic, $G$ has a cut edge, say $e$, incident to $v$. Let $K$ be a component of $G - e$ with $v \in V(K)$. Then $K$ is bipartite, and all vertices of $K$ are of degree three, except for $v$ which is of degree two. Let $V(K) = B \cup W$ be the bipartition of $K$ with $v \in W$. Since $K$ is bipartite and all vertices in $B$ have degree three, we have $|E(K)| = \sum_{x \in B} \deg(x) \equiv 0 \pmod{3}$. On the other hand, since $v$ has degree 2, $|E(K)| = \sum_{x \in W} \deg(x) \equiv 2 \pmod{3}$, a contradiction. ■

Now we may now turn to the proof of Theorem 1.

Proof of Theorem 1. Assume for a contradiction that the claim is false, and let $G$ be a counterexample to Theorem 1 with $|V(G)|$ minimum. Clearly, We may assume that $G$ is connected (for, all components of $G$ have polychromatic 4-colorings, then so does $G$.) By Proposition 4, $G \in G_2$. By Theorem 3, if $G \in G_3$, then we are done, and hence we suppose $G \in G_2 \setminus G_3$.

Hence $G$ contains two distinct vertices removal of which disconnects $G$. Since $G$ is cubic, there exist two edges $e = ab$ and $e' = cd$ such that $G - \{e, e'\}$ is disconnected, where we suppose that $a$ and $c$ are contained in the same component, say $L$, of $G - \{e, e'\}$.
We claim that $a$ and $c$ are contained in distinct partite sets of $L$. For otherwise, $L$ admits a bipartition $V(L) = B \cup W$ with $a, c \in B$. Now, by an argument similar to the one presented in the proof of Proposition 4, the assumption that $L$ is bipartite implies that $|E(L)| = \sum_{v \in B} \deg(v) = 0 + 2 + 2 \equiv 1 \pmod{3}$, but $|E(L)| = \sum_{v \in W} \deg(v) \equiv 0 \pmod{3}$, a contradiction.

It follows, that by deleting the $e$ and $e'$ from $G$ and adding the edges $ac$ and $bd$, we obtain two cubic bipartite plane graphs $K$ and $H$ containing $\{a, c\}$ and $\{b, d\}$, respectively, possibly with parallel edges (i.e., edges with the same endpoints). See Figure 5(1).

If both $K$ and $H$ have no parallel edges, then, by the minimality of $G$, $K$ and $H$ have polychromatic proper 4-colorings $\rho_K$ and $\rho_H$, which are easily combined to a polychromatic proper 4-colorings of $G$ as follows. Permute the four colors of $H$ so that $a$ and $b$ have distinct colors, and so do $c$ and $d$. Now removing $ac$ and $bd$ and adding $e$ and $e'$, we get a polychromatic proper 4-coloring of $G$. Note that the two faces of $G$ incident to both $e$ and $e'$ are polychromatic since $\rho_K$ and $\rho_H$ are polychromatic.

Figure 5: Reduction of $G$ into smaller graphs

Suppose that $K$ has parallel edges between $a$ and $c$. Let $p$ be the neighbor of $a$ in $G$ distinct from $b$ and $c$, and let $q$ be the neighbor of $c$ in $G$ distinct from $a$ and $d$. Consider whether $p$ and $q$ are adjacent in $G$ or not. If they are not, then we join $p$ and $q$ to get a graph corresponding to the above $K$. If $p$ and $q$ are adjacent in $G$, then we look at the neighbor, say $p'$, of $p$ distinct from $q$ and $a$, and the neighbor, $q'$, of $q$ distinct from $p$ and $c$, and consider whether $p'$ and $q'$ are adjacent in $G$. However, this argument does not continue infinitely, since $G$ is finite. Consequently, after relabeling of vertices in $G$, we can find two disjoint paths $av_1 \cdots v_mb$ and $cu_1 \cdots u_md$ with $ac, bd \notin E(G)$ and $v_iu_i \in E(G)$ for $i = 1, \ldots, m$, as shown in Figure 5(2). Then, removing $\{v_1, \ldots, v_m, u_1, \ldots, u_m\}$ from $G$, and adding edges $ac$ and $bd$, we have two components, say $K$ and $H$, each of which belongs to $\mathcal{G}_2$, where $a, c \in V(K)$ and $b, d \in V(H)$. By the minimality of $G$, $K$ and $H$ have polychromatic proper 4-colorings $\rho_K$ and $\rho_H$, respectively, where we may suppose that $1 = \rho_K(a) \neq \rho_K(c) = 2$. Color $v_1, \ldots, v_m$ (resp., $u_1, \ldots, u_m$) by 3, 1, 3, 1, … (resp., 4, 2, 4, 2, …) alternately. Note that then each quadrilateral face $v_iu_iu_{i+1}v_{i+1}$ of $G$ is polychromatic. Next, permute the four colors in $\rho_H$ so that the colors of $b$ and $d$ do not collide with those of $v_m$ and $u_m$. Similarly to the previous case, we can verify that it is a polychromatic proper 4-coloring. ■
References


