A Note on the Complexity of Longest Path Problems Related to Graph Coloring

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Abstract—In this note, we show that some problems related to the length of the longest simple path from a given vertex in a graph are NP-complete. We also discuss an extension to the graph coloring problem. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Let $G = (V, E)$ be a connected graph and let $p(v)$ denote the length of a longest simple path in $G$ starting from vertex $v \in V$. In this note, we mainly address the following questions.

1. What is the complexity of computing $p(v)$?
2. What is the complexity of computing $\Gamma = \max_{v \in V} p(v)$?
3. What is the complexity of computing $\gamma = \min_{v \in V} p(v)$?

The interest these problems arises from the fact that if $\chi(G)$ denotes the chromatic number of $G$, then $\chi(G) \leq 1 + \gamma$. This generalizes the well-known result of Gallai, which states that $\chi(G) < 1 + m(G)$, where $m(G)$ denotes the length of a longest path in $G$ [1]. In addition, the minimum degree $d$ of the graph is related to paths of length $d$ in the graph.

Indeed, if the graph $G$ has minimum degree $d$ then for every vertex $v \in V$, there is a path of length $d$ starting at $v$.

To prove this assertion, assume that $v, w \in V$, $v \neq w$ and let $v - \rightarrow w$ be the longest path starting at vertex $v$. Now suppose the contrary, the length of this path is less than $d$. Consider the vertex $w$. If it is adjacent only to vertices on the path $v - \rightarrow w$ then its degree is less than $d$ and it is a contradiction. Opposite, if the vertex $w$ has also some neighbor not on the path $v - \rightarrow w$ then the contradiction comes immediately from the fact that $v - w$ is the longest path starting at $v$.

Next, we show that all the above-stated problems are NP-complete. Results for related problems are given in [2].
2. THE RESULTS

2.1. Longest Path

We give proofs for the corresponding decision problems, assuming that \( k \) is a positive integer and using the fact that the Hamiltonian path (HP) problem is \( \mathcal{NP} \)-complete [3].

**Theorem 1.** All the above-mentioned Problems 1–3 are \( \mathcal{NP} \)-hard, and the corresponding decision problems are \( \mathcal{NP} \)-complete.

**Proof.**

**Part 1.** We first show that checking if \( p(v) \geq k \) is an \( \mathcal{NP} \)-complete problem. We use a reduction from the (HP) problem. Clearly, the problem is in \( \mathcal{NP} \). To see that the problem is \( \mathcal{NP} \)-complete, we carry out the following reduction. Let \( \{G = (V,E), v_A, v_B\} \) be an instance of the (HP) problem, where \( v_A, v_B \in V \) and we have to determine if there is a Hamiltonian path from \( v_A \) to \( v_B \). We now construct the graph \( G' \) as follows. Define \( G' = (V \cup \{v'_A, v'_B\}, E \cup \{(v'_A, v'_B), (v_B, v'_B)\}) \). It is easy to see that \( p(v'_A) \geq |V| + 1 \) iff \( G \) has a Hamiltonian path from \( v_A \) to \( v_B \).

**Part 2.** Consider now the problem of proving if \( \gamma = \max_{v \in V} p(v) \geq k \). Again, the problem is clearly in \( \mathcal{NP} \). All that we have to do is to guess a path and verifying that the path is simple and has length at least \( k \) can be done in polynomial time. To prove that the problem is \( \mathcal{NP} \)-complete, we carry out the same reduction as before. Let \( \{G = (V,E), v_A, v_B\} \) be an instance of the (HP) problem, where \( v_A, v_B \in V \) and we have to determine if there is a Hamiltonian path from \( v_A \) to \( v_B \). Define \( G' = (V', E') \) where \( V' = V \cup \{v'_A, v'_B\}, E' = E \cup \{(v'_A, v'_B), (v_B, v'_B)\} \). It follows that \( \gamma = \max_{(v \in V')} p(v) \geq |V| + 1 \) iff there is a Hamiltonian path in \( G \) from \( v_A \) to \( v_B \).

**Part 3.** Finally, we prove that determining if \( \gamma = \min_{v \in V} p(v) \geq k \) is \( \mathcal{NP} \)-complete. We can guess \( |V| \) paths beginning from each of the vertices in the graph and verify that each path has length at least \( k \) in polynomial time. So the problem is in \( \mathcal{NP} \). To show that the problem is \( \mathcal{NP} \)-complete, we carry out a reduction from (HP). Let \( \{G = (V,E), v_A, v_B\} \) be an instance of the Hamiltonian path problem. Define \( G' = (V', E') \) where \( V' = V \cup \{v_L, v_M, v_R\} \) and \( E' = E \cup \{(v_A, v_L), (v_L, v_M), (v_M, v_R), (v_R, v_B)\} \). Now, we claim that \( \gamma = \min_{(v \in V')} p(v) \geq |V| + 2 \) iff there is a Hamiltonian path in \( G \) from \( v_A \) to \( v_B \). One can see that if \( G \) has a Hamiltonian path from \( v_A \) to \( v_B \), then \( G' \) has a Hamiltonian circuit and thus \( \gamma \geq |V| + 2 \). If \( G \) does not have a Hamiltonian path from \( v_A \) to \( v_B \), notice that the length of the longest path from any of the vertices \( v_L, v_M, \) or \( v_R \) in \( G' \) can be at most \( |V| + 1 \). This completes the proof.

2.2. Graph Coloring

Here we state some questions about the longest path problem related to graph coloring.

Let \( \chi(G) \) be the chromatic number of graph \( G = (V,E) \). Is there a coloring of \( G \), using \( \chi(G) \) colors, such that for a given vertex \( v \in V \) there exists a path starting at \( v \) which represents all \( \chi(G) \) colors?

**Theorem 2.** Let \( \chi(G) \) be the chromatic number of graph \( G = (V,E) \). To decide whether \( G \) is colorable with \( \chi(G) \) colors in such a way that for a given vertex \( v \in V \) there is a path starting at \( v \) representing all \( \chi(G) \) colors is an \( \mathcal{NP} \)-complete problem.

**Proof.** Suppose that graph \( G = (V,E) \) has cardinality of the vertex set \( |V| = n \). Let vertex \( v \in V \) be the vertex from which the required path should exist. To show that the problem belongs to \( \mathcal{NP} \), we can check in polynomial time if it contains all \( \chi(G) \) colors or not. To see that the problem is \( \mathcal{NP} \)-complete we will do the following reduction. Let \( G' = (V', E') \) such that \( V' = V \cup V'' \) and \( E' = E \cup E'' \) where \( V'' = \{u_1, u_2, \ldots, u_n\} \) and \( E'' \) contains all \( \binom{n}{2} \) edges between vertices \( u_1, u_2, \ldots, u_n \). The graph \( K_n = (V'', E'') \) is a complete graph on \( n \) vertices. As the chromatic number \( \chi(K_n) = n \), we can easily color the graph \( G' \) using \( n \) colors in such a way that each vertex of graph \( G \) has different color. Now it is easy to see that
a path starting at vertex $v$ represents all $\chi(G')$ colors iff graph $G$ contains a Hamiltonian path starting at $v$.

**Problem.** Let $\chi(G)$ be the chromatic number of graph $G = (V, E)$ [4–8]. Suppose that $G$ is properly colored with $\chi(G)$ colors and we fix this coloring. Let $v$ be a vertex of $G$. Is there a path starting at $v$ which represents all $\chi(G)$ colors?

**References**

2. S. Fajtlowicz, List of graffiti conjectures (private communication), (1994).