Synchronizing Hyperchaotic Circuits

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Abstract. With regard to possible applications to secure communications, the applicability of synchronizing hyperchaotic circuits with a single dynamical variable is discussed. Several specific examples are reviewed, including the fourth-order circuits with two positive Lyapunov exponents as well as the oscillator with a delay line characterized by multiple positive Lyapunov exponents.

INTRODUCTION

Synchronization of chaotic systems has attracted in this decade much attention (see [1–5] and references therein). The most comprehensive bibliography on this topic has been compiled by Guanrong Chen [6]. The last updated version of the bibliography includes more than 750 entries.

Chaotic synchronization is believed to have interesting applications in secure communications [7–9]. However, nearly all investigations in this field deal with low-dimensional chaotic systems characterized by only one positive Lyapunov exponent. Meanwhile, masking signals with comparatively simple chaos does not ensure sufficient security. In some cases extracting of the messages can be performed by means of common signal processing techniques [10].

For enhanced security hyperchaotic systems with two or more positive Lyapunov exponents seem to be promising. It was, however, believed that hyperchaos could not be synchronized by a single variable and one needed to transmit as many variables as there were positive Lyapunov exponents. If this was the case, the very use of hyperchaos would seem rather doubtful since most communication systems operate with just one signal.

Fortunately, several advanced methods have been recently described suggesting either scalar transmitted signal techniques [11,12] or even simpler strategy [13] employing a single dynamical variable to synchronize hyperchaotic systems.
The objective of this paper is to demonstrate that a large number of hyperchaotic circuits with more than one positive Lyapunov exponents described in literature so far can be synchronized with only one variable.

SINGLE DYNAMICAL VARIABLE STRATEGY

General remarks

The advantage of the scalar transmitted signal technique [12] is evident. From a practical point of view, however, this approach leads to some inconvenience since it requires direct access to all or at least several dynamical variables in the transmitter as well as in the receiver. In addition, large sets of parameters should be adjusted empirically for a specific hyperchaotic system.

Let us recall here that considering synchronization of hyperchaotic systems we regard possible applications to secure communications. In communications one deals, as a rule, with artificial systems including specially designed electronic circuits. In other words, we are not restricted to either existing natural oscillators or to mathematical models, such as the hyperchaotic Rössler equations considered in [12]. Instead, we can make free use of any hyperchaotic circuit that admits simple one-variable synchronization technique [13].

Given a hyperchaotic oscillator as a transmitter

\[
d\vec{v}^*/dt = \vec{F}(v^*), \tag{1}
\]

where \(\vec{v}^* = (v_1^*, v_2^*, ... v_s^*, ... v_m^*)\) is an \(m\)-dimensional \((m \geq 4)\) state vector and \(v_s^*\) is a selected accessible variable, in some cases the receiver can be synchronized by adding the one-variable signal (shown in braces \(\{\ldots\}\)) to the corresponding input:

\[
\begin{align*}
    d\vec{v}'/dt &= \vec{F}'(v'), \\
    dv_s'/dt &= F_s(\vec{v}) - \{k(v_s - v_s^*)\}. \tag{2}
\end{align*}
\]

Here \(\vec{v}' = (v_1, v_2, ..., v_{m-1})\), so that \(\vec{v} = (\vec{v}', v_s)\) is a corresponding \(m\)-dimensional state vector of the receiver, and \(k\) is a control parameter. The asterisks "\(*\)" denote the variables attributed to the transmitter.

Specific examples

In this subsection four specific examples are presented to illustrate the performance of the method. The examples include three fourth-order hyperchaotic circuits with two positive Lyapunov exponents also a delay line circuit with multiple positive Lyapunov exponents. The following notations are used:

\[
\begin{align*}
x &= U_{C1}/U_0, & y &= \rho I_{L1}/U_0, & z &= U_{C2}/U_0, & w &= \rho I_{L2}/U_0, & t &= t/\tau, \\
\rho &= \sqrt{L_1/C_1}, & \tau &= \sqrt{L_1C_1}, & a &= \rho/R, & b &= R/\rho, & \epsilon &= C_2/C_1, & \mu &= L_2/L_1.
\end{align*}
\]
Hyperchaotic circuit with a combined parallel-series resonance loop

The circuit shown in Figure 1 has been first proposed in [14], later investigated numerically [15] and studied experimentally [16]. General purpose diode can be used for the nonlinear element G.

\[ \frac{dx}{dt} = ax - y - w - \{k(x - x^*)\}, \]
\[ \frac{dy}{dt} = x, \]
\[ \frac{dz}{dt} = w - c(z - 1)H(z - 1), \]
\[ \mu \frac{dw}{dt} = x - z. \]

(3)

Here \( H(u) \) is the Heaviside function; \( H(u < 0) = 0, \ H(u \geq 0) = 1 \). The circuit has been confirmed to be a hyperchaotic one with two positive Lyapunov exponents, \( \lambda_1 = 0.44, \ \lambda_2 = 0.15 \) at \( a = 0.6, \ \varepsilon \approx 0.3, \ \mu = 0.2, \ c \to \infty \) [14]. Hyperchaos has been found to exist over wide range of the parameter \( a \), e.g. \( 0.25 < a < 0.8 \) with the other parameters fixed at \( c = 10, \ \varepsilon \approx \mu \approx 0.3 \) [15,16].

The receiver can be synchronized to the transmitter, \( x = x^* \) at \( k \gtrsim 0.3 \) [17]. It is interesting to note that robust synchronization can be achieved via other variables, \( y \) or \( z \) as well.

Related experiments with a passive nonlinear resonator used for the receiver and containing no negative resistor -R have been described in [18].

Hyperchaotic circuit with two parallel resonance loops

The circuit contains two parallel LC loops (see Figure 2) coupled by means of a diode [19]. In contrast to the previous circuit this oscillator includes a two-variable nonlinearity \( G(x, z) = c(x - z - 1)H(x - z - 1) \).
The oscillator exhibits hyperchaotic behavior with two positive Lyapunov exponents in the certain ranges of the parameter $a$ for a common values of the others, $c = 40$, and $\varepsilon \approx \mu \approx 0.3$ [19], e.g. $\lambda_1 = 0.12$ and $\lambda_2 = 0.07$. Synchronization of the circuit has been observed both numerically in the equations

\[
\frac{dx}{dt} = ax - y - c(x - z - 1)H(x - z - 1) - \{k(x - x^*)\},
\]
\[
\frac{dy}{dt} = x,
\]
\[
\varepsilon \frac{dz}{dt} = -w + c(x - z - 1)H(x - z - 1),
\]
\[
\frac{dw}{dt} = z.
\]

(4)

via $x$ and experimentally by adding an appropriate control current to the capacitor $C_1$. However, in contrast to the first example it seems that synchronization can not be achieved via other variables.

**Matsumoto-Chua-Kobayashi hyperchaotic circuit**

The first experimental observation of hyperchaotic oscillations has been described by Matsumoto, Chua and Kobayashi [20] in a circuit shown in Figure 3. In addition to the linear negative resistor $-R$ the circuit contains the second, N-type negative resistor implemented by means of a negative impedance converter shunted with two diodes [20].

**FIGURE 3.** Hyperchaotic circuit and I/V characteristic of the nonlinear element.
So, in contrast to the two above circuits this one contains two active elements. The equations for the oscillator can be given in the form

\[
\begin{align*}
\frac{dx}{dt} &= -y - N(x, z) \\
\frac{dy}{dt} &= x + by - \{k(y - y^*)\}, \\
\frac{dz}{dt} &= -w + N(x, z), \\
\frac{dw}{dt} &= z,
\end{align*}
\]

(5)

where \(N(x, z)\) is a two-variable three-segment function, thus more complicated one than \(G(z)\) or \(G(x, z)\) in the above examples. The authors of the paper [20] found two positive Lyapunov exponents, \(\lambda_1 = 0.24\) and \(\lambda_2 = 0.06\).

The synchronization phenomenon of this circuit has been studied numerically in [13]. Though the circuit is rather complicated and involves two active elements it can be still synchronized via one particular variable, namely \(y\). The largest transversal Lyapunov exponent, calculated for \(x = x^*, y = y^*, z = z^*, w = w^*\) becomes negative at \(k > 0.6\) [13] indicating synchronism between the transmitter and the receiver. Synchronization, however, can not be achieved via other variable, \(x, z\) or \(w\).

**Hyperchaotic circuit with delay line**

The circuit presented in Figure 4 is an electronic analog [21] of the physiological Mackey-Glass system [22].

![Hyperchaotic circuit and I/V characteristic of the nonlinear element.](image)

**FIGURE 4.** Hyperchaotic circuit and I/V characteristic of the nonlinear element.

The dynamics is given by the following delay differential equation

\[
\frac{dx}{dt} = -x + N(x, \tau) - \{k(x - x^*)\}, \quad N(x, \tau) = \frac{2x(t - \tau)}{1 + x^{10}(t - \tau)},
\]

(6)

Here \(x = U_C/U_0\), where \(U_0\) is the stationary output value derived from \(N(U_0) = U_0\), \(t = t/RC\), and \(\tau = T_{\text{del}}/RC\) is the dimensionless time and the dimensionless delay, respectively. Though the circuit is described by one-variable equation, it can be considered as an infinite-dimensional one [22]
because of the time delay $\tau$. The number of positive Lyapunov exponents increases with the parameter $\tau$. For example, at $\tau=6$ there are three positive Lyapunov exponents, $\lambda_1 = 0.05$, $\lambda_2 = 0.033$, and $\lambda_3 = 0.01$ [23]. Nevertheless, the receiver can be synchronized via single variable, as given in Eq. (6) provided $k > 0.65$. The number of positive Lyapunov exponents can be increased further by means of the delay $\tau$ [22], meanwhile synchronization is still easily achieved with only one variable $x$ as shown both numerically and experimentally in [23].

REFERENCES