Open Journal of Discrete Mathematics, 2013, 3, 143-150
doi:10.4236/ojdm.2013.33026 Published Online July 2013 (http://www.scirp.org/journal/ojdm)

Some Results on Generalized Degree Distance

Asma Hamzeh, Ali Iranmanesh*, Samaneh Hossein-Zadeh
Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Tehran, Iran
Email: *iranmanesh@modares.ac.ir

Received January 14, 2013; revised February 15, 2013; accepted May 14, 2013

ABSTRACT

In [1], Hamzeh, Iranmanesh Hossein-Zadeh and M. V. Diudea recently introduced the generalized degree distance of graphs. In this paper, we present explicit formulas for this new graph invariant of the Cartesian product, composition, join, disjunction and symmetric difference of graphs and introduce generalized and modified generalized degree distance polynomials of graphs, such that their first derivatives at $x = 1$ are respectively, equal to the generalized degree distance and the modified generalized degree distance. These polynomials are related to Wiener-type invariant polynomial of graphs.

Keywords: Generalized Degree Distance; Cartesian Product; Join; Symmetric Difference; Composition; Disjunction

1. Introduction

A graph invariant is any function on a graph that does not depend on a labeling of its vertices. Topological indices and graph invariants based on the distances between the vertices of a graph are widely used in theoretical chemistry to establish relations between the structures and the properties of molecules. Topological indices provide correlations with physical, chemical and thermodynamic parameters of chemical compounds [2]. In this paper, we only consider simple and connected graphs. Let G be a graph on n vertices and m edges. We denote the vertex and the edge set of G by $V(G)$ and $E(G)$, respectively. As usual, the distance between the vertices u and v of G, denoted by $d_G(u,v)$ ($d(u,v)$ for short), is defined as the length of a minimum path connecting them. We let $d_G(v)$ be the degree of a vertex v in G. The eccentricity, denoted by $\varepsilon(v)$, is defined as the maximum distance from vertex v to any other vertex. The diameter of a graph G, denoted by $\operatorname{diam}(G)$, is the maximum eccentricity over all vertices in a graph G.

The Cartesian product $G\times H$ of graphs G and H is a graph such that $V(G\times H)=V(G)\times V(H)$, and any two vertices (a,b) and (u,v) are adjacent in $G\times H$ if and only if either $a=u$ and b is adjacent to v, or $b=v$ and a is adjacent to u, see [3] for details. Let G_1 and G_2 be two graphs with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2. The join G_1+G_2 is the graph union $G_1\cup G_2$ together with all the edges joining V_1 and V_2. The composition $G_1[G_2]$ is the graph with vertex set $V_1\times V_2$ and $u=(u_1,v_1)$ is adjacent to $v=(u_2,v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1=u_2$ and v_1 is adjacent to v_2), [3, p. 185]. The disjunction $G\vee H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ and (u_1,v_1) is adjacent to (u_2,v_2) whenever $u_1u_2\in E(G)$ or $v_1v_2\in E(H)$. The symmetric difference $G\oplus H$ of two graphs G and H is the graph with vertex set $V(G)\times V(H)$ and

$$E(G\oplus H)=\{(u_1,u_2)(v_1,v_2)\,|\,u_1v_1\in E(G) \text{ or } u_2v_2\in E(H) \text{ but not both}\}.$$

The first Zagreb index was originally defined as $M_1(G)=\sum_{u\in V(G)} d_G(u)^2$ [4]. The first Zagreb index can be also expressed as a sum over edges of G, $M_1(G)=\sum_{uv\in E(G)}\left[d_G(u)+d_G(v)\right]$. We refer readers to [5] for the proof of this fact and for more information on Zagreb index. The first Zagreb coindex of a graph G is defined in [6] as:

$$\bar{M}_1(G)=\sum_{uv\notin E(G)}\left[d_G(u)+d_G(v)\right].$$

Let $d(G,k)$ be the number of pairs of vertices of a graph G that are at distance k, λ be a real number, and $W_\lambda(G)=\sum_{k\ge 1} d(G,k)k^\lambda$, that is called the Wiener-type invariant of G associated to λ, see [7,8] for details. Additively weighted Harary index is defined in [9] as

*Corresponding author.

$$H_A(G)=\sum_{\{u,v\}\subseteq V(G)} d^{-1}(u,v)\left(d_G(u)+d_G(v)\right).$$

Dobrynin and Kochetova in [10] and Gutman in [11] introduced a new graph invariant with the name degree distance that is defined as follows:

$$D'(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v)\left(d_G(u)+d_G(v)\right).$$

In [12], the modified degree distance was defined as follows:

$$S^*(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)\left(d_G(u)d_G(v)\right).$$

The generalized degree distance, denoted by $H_\lambda(G)$, is defined as follows in [1].

For every vertex x and real number λ, $H_\lambda(x)$ is defined by $H_\lambda(x)=D^\lambda(x)d_G(x)$, where $D^\lambda(x)=\sum_{y\in V(G)} d^\lambda(x,y)$. We then define

$$H_\lambda(G)=\sum_{x\in V(G)} H_\lambda(x)=\sum_{x\in V(G)} D^\lambda(x)d_G(x)$$
$$=\sum_{\{u,v\}\subseteq V(G)} d^\lambda(u,v)\left(d_G(u)+d_G(v)\right).$$

If $\lambda=0$, then $H_\lambda(G)=4\,\mathrm{m}$. When $\lambda=1$, this new topological index $\left(H_\lambda(G)\right)$ is equal to the degree distance (or Schultz index). There are many papers for studying this topological index, for example see [13-16]. Also if $\lambda=-1$, then $H_\lambda(G)=H_A(G)$. Therefore the study of this new topological index is important and we try to obtain some new results related to this topological index. The modified generalized degree distance, denoted by $H^*_\lambda(G)$, is defined in [1] as:

$$H^*_\lambda(G)=\sum_{\{u,v\}\subseteq V(G)} d^\lambda(u,v)\left(d_G(u)d_G(v)\right).$$

If $\lambda=1$, then $H^*_\lambda(G)=S^*(G)$.

We construct graph polynomials having the property such that their first derivatives at $x=1$ are equal to the generalized degree distance, the modified generalized degree distance and Wiener-type invariant respectively. These polynomials are defined as follows:

$$H_\lambda(G,x)=\sum_{\{u,v\}\subseteq V(G)} \left(d_G(u)+d_G(v)\right)x^{d^\lambda(u,v)},$$

$$H^*_\lambda(G,x)=\sum_{\{u,v\}\subseteq V(G)} \left(d_G(u)d_G(v)\right)x^{d^\lambda(u,v)},$$

and

$$W_\lambda(G,x)=\sum_{\{u,v\}\subseteq V(G)} x^{d^\lambda(u,v)}.$$

The Wiener index of the Cartesian product of graphs was studied in [17,18]. In [19], Klavžar, Rajapakse and Gutman computed the Szeged index of the Cartesian product of graphs. In [9,20-24], exact formulae for the hyper-Wiener, the first Zagreb index, the second Zagreb index and Schultz polynomials of some graph operations were computed.

Throughout this paper, C_n, P_n, K_n and S_n denote the cycle, path, complete graph and star on n vertices. The complement of a graph G is a graph H on the same vertices such that two vertices of H are adjacent if and only if they are not adjacent in G. The graph H is usually denoted by $\overline{G}$. Our other notations are standard and taken mainly from [2,25,26].

In this paper we present explicit formulas for $H_\lambda(G)$ of graph operations containing the Cartesian product, composition, join, disjunction and symmetric difference of graphs and introduce generalized and modified generalized degree distance polynomials of graphs, such that their first derivatives at $x=1$ are respectively, equal to the generalized degree distance and the modified generalized degree distance. These polynomials are related with Wiener-type invariant polynomial of graphs.

2. Main Results

The aim of this section is to compute the generalized degree distance for five graph operations. We start with a lemma which gives some information about the number of vertices and edges of operations on two arbitrary graphs. For a given graph G_i, the number of vertices and edges will be denoted by n_i and m_i, respectively.

Lemma 2.1. [3,20] Let G and H be graphs. Then we have:

a)

$$|V(G\times H)|=|V(G\vee H)|=|V(G[H])|=|V(G\oplus H)|$$
$$=|V(G)|\cdot|V(H)|,$$
$$|E(G\times H)|=|E(G)|\cdot|V(H)|+|V(G)|\cdot|E(H)|,$$
$$|E(G+H)|=|E(G)|+|E(H)|+|V(G)|\cdot|V(H)|,$$
$$|E(G[H])|=|E(G)|\cdot|V(H)|^2+|E(H)|\cdot|V(G)|,$$
$$|E(G\vee H)|=|E(G)|\cdot|V(H)|^2+|E(H)|\cdot|V(G)|^2$$
$$-2|E(G)|\cdot|E(H)|,$$

and

$$|E(G\oplus H)|=|E(G)|\cdot|V(H)|^2+|E(H)|\cdot|V(G)|^2$$
$$-4|E(G)|\cdot|E(H)|.$$

b) The graph $G\times H$ is connected if and only if G and H are connected.

c) If (a,c) and (b,d) are vertices of $G\times H$, then $d_{G\times H}\left((a,c),(b,d)\right)=d_G(a,b)+d_H(c,d)$.

d) The Cartesian product, join, composition, disjunction and symmetric difference of graphs are associative and all of them are commutative except the composition of graphs.

e)

$$d_{G+H}(u,v) = \begin{cases} 0, u=v \\ 1, uv\in E(G) \text{ or } uv\in E(H) \text{ or } (u\in V(G)\ v\in V(H)). \\ 2, \text{otherwise} \end{cases}$$

f)

$$d_{G[H]}((a,b),(c,d)) = \begin{cases} d_G(a,c), a\neq c \\ 0, \quad a=c\ \&\ b=d \\ 1, \quad a=c\ \&\ bd\in E(H) \\ 2, \quad a=c\ \&\ bd\notin E(H) \end{cases}.$$

g)

$$d_{G\vee H}((a,b),(c,d)) = \begin{cases} 0, a=c\ \&\ b=d \\ 1, ac\in E(G) \text{ or } bd\in E(H). \\ 2, \text{otherwise} \end{cases}$$

h)

$$d_{G\oplus H}((a,b),(c,d)) = \begin{cases} 0, a=c\ \&\ b=d \\ 1, ac\in E(G) \text{ or } bd\in E(H) \text{ but not both}. \\ 2, \text{otherwise} \end{cases}$$

i) $d_{G\times H}((a,b)) = d_G(a)+d_H(b)$.

j) $d_{G[H]}((a,b)) = |V(H)|d_G(a)+d_H(b)$.

k) $d_{G+H}(a) = \begin{cases} d_G(a)+|V(H)|, a\in VG \\ d_H(a)+|V(G)|, a\in V(H) \end{cases}$.

l)

$$d_{G\vee H}((a,b)) = |V(H)|d_G(a)+|V(G)|d_H(b) - d_G(a)d_H(b).$$

m)

$$d_{G\oplus H}((a,b)) = |V(H)|d_G(a)+|V(G)|d_H(b) - 2d_G(a)d_H(b).$$

In Theorem 2.2, we give a formula for the generalized degree distance of the join of two graphs.

Theorem 2.2. Let G_1 and G_2 be two graphs. Then

$$H_\lambda(G_1+G_2) = n_1n_2(n_1+n_2)+4n_2m_1+4n_1m_2+M_1(G_1)+M_1(G_2) + 2^\lambda(\bar{M}_1(G_1)+\bar{M}_1(G_2))+2^{\lambda+1}(n_2\bar{m}_1+n_1\bar{m}_2).$$

Proof. It is obvious from definition that for any $u,v\in V(G_1+G_2)$, the distance $d_{G_1+G_2}(u,v)$ is either 1 or 2. In the formula for $H_\lambda(G_1+G_2)$, we partition the set of pairs of vertices of G_1+G_2 into three subsets A_0, A_1 and A_2. In A_0 we collect all pairs of vertices *u* and *v* such that *u* is in G_1 and *v* is in G_2. Hence, they are adjacent in G_1+G_2. The set $A_i, i=1,2$ is the set of pairs of vertices *u* and *v* which are in G_i. Also we partition the sum in the formula of $H_\lambda(G_1+G_2)$ into three sums S_i so that S_i is over A_i for $i=0,1,2$. For S_0 we obtain

$$S_0 = \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}(d_{G_1}(u)+d_{G_2}(v)+n_1+n_2) = n_1n_2(n_1+n_2)+2n_2m_1+2n_1m_2,$$

and

$$\begin{aligned} S_1 &= \sum_{\{u,v\}\subseteq V(G_1)}(d_{G_1}(u)+d_{G_1}(v)+2n_2)d^\lambda_{G_1+G_2}(u,v) \\ &= \sum_{uv\in E(G_1)}(d_{G_1}(u)+d_{G_2}(v)+2n_2) + \sum_{uv\notin E(G_1)}2^\lambda(d_{G_1}(u)+d_{G_2}(v)+2n_2) \\ &= M_1(G_1)+2n_2m_1+2^\lambda\bar{M}_1(G_1)+2^{\lambda+1}n_2\bar{m}_1. \end{aligned}$$

Similarly,

$$S_2 = M_1(G_2)+2n_1m_2+2^\lambda\bar{M}_1(G_2)+2^{\lambda+1}n_1\bar{m}_2.$$

Therefore

$$\begin{aligned} &H_\lambda(G_1+G_2) \\ &= S_0+S_1+S_2 \\ &= n_1n_2(n_1+n_2)+4n_2m_1+4n_1m_2+M_1(G_1)+M_1(G_2) \\ &+2^\lambda(\bar{M}_1(G_1)+\bar{M}_1(G_2))+2^{\lambda+1}(n_2\bar{m}_1+n_1\bar{m}_2). \end{aligned}$$ □

Corollary 2.3. Let *G* be a connected graph with *n* vertices and *m* edges. Then

$$H_\lambda(K_1+G) = n(n+1)+4m+M_1(G) + 2^\lambda\bar{M}_1(G)+2^{\lambda+1}\bar{m}.$$

The exact formulas $H_\lambda(G)$ for the fan graph K_1+P_n and for the wheel graph $W_n = K_1+C_n$ are given in the following Corollary.

Corollary 2.4.

$$H_\lambda(K_1+P_n) = n_2+9n-10+2^\lambda(n-2)(5n-9),$$

and

$$H_\lambda(K_1+C_n) = n_2+9n+3\times 2^\lambda n(n-3).$$

Remark 2.5. In the above theorem, if $\lambda=1$, then we obtain $D'(G_1+G_2)$, which gives first derivatives formula Theorem 3 in [22] at $x=1$.

In the next theorem we obtain the exact formula for the generalized degree distance of the composition of two graphs.

Theorem 2.6. Let G_1 and G_2 be two graphs. Then

$$H_\lambda(G_1[G_2]) = 4m_2m_1n_2 + n_1M_1(G_2) + 2^{\lambda+2}\bar{m}_2n_2m_1 + 2^\lambda n_1\bar{M}_1(G_2) + n_2^3H_\lambda(G_1) + 4n_2m_2W_\lambda(G_1)$$

Proof. Suppose $\{u_1,\cdots,u_{n_1}\}$ and $\{v_1,\cdots,v_{n_2}\}$ are two set of vertices of G_1 and G_2, respectively. Then by Lemma 2.1 and definition of H_λ, we have:

$$\begin{aligned}
H_\lambda(G_1[G_2]) &= \sum_{\{u,v\}\subseteq V(G_1[G_2])} d^\lambda_{G_1[G_2]}(u,v)\left(d_{G_1[G_2]}(u)+d_{G_1[G_2]}(v)\right)\\
&= \frac{1}{2}\sum_{(u_i,v_k)}\sum_{(u_j,v_l)} d^\lambda_{G_1[G_2]}\left((u_i,v_k),(u_j,v_l)\right)\left(nd_{G_1}(u_i)+d_{G_2}(v_k)+nd_{G_1}(u_j)+d_{G_2}(v_l)\right)\\
&= \sum_{p=1}^{n_1}\sum_{k,l=1}^{n_2} d^\lambda_{G_1[G_2]}\left((u_p,v_k),(u_p,v_l)\right)\left(2n_2d_{G_1}(u_p)+d_{G_2}(v_k)+d_{G_2}(v_l)\right)\\
&\quad+\sum_{k,l=1}^{n_2}\sum_{i,j=1,i\neq j}^{n_1} d^\lambda_{G_1[G_2]}\left((u_i,v_k),(u_j,v_l)\right)\left(n_2\left(d_{G_1}(u_i)+d_{G_1}(u_j)\right)+d_{G_2}(v_k)+d_{G_2}(v_l)\right)\\
&\quad+\sum_{p=1}^{n_1}\sum_{k,l=1,u_kv_l\notin E(G_2)}^{n_2} 2^\lambda\left(2n_2d_{G_1}(u_p)+d_{G_2}(v_k)+d_{G_2}(v_l)\right)+n_2^3H_\lambda(G_1)+4n_2m_2W_\lambda(G_1)\\
&= 4m_2m_1n_2+n_1M_1(G_2)+2^{\lambda+2}\bar{m}_2n_2m_1+2^\lambda n_1\bar{M}_1(G_2)+n_2^3H_\lambda(G_1)+4n_2m_2W_\lambda(G_1).
\end{aligned}$$

So the proof of theorem is now completed. □

By composing paths and cycles with various small graphs we can obtain classes of polymer-like graphs. Now we give the formula of the H_λ index for the fence graph $P_n[K_2]$ and the closed fence $C_n(K_2)$.

Corollary 2.7.

$$H_\lambda(P_n[K_2]) = 9n-8+8\left(H_\lambda(P_n)+W_\lambda(P_n)\right),$$

and

$$H_\lambda(C_n[K_2]) = 9n+8\left(H_\lambda(C_n)+W_\lambda(C_n)\right).$$

Remark 2.8. In Theorem 2.6, if $\lambda=1$, then we obtain $D'(G_1[G_2])$, which gives first derivatives formula Theorem 5 in [22] at $x=1$.

Now we prove the theorem that characterizes the generalized degree distance of the disjunction of two graphs.

Theorem 2.9. Let G_1 and G_2 be two graphs. Then

$$\begin{aligned}
H_\lambda(G_1\vee G_2) &= 8n_1n_2m_1m_2+\left(n_1^3-4n_1m_1\right)M_1(G_2)+\left(n_2^3-4n_2m_2\right)M_1(G_1)+M_1(G_1)M_1(G_2)\\
&\quad+2^\lambda\left[\left(2n_2\bar{m}_2+n_2^2-2m_2\right)\bar{M}_1(G_1)+\left(2n_1\bar{m}_1+n_1^2-2m_1\right)\bar{M}_1(G_2)\right]\\
&\quad-2^\lambda\bar{M}_1(G_1)\bar{M}_1(G_2)+2^{\lambda+2}\left(n_1\bar{m}_1m_2+n_2\bar{m}_2m_1\right)
\end{aligned}$$

Proof. According to definition of $G_1\vee G_2$, we have the following relations:

$$\begin{aligned}
S_1 &= \sum_{\{u,v\}\subseteq V(G_1)}\sum_{uv\in E(G_2)}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= \sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)}\left(n_1\left(d_{G_2}(u)+d_{G_2}(v)\right)+n_2\left(d_{G_1}(y)+d_{G_1}(x)\right)-d_{G_1}(x)d_{G_2}(u)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= n_1^3M_1(G_2)+4n_1m_2m_1n_2-2n_1m_1M_1(G_2),
\end{aligned}$$

$$\begin{aligned}
S_2 &= \sum_{xy\in E(G_1)}\sum_{\{u,v\}\subseteq V(G_2)}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= n_2^3M_1(G_1)+4n_1m_2m_1n_2-2n_2m_2M_1(G_1),
\end{aligned}$$

$$\begin{aligned}
S_3 &= \sum_{xy\in E(G_1)}\sum_{uv\in E(G_1)}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)\\
&= 2n_1m_1M_1(G_2)+2n_2m_2M_1(G_1)-M_1(G_1)M_1(G_2),
\end{aligned}$$

and

$$S_4=\sum_{xy\notin E(G_1),x\neq y}\sum_{uv\notin E(G_2),u\neq v}2^{\lambda}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\right)$$
$$+2^{\lambda}\sum_{xy\notin E(G_1)}\sum_{u\in V(G_2)}\left(2n_1d_{G_2}(u)+\left(n_2-d_{G_2}(u)\right)\left(d_{G_1}(x)+d_{G_1}(y)\right)\right)$$
$$+2^{\lambda}\sum_{x\in V(G_1)}\sum_{uv\notin E(G_2)}\left(2n_2d_{G_1}(x)+\left(n_1-d_{G_1}(x)\right)\left(d_{G_2}(u)+d_{G_2}(v)\right)\right)$$
$$=2^{\lambda}\left[\left(2n_2\bar{m}_2+n_2^2-2m_2\right)\bar{M}_1(G_1)+\left(2n_1\bar{m}_1+n_1^2-2m_1\right)\bar{M}_1(G_2)-\bar{M}_1(G_1)\bar{M}_1(G_2)\right]+2^{\lambda+2}\left(n_1\bar{m}_1m_2+n_2\bar{m}_2m_1\right).$$

So we have:

$$H_{\lambda}(G_1\vee G_2)$$
$$=S_1+S_2+S_4-S_3=8n_1n_2m_1m_2+\left(n_1^3-4n_1m_1\right)M_1(G_2)+\left(n_2^3-4n_2m_2\right)M_1(G_1)+M_1(G_1)M_1(G_2)$$
$$+2^{\lambda}\left[\left(2n_2\bar{m}_2+n_2^2-2m_2\right)\bar{M}_1(G_1)+\left(2n_1\bar{m}_1+n_1^2-2m_1\right)\bar{M}_1(G_2)\right]-2^{\lambda}\bar{M}_1(G_1)\bar{M}_1(G_2)+2^{\lambda+2}\left(n_1\bar{m}_1m_2+n_2\bar{m}_2m_1\right).$$

This completes the proof. □

Now we prove the theorem that characterizes the generalized degree distance of the symmetric difference of two graphs.

Theorem 2.10. Let G_1 and G_2 be two graphs. Then

$$H_{\lambda}(G_1\oplus G_2)=8n_1n_2m_1m_2+\left(n_1{}^3-8n_1m_1\right)M_1(G_2)+\left(n_2{}^3-8n_2m_2\right)M_1(G_1)+4M_1(G_1)M_1(G_2)$$
$$+2^{\lambda}\left[\left(2n_2\bar{m}_2+n_2{}^2-4m_2\right)\bar{M}_1(G_1)+\left(2n_1\bar{m}_1+n_1{}^2-4m_1\right)\bar{M}_1(G_2)\right]$$
$$-2^{\lambda}\bar{M}_1(G_1)\bar{M}_1(G_2)+2^{\lambda+2}\left(n_1\bar{m}_1m_2+n_2\bar{m}_2m_1\right)$$

Proof. We consider four sums $S_1,\cdots,S_4$ as follows:

$$S_1=\sum_{\{x,y\}\subseteq V(G_1)}\sum_{uv\in E(G_2)}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right)$$
$$=n_1^3M_1(G_2)+4n_1m_2m_1n_2-4n_1m_1M_1(G_2),$$

similarly to S_1

$$S_2=n_2^3M_1(G_1)+4n_1m_2m_1n_2-4n_2m_2M_1(G_1),$$
$$S_3=\sum_{xy\in E(G_1)}\sum_{uv\in E(G_2)}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right)$$
$$=2n_1m_1M_1(G_2)+2n_2m_2M_1(G_1)-2M_1(G_1)M_1(G_2),$$

and

$$S_4=\sum_{xy\notin E(G_1),x\neq y}\sum_{uv\notin E(G_2),u\neq v}2^{\lambda}\left(n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)+n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\right)$$
$$+2^{\lambda}\sum_{xy\notin E(G_1)}\sum_{u\in V(G_2)}\left(2n_1d_{G_2}(u)+\left(n_2-2d_{G_2}(u)\right)\left(d_{G_1}(x)+d_{G_1}(y)\right)\right)$$
$$+2^{\lambda}\sum_{x\in V(G_1)}\sum_{uv\notin E(G_2)}\left(2n_2d_{G_1}(x)+\left(n_1-2d_{G_1}(x)\right)\left(d_{G_2}(u)+d_{G_2}(v)\right)\right)$$
$$=2^{\lambda}\left[\left(2n_2\bar{m}_2+n_2^2-4m_2\right)\bar{M}_1(G_1)+\left(2n_1\bar{m}_1+n_1^2-4m_1\right)\bar{M}_1(G_2)-2\bar{M}_1(G_1)\bar{M}_1(G_2)\right]$$
$$+2^{\lambda+2}\left(n_1\bar{m}_1m_2+n_2\bar{m}_2m_1\right).$$

By the definition of $G_1\oplus G_2$, we have:

$$H_{\lambda}(G_1\oplus G_2)=S_1+S_2+S_4-2S_3=8n_1n_2m_1m_2+\left(n_1^3-8n_1m_1\right)M_1(G_2)+\left(n_2^3-8n_2m_2\right)M_1(G_1)+4M_1(G_1)M_1(G_2)$$
$$+2^{\lambda}\left[\left(2n_2\bar{m}_2+n_2^2-4m_2\right)\bar{M}_1(G_1)+\left(2n_1\bar{m}_1+n_1^2-4m_1\right)\bar{M}_1(G_2)-2\bar{M}_1(G_1)\bar{M}_1(G_2)\right]$$
$$+2^{\lambda+2}\left(n_1\bar{m}_1m_2+n_2\bar{m}_2m_1\right).$$

So the proof is now completed. □

In the next theorem we find the generalized degree distance of the Cartesian product of two graphs.

Theorem 2.11. Let G_1 and G_2 be two graphs. Then

$$H_\lambda(G_1\times G_2)=n_1^2H_\lambda(G_2)4m_1n_1W_\lambda(G_2)+2\binom{\lambda}{1}\big(H_1(G_1)W_{\lambda-1}(G_2)+W(G_1)H_{\lambda-1}(G_2)\big)$$
$$+2\binom{\lambda}{2}\big(H_2(G_1)W_{\lambda-2}(G_2)+W_2(G_1)H_{\lambda-2}(G_2)\big)+\cdots+2\binom{\lambda}{\lambda-1}\big(H_{\lambda-1}(G_1)W(G_2)+W_{\lambda-1}(G_1)H_1(G_2)\big)$$
$$+n_2^2H_\lambda(G_1)+4m_2n_2W_\lambda(G_1).$$

Proof. Suppose $\{u_1,\cdots,u_{n_1}\}$ and $\{v_1,\cdots,v_{n_2}\}$ are two set of vertices of G_1 and G_2, respectively. Then by Lemma 2.1 and definition of H_λ, we have:

$$H_\lambda(G_1\times G_2)=\sum_{\{u,v\}\subseteq V(G_1\times G_2)} d^\lambda_{G_1\times G_2}(u,v)\big(d_{G_1\times G_2}(u)+d_{G_1\times G_2}(v)\big)$$
$$=\frac{1}{2}\sum_{(u_i,v_k)}\sum_{(u_j,v_l)} d^\lambda_{G_1\times G_2}\big((u_i,v_k),(u_j,v_l)\big)\big(d_{G_1}(u_i)+d_{G_2}(v_k)+d_{G_1}(u_j)+d_{G_2}(v_l)\big)$$
$$=\frac{1}{2}\sum_{k,l=1}^{n_2}\sum_{i,j=1}^{n_1}\big(d_{G_1}(u_i,u_j)+d_{G_2}(v_k,v_l)\big)^\lambda\big(d_{G_1}(u_i)+d_{G_2}(v_k)+d_{G_1}(u_j)+d_{G_2}(v_l)\big)$$
$$=\frac{1}{2}\sum_{k,l=1}^{n_2}\sum_{i,j=1}^{n_1}\left(\sum_{r=0}^{\lambda}\lambda_r d^r_{G_1}(u_i,u_j)d^{\lambda-r}_{G_2}(v_k,v_l)\right)\big(d_{G_1}(u_i)+d_{G_2}(v_k)+d_{G_1}(u_j)+d_{G_2}(v_l)\big)$$
$$=n_1^2H_\lambda(G_2)+4m_1n_1W_\lambda(G_2)+2\binom{\lambda}{1}\big(H_1(G_1)W_{\lambda-1}(G_2)+W(G_1)H_{\lambda-1}(G_2)\big)$$
$$+2\binom{\lambda}{2}\big(H_2(G_1)W_{\lambda-2}(G_2)+W_2(G_1)H_{\lambda-2}(G_2)\big)$$
$$+\cdots+2\binom{\lambda}{\lambda-1}\big(H_{\lambda-1}(G_1)W(G_2)+W_{\lambda-1}(G_1)H_1(G_2)\big)+n_2^2H_\lambda(G_1)+4m_2n_2W_\lambda(G_1).$$

So the proof is now completed. □

As an application of the above theorem, we list explicit formulae for the generalized degree distance of $P_n\times P_m, P_n\times C_m$ and $C_n\times C_m$. These graphs are known as the rectangular grid, the C_4 nanotube, and the C_4 nanotorus, respectively.

Lemma 2.12. Define $\alpha(k,r)=\sum_{i=1}^{k} i^r$. By [1,23], we have:

$$W_\lambda(P_n)=n\alpha(n-1,\lambda)-\alpha(n-1,\lambda+1),$$

$$W_\lambda(C_n)=\begin{cases} n\alpha\left(\frac{n}{2}-1,\lambda\right)+\left(\frac{n}{2}\right)^\lambda\frac{n}{2}, & n \text{ is even}\\ n\alpha\left(\frac{n-1}{2},\lambda\right), & n \text{ is odd}\end{cases},$$

$$H_\lambda(P_n)=-2\alpha(n-3,\lambda)+4n\alpha(n-3,\lambda)$$
$$-4\alpha(n-3,\lambda+1)+2(n-1)^\lambda+6(n-2)^\lambda,$$

$$H_\lambda(C_n)=\begin{cases} 4n\alpha\left(\frac{n}{2}-1,\lambda\right)+\left(\frac{n}{2}\right)^{\lambda+1}, & n \text{ is even}\\ 4n\alpha\left(\frac{n-1}{2},\lambda\right), & n \text{ is odd}\end{cases}.$$

Corollary 2.13. By Theorem 2.9 and Lemma 2.12 we have:

$$H_\lambda(C_n\times C_m)$$
$$=n^2H_\lambda(C_m)+4n^2W_\lambda(C_m)+m^2H_\lambda(C_n)+4m^2W_\lambda(C_n)$$
$$+2\sum_{i=1}^{\lambda-1}\binom{\lambda}{i}\big(H_i(C_n)W_{\lambda-i}(C_m)+H_{\lambda-i}(C_n)W_i(C_m)\big),$$

$$H_\lambda(P_n\times P_m)$$
$$=n^2H_\lambda(P_m)+4n(n-1)W_\lambda(P_m)+m^2H_\lambda(P_n)$$
$$+4m(m-1)W_\lambda(P_n)$$
$$+2\sum_{i=1}^{\lambda-1}\binom{\lambda}{i}\big(H_i(P_n)W_{\lambda-i}(P_m)+H_{\lambda-i}(P_n)W_i(P_m)\big),$$

and

$$H_\lambda(P_n\times C_m)$$
$$=n^2H_\lambda(C_m)+4n(n-1)W_\lambda(C_m)$$
$$+m^2H_\lambda(P_n)+4m^2W_\lambda(P_n)$$
$$+2\sum_{i=1}^{\lambda-1}\binom{\lambda}{i}\big(H_i(P_n)W_{\lambda-i}(C_m)+H_{\lambda-i}(P_n)W_i(C_m)\big).$$

Remark 2.14. In the above theorem, if $\lambda=1$, then we obtain $D'(G_1\times G_2)$, which gives first derivatives formula Theorem 1 in [22] at $x=1$.

Now we obtain the relation between the generalized degree distance polynomial and Wiener-type invariant polynomial and the relation between the modified generalized degree distance polynomial and Wiener-type invariant polynomial for graphs.

Theorem 2.15. If G is a graph with n vertices and m edges, then

$$H_\lambda(G,x)=\sum_{u\neq v\in V(G)}\left(d_G(u)+d_G(v)-2\right)x^{d^\lambda(u,v)}+4m-2n+2W_\lambda(G,x).$$

Proof. By definition, we have

$$\begin{aligned}H_\lambda(G,x)&=\sum_{\{u,v\}\subseteq V(G)}\left(d_G(u)+d_G(v)\right)x^{d^\lambda(u,v)}\\&=\sum_{u\neq v\in V(G)}\left(d_G(u)+d_G(v)-2\right)x^{d^\lambda(u,v)}\\&\quad+\sum_{u=v\in V(G)}\left(d_G(u)+d_G(v)-2\right)x^{d^\lambda(u,v)}\\&\quad+2\sum_{\{u,v\}\subseteq V(G)}x^{d^\lambda(u,v)}\\&=\sum_{u\neq v\in V(G)}\left(d_G(u)+d_G(v)-2\right)x^{d^\lambda(u,v)}\\&\quad+4m-2n+2W_\lambda(G,x).\end{aligned}$$

This completes the proof. □

Theorem 2.16. If G is a graph with n vertices and m edges, then

$$H^*_\lambda(G,x)=\sum_{u\neq v\in V(G)}\left(d_G(u)-1\right)\left(d_G(v)-1\right)x^{d^\lambda(u,v)}+M_1(G)-4m+n+H_\lambda(G,x)-W_\lambda(G,x).$$

Proof. By definition, we have

$$\begin{aligned}&H^*_\lambda(G,x)\\&=\sum_{\{u,v\}\subseteq V(G)}\left(d_G(u)d_G(v)\right)x^{d^\lambda(u,v)}\\&=\sum_{u\neq v\in V(G)}\left(d_G(u)-1\right)\left(d_G(v)-1\right)x^{d^\lambda(u,v)}\\&+\sum_{u=v\in V(G)}\left(d_G(u)-1\right)\left(d_G(v)-1\right)x^{d^\lambda(u,v)}\\&+\sum_{\{u,v\}\subseteq V(G)}\left(d_G(u)+d_G(v)\right)x^{d^\lambda(u,v)}-\sum_{\{u,v\}\subseteq V(G)}x^{d^\lambda(u,v)}\\&=\sum_{u\neq v\in V(G)}\left(d_G(u)-1\right)\left(d_G(v)-1\right)x^{d^\lambda(u,v)}\\&+\sum_{u\in V(G)}d_G^2(u)-4m+n+H_\lambda(G,x)-W_\lambda(G,x)\\&=\sum_{u\neq v\in V(G)}\left(d_G(u)-1\right)\left(d_G(v)-1\right)x^{d^\lambda(u,v)}\\&+M_1(G)-4m+n+H_\lambda(G,x)-W_\lambda(G,x).\end{aligned}$$

This completes the proof. □

REFERENCES

[1] A. Hamzeh, A. Iranmanesh, S. Hossein-Zadeh and M. V. Diudea, "Generalized Degree Distance of Trees, Unicyclic and Bicyclic Graphs," *Studia Ubb Chemia*, *LVII*, Vol. 4, 2012, pp. 73-85.

[2] M. V. Diudea, I. Gutman and L. Jantschi, "Molecular Topology," Nova Science, Huntington, 2001.

[3] W. Imrich and S. Klavžar, "Product Graphs: Structure and Recognition," John Wiley & Sons, New York, 2000.

[4] I. Gutman and N. Trinajstic, "Graph Theory and Molecular Orbitals. Total π-Electron Energy of Alternant Hydrocarbons," *Chemical Physics Letters*, Vol. 17, No. 4, 1972, pp. 535-538. doi:10.1016/0009-2614(72)85099-1

[5] S. Nikolic, G. Kovacevic, A. Milicevic and N. Trinajstic, "The Zagreb Indices 30 Years after," *Croatica Chemica Acta*, Vol. 76, No. 2, 2003, pp. 113-124.

[6] T. Došlic, "Vertex-Weighted Wiener Polynomials for Composite Graphs," *Ars Mathematica Contemporanea*, Vol. 1, No. 1, 2008, pp. 66-80.

[7] I. Gutman, "A Property of the Wiener Number and Its Modifications," *Indian Journal of Chemistry*, Vol. 36A, No. 2, 1997, pp. 128-132.

[8] I. Gutman, A. A. Dobrynin, S. Klavžar and L. Pavlovic, "Wiener-Type Invariants of Trees and Their Relation," *Bulletin of the Institute of Combinatorics and Its Applications*, Vol. 40, No. 1, 2004, pp. 23-30.

[9] Y. Alizadeh, A. Iranmanesh and T. Došlic, "Additively Weighted Harary Index of Some Composite Graphs," *Discrete Mathematics*, Vol. 313, No. 1, 2013, pp. 26-34. doi:10.1016/j.disc.2012.09.011

[10] A. A. Dobrynin and A. A. Kochetova, "Degree Distance of a Graph: A Degree Analogue of the Wiener Index," *Journal of Chemical Information and Computer Sciences*, Vol. 34, No. 5, 1994, pp. 1082-1086. doi:10.1021/ci00021a008

[11] I. Gutman, "Selected Properties of the Schultz Molecular Topological Index," *Journal of Chemical Information and Computer Sciences*, Vol. 34, No. 5, 1994, pp. 1087-1089. doi:10.1021/ci00021a009

[12] I. Gutman, and S. Klavžar, "Wiener Number of Vertex-Weighted Graphs and a Chemical Applications," *Discrete Applied Mathematics*, Vol. 80, No. 1, 1997, pp. 73-81. doi:10.1016/S0166-218X(97)00070-X

[13] H. Hua and S. Zhang, "On the Reciprocal Degree Distance of Graphs," *Discrete Applied Mathematics*, Vol. 160, No. 7-8, 2012, pp. 1152-1163. doi:10.1016/j.dam.2011.11.032

[14] S. Hossein-Zadeh, A. Hamzeh and A. R. Ashrafi, "Extremal Properties of Zagreb Coindices and Degree Distance of Graphs," *Miskolc Mathematical Notes*, Vol. 11, No. 2, 2010, pp. 129-137.

[15] I. Tomescu, "Unicyclic and Bicyclic Graphs Having Minimum Degree Distance," *Discrete Applied Mathematics*, Vol. 156, No. 1, 2008, pp. 125-130. doi:10.1016/j.dam.2007.09.010

[16] I. Tomescu, "Some Extremal Properties of the Degree Distance of a Graph," *Discrete Applied Mathematics*, Vol. 98, No. 1-2, 1999, pp. 159-163. doi:10.1016/S0166-218X(99)00117-1

[17] A. Graovac and T. Pisanski, "On the Wiener Index of a Graph," *Journal of Mathematical Chemistry*, Vol. 8, No. 1, 1991, pp. 53-62. doi:10.1007/BF01166923

[18] B. E. Sagan, Y.-N. Yeh and P. Zhang, "The Wiener Polynomial of a Graph," *International Journal of Quantum Chemistry*, Vol. 60, No. 5, 1996, pp. 959-969. doi:10.1002/(SICI)1097-461X(1996)60:5<959::AID-QUA2>3.0.CO;2-W

[19] S. Klavžar, A. Rajapakse and I. Gutman, "The Szeged and the Wiener Index of Graphs," *Applied Mathematics Letters*, Vol. 9, No. 5, 1996, pp. 45-49. doi:10.1016/0893-9659(96)00071-7

[20] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, "The Hyper-Wiener Index of Graph Operations," *Computers & Mathematics with Applications*, Vol. 56, No. 5, 2008, pp. 1402-1407. doi:10.1016/j.camwa.2008.03.003

[21] M. Eliasi and A. Iranmanesh, "The Hyper-Wiener Index of the Generalized Hierarchical Product of Graphs," *Discrete Applied Mathematics*, Vol. 159, No. 8, 2011, pp. 866-871. doi:10.1016/j.dam.2010.12.020

[22] M. Eliasi and B. Taeri, "Schultz Polynomials of Composite Graphs," *Applicable Analysis and Discrete Mathematics*, Vol. 2, No. 2, 2008, pp. 285-296. doi:10.2298/AADM0802285E

[23] S. Hossein-Zadeh, A. Hamzeh and A. R. Ashrafi, "Wiener-Type Invariants of Some Graph Operations," *Filomat*, Vol. 29, No. 3, 2009, pp. 103-113. doi:10.2298/FIL0903103H

[24] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, "The First and Second Zagreb Indices of Some Graph Operations," *Discrete Applied Mathematics*, Vol. 157, No. 4, 2009, pp. 804-811. doi:10.1016/j.dam.2008.06.015

[25] F. Harary, "Graph Theory," Addison-Wesley, Reading, 1969.

[26] N. Trinajstic, "Chemical Graph Theory," CRC Press, Boca Raton, 1992.

